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ON P-KRULL MONOIDS

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Abstract: In this note we introduced a P-Krull monoid, which is a weak but parallel to Krull monoid, we also characterized a P-Krull monoid as a BFM.

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1. Introduction and Preliminaries

A monoid H is called a valuation monoid if for all $a, b \in H$, either $a \mid_H b$ or $b \mid_H a$ [3, Definition 15.1]. Similarly H is a pseudo-valuation monoid if $x \in G \backslash H$ and $a \in H \backslash H^{\times}$(where $H^{\times}$ is a set of invertible elements of H) implies $x^{-1}a \in H$ [3, Definition 16.7]. An integral domain R is called a pseudo valuation domain if it is pseudo-valuation monoid and vice versa. For the definitions and terminologies one may consult [3].

For a monoid H, $H^* = H \backslash \{0\}$ and $a, b \in H$ are associates if $a \mid_H b$ and $b \mid_H a$. Associates of 1 in H are called units (invertible elements) and the set of units of H is denoted by $H^{\times}$. Furthermore H is said to be reduced if $H^{\times} = \{1\}$. Thus $H^{\times}$ is a subgroup of H and we can consider the quotient monoid $H/H^{\times}$ which is obviously reduced and it is denoted by H_{red}. H is said to be a groupoid if H^* is a group (equivalently: every nonzero element of H is invertible, or $H^* = H^{\times}$). We have a quotient groupoid of a cancellative

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monoid H in the place of quotient field as in integral domain. By a quotient groupoid of H we mean a groupoid $G(H)$ such that $H \subset G(H)$ is a submonoid and $G(H) = \{c^{-1}h : h \in H \text{ and } c \in H^*\}$.

An ideal system r of a monoid H is a map on $P(H)$, the power set of H, defined as $X \mapsto X_r$ such that for all $X, Y \in P(H)$ and $c \in H$, the following conditions hold:

(1) $X \cup \{0\} \subseteq X_r$,
(2) $X \subseteq Y_r$ implies $X_r \subseteq Y_r$,
(3) $cH \subseteq \{c\}_r$, and
(4) $(cX)_r = cX_r$.

An ideal I is an $r-$ideal if $I = I_r$ and is $r-$finitely generated if $I = J_r$ for a finitely generated ideal J of H.

An r-ideal $M \in I_r(H)$ is called r-maximal if $M \neq H$ and there is no r-ideal J such that $M \subseteq J \subseteq H$ and monoid H is called r-local, if H possesses exactly one r-maximal r-ideal [3, Definitions 6.4, 6.5]. Let S be a monoid, r a finitary ideal system on S and $G(H)$(a quotient groupoid of S) S is called r-closed if $(J : J) = S$ for all nonzero r-finitely generated r-ideal J of S, and S is called root closed, if for all $x \in G(H)$ and $n \geq 1$, $x^n \in S$ implies $x \in S$ [4, Page 2] .

Throughout by a monoid we mean a commutative cancellative semigroup having identity, with adjoined zero and the semigroup operation is represented by ordinary multiplication. By [3] a zero element 0 of a monoid H with the property that $0x = 0$; yet $xy = 0$ implies $x = 0$ or $y = 0$, $x, y \in H$.

In this note first, we construct the pseudo-valuation maps and define a P-Krull monoid with the help of these maps.

2. P-Krull Monoids

Let H be a cancellative monoid and $G(H)$ be its quotient groupoid (i.e. $G(H) = \{c^{-1}a : a \in H \text{ and } c \in H^*\}$ [3, Definition 4.2]) with partial order $\leq$ on $G(H)/H^\times$ by $xH^\times \leq yH^\times$ if $x^{-1}y \in H$, where $x \in G(H)\backslash H$ and $y \in H\backslash H^\times$. Hereafter we consider any partially ordered group G with the property that each $g \in G$, either $g \geq 0$ or $g < h$ for all $h \in G$ with $h > 0$ and we denote such a group by G°. If G° is totally ordered, then clearly H is a valuation monoid.

Let $G(H)$be a quotient groupoid of a cancellative monoid H and G° is a partially ordered group (having property that each $g \in G$, either $g \geq 0$ or $g < h$ for all $h \in G$ with $h > 0$). We initiate with the following definition.

Definition 1. Let ω: $G(H) \to G$ be an onto map, which has the following properties, for $x, y \in G^*$;

(a) $\omega(xy) = \omega(x) + \omega(y)$.
(b) $\omega(x) < \omega(y)$ implies $\omega(x + y) = \omega(x)$.
(c) $\omega(x) = g \geq 0$ or $\omega(x) < \omega(y) = h$, where g, $h \in G$ and $h > 0$.

Since a monoid H is contained in $G(H)$, so definition1 (b) reflects that $H \backslash H^{\times}$ is an $r-$ideal of H and is $r-$local. Also if r is a finitary weak ideal system then H is $r-$local and $H \backslash H^{\times}$ is the unique $r-$maximal $r-$ideal of H, as proved in [2, Page 180] for domains. Moreover condition (c) in definition1 plays an important role and it induces a specific property in G, now we will denote G by G°(pseudo-value group of a monoid S). Hereafter we call ω, the pseudo-valuation map and clearly $H_{\omega} = \{x \in G(H) : \omega(x) \geq 0\}$, a pseudo-valuation monoid.

Remark 1. Let G be a partially ordered group and X be a set of bounded elements of G, then X is convex subsemigroup of G. The subgroup $B(G)$ of G generated by X is a convex subgroup of G. If G is lattice ordered, then $B(G)$ is a sublattice subgroup of G [2, Proposition 19.10].

Remark 2. Let P be a strongly r-prime ideal in a monoid S and G° be a r-group of divisibility of S, then there is one to one correspondence between strongly r-prime ideal and a convex subset X in G° which generate the convex subgroup $B(G^{\circ})$ as in remark 1. By the definition of strongly r-prime ideal (i.e. P is strongly r-prime if and only if $x^{-1}P \subset P$ whenever $x \in G^{*} \backslash S$), we have convex set C and for $x \in G^{\circ} \backslash C$ such that $-x + C \subset C$. We call such C a strongly convex set.

In the following we adjust our terminology which shall be helpful for constructing P-Krull domain.

Definition 2. (a) A family Ω of pseudo valuations of the quotient groupoid $G(H)$ is said to be of finite P-character if for every $x \in G(H), x \neq 0$, the set $\{\omega \in \Omega : w(x) \neq 0\}$ is finite.

(b) Corresponding to $\omega \in \Omega$, the pseudo-valuation monoid S_{ω} with a maximal ideal(which is strongly r-prime) P, the semigroup $S = \cap_{\omega \in \Omega} S_{\omega}$ is said to be defined by Ω, and $P \cap S$ is a (strongly) r-prime ideal in S, called centre of ω on S and we denote it by $Z(\omega)$. If $S_{\omega} = S_{Z(\omega)}$, then ω is said to be essential pseudo valuation for S.

(c) Rank of a pseudo-valuation monoid S is the rank of pseudo-value group of S, i.e., G°. Rank of pseudo-value group G° depend upon the existence of ordinal type of set of proper strongly convex sets which are described in remark2 under inclusion $\subseteq$.

Remark 3. Let $F = \{V_{\lambda}\}_{\lambda \in \Lambda}$ be the family of pseudo-valuation over monoids of S such that each V_λ is root closed and $S = \cap V_\lambda$. Clearly S is a root closed pseudo-valuation monoid.

We also define.

Definition 3. Rank of a pseudo-valuation monoid S is the rank of pseudo-value group of S (i.e., Rank of a pseudo-valuation monoid S is the rank of pseudo-value group of S i.e., G°). Rank of pseudo-value group G° depend upon the existence of ordinal type of set of proper strongly convex sets which are described in remark2 under inclusion $\subseteq$. If pseudo-value group $G^\circ$is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ then it is P-rank one discrete.

Remark 4. After above definition we will call now V_λ pseudo-discrete valuation over monoids.

Finally we define a P-Krull monoid.

Definition 4. Let $F = \{V_\lambda\}_{\lambda \in \Lambda}$ be the family of pseudo discrete valuation (P-discrete valuation) over monoids of a root closed monoid S such that each V_λ is root closed,

(a) $S = \cap\, V_\lambda$.

(b) Each V_λ is of finite P-character (i.e. every non-zero element is contain in at most finitely many maximal ideals of V_λ $\forall\ \lambda \in \Lambda$).

(c) Each V_λ is P-rank one discrete.

(d) Each V_λ is essential for S (pseudo-valuation monoid is essential for S if it is quotient groupoid of S).

Example 1. Let S be Noetherian root closed monoid then it has discrete valuation over monoids as for domains in [1]. Let these valuation over monoids of the form $B = G(H) + M$ where M is r-maximal ideal of S. Also $G_1 \subset G_2 \subset \ldots \subset G(H)$ where each G_i is root closed in $G(H)$. Let G_1 be a root closed sub quotient groupoid of $G(H)$, then the submonoid $S_1 = G_1 + M$, clearly S_1 is a Noetherian root closed pseudo-valuation monoid and can be written as an intersection of $G_i + M$ where $i > 1$. Clearly S_1 is a P-Krull monoid.

Remark 5. P-Krull monoid may or may not be a pseudo-valuation monoid (PVM).

The new established monoid in Definition4 is a weak but parallel to a Krull monoid as.

Proposition 1. *Krull monoid having every prime ideal a strongly prime is a P-Krull monoid (a PVM).*

Proof. Let S be a Krull monoid no doubt it is integrally closed hence root closed, also each Krull monoid can be written as an intersection of discrete valuation over monoids. As discrete valuation over monoids implies valuation over monoids which implies pseudo-valuation over monoids thus definition4(1) is satisfied, similarly easy to prove (2), (3) and (4), which completes the proof. □

Remark 6. In general from proposition1, every Krull monoid (having each of its prime ideal is a strongly prime)⇒P-Krull monoid (PVM) but converse is not true.

Since a P-Krull monoid (PVM) is clearly an HFD, thus we characterize P-Krull monoid (not a PVM) by generalizing [5, Proposition 2] as.

Proposition 2. *Let S be an atomic P-Krull monoid (not a PVM) and $X^{(1)}(S)$, the set of height one strongly prime ideals with family $\{v_P : P \in X^{(1)}(S)\}$ of essential pseudo-discrete pseudo-rank one valuation monoids. Then $V : S^* \to \mathbb{Z}_+$ by $V(x) = \sum v_P(x)$, where $x \in S^*$ characterize P-Krull monoid as a BFD.*

Proof. Let us define

$$V \quad : \quad S^* \to \mathbb{Z}_+ \text{ by } V(x) = \sum v_P(x), \text{ where } x \in S^*.$$

$$\text{Thus } V(x) \quad = \quad n \geq 1 \text{ if and only if } \; xS = (P_1 ... P_n) \; \text{ for some } \; P_i \in X^{(1)}(S).$$

Then V defines a length function on S such that $V(x) = 1$ if and only if x is irreducible. Note that $L_S(x) \leq V(x)$ for each $x \in S^*$, so we can say there exist a bound on the factorization into irreducibles of the elements of the P-Krull monoid S (not a PVD). Hence a P-Krull monoid (not a PVM) is a BFM. □

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