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GENERALIZED HYERS-ULAM-RASSIAS THEOREM OF QUADRATIC FUNCTIONAL EQUATION IN MENGER PROBABLISTIC NORMED SPACES

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Abstract: In this paper, the authors investigate the generalized Hyers-Ulam-Rassias stabitity of a quadratic functional equation $f(2x+y)-4f(x)-f(y) = f(x+y)-f(x-y)$ in Menger Probablistic Normmed Spaces.

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1. Introduction

In 1940, Ulam gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. The stability problem of functional equations originated from a question of Ulam [27] concerning the stability of group homomorphisms.

In 1941, Hyers [8] considered the case of approximately additive functions $f : X \longrightarrow Y$ such that

$$\|f(x+y)-f(x)-f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon > 0$, where X and Y are Banach spaces. Then

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there exists a unique additive function $A : X \longrightarrow Y$ such that

$$\|f(x) - A(x)\| \leq \varepsilon$$

for all $x \in X$.

Aoki [2] and Rassias [14] provided a generalization of the Hyers theorem for additive and linear functions, respectively, by allowing the Cauchy difference to be unbounded.

Theorem 1.1. *Let $f : X \to Y$ be a function from a normed vector space X into a Banach space Y subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in X$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive function $A : X \to Y$ defined by $A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p \tag{1.2}$$

for all $x \in X$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each fixed $x \in X$ the function $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then A is linear.

The above Theorem has provided a lot of influence during the last three decades in the development of a generalization of the Hyers–Ulam stability concept. This new concept is known as generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations (see [4, 9]).

In 1994, a generalization of Rassias theorem was obtained by Gavruta [7] by replacing the bound $\epsilon\left(\|x\|^p + \|y\|^p\right)$ by a general control function $\varphi(x, y)$. Several stability results have been recently obtained for various equations, also for mapping with more general domains and ranges (see [4, 9, 11]).

During the last three decades a number of papers and research monographs have been published on various generalizations and applications of the generalized Hyers–Ulam stability to a number of functional equations and functions (see [3]–[10], [12, 13] and [17]–[23]).

Menger prposed the transferring some processes studied in applied sciences represented by differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics, theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology, medicine and so on.

PN spaces were first defined and their definition was generalized by Serstnev in 1963 (see [26]). We recall and apply the definition of probabilistic space briefly as given in [24]

Definition 1.1. A Prpbabilistic Normed space (briefly, PN space) is a quadruple $(X, \nu, \tau, \tau*)$, where X is a real vector space, τ and $\tau*$ are continuous triangle functions with $\tau \leq \tau*$ and ν is a mapping (the prpbabilistic norm) from V into Δ^+, such that for every choice of p and q in V the following hold:

(N1) $\nu_p = \epsilon_0$ iff $p = \theta$ (θ is the null vector in X);

(N2) $\nu_{-p} = \nu_p$;

(N3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;

(N4) $\nu_p \leq \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p}$ for every $\lambda \in [0, 1]$.

A PN space is called a Serstnev space if it satisfies (N1), (N3) and the following condition:

$$\nu_\alpha p(x) = \nu_p \left(\frac{x}{|\alpha|} \right),$$

holds for every $\alpha \neq 0 \in \Re$ and $x > 0$. When T is a continuos t-norm T such that $\tau = \prod_T$ and $\tau^* = \prod_T *$, the PN space $(X, \nu, \tau, \tau*)$ is called Menger PN space (briefly, MPN space), and is denoted by (X, ν, τ).

Let (X, ν, τ) be an MPN space let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$.

The sequence x_n in MPN space (X, ν, τ) is called Cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exists some n_0 such that $\nu(x_n - x_m)(\delta) > 1 - \epsilon$ for all $m, n \geq n_0$.

Clearly, every convergent sequence in a MPN sapce is Cauchy. If eqch Cauchy sequence is convergent in a MPN space (X, ν, τ), then (X, ν, τ) is called Menger Probabiistic Banach space (briefly, MPB space).

Recently, the stability of functional equations on PN spaces and MPN spaces have been investigated by some authors; (see [6, 25]). and refrences therein. In this paper, we invetigate the stability of qudratic functional equations on Serstnev probabilistic normed space endowed with $\prod_M$ triangle function.

2. Main Results

We begin our work with uniform version of the Hyers-Ulam-Rassias stability in a Serstnev PN space in which we uniformly approximate a uniform quadratic mapping.

Theorem 2.1. *Let X be a linear space and $(\gamma, \nu, \prod_M)$ be a Serstnev PB space. Let $\varphi : X \times X \to [0, \infty)$ be a control function such that*

$$\widetilde{\varphi_n}(x, 0) = \left\{4^{-n-1}\varphi(2^n x, 0)\right\} \quad (x \in X) \tag{2.1}$$

converges to zero. Let $f : X \to \gamma$ be a uniformly approximately quadratic function with respect to φ in the sense that

$$\lim_{t\to\infty} \nu\left(f(2x+y) - 4f(x) - f(y) - f(x+y) + f(x-y)\right)(t\varphi(x,y)) = 1 \tag{2.2}$$

uniformly on $X \times X$. Then $T(x) := \lim_{n\to\infty} 2^{-n} f(2^n x)$ for any $x \in X$ exists and defines an additive mapping $T : X \to \gamma$ such that if for some $\delta > 0, \alpha > 0$

$$\nu\left(f(2x+y) - 4f(x) - f(y) - f(x+y) + f(x-y)\right)(\delta\varphi(x,0)) > \alpha \quad (x, y \in X) \tag{2.3}$$

then

$$\nu\left(T(x) - f(x)\right)(\delta\widetilde{\varphi_n}(x,0)) > \alpha \quad (x \in X). \tag{2.4}$$

Proof. Given $\epsilon > 0$, by (2.2) we can choose some t_0 such that

$$\nu\left(f(2x+y) - 4f(x) - f(y) - f(x+y) + f(x-y)\right)(t\varphi(x,0)) > 1 - \epsilon \tag{2.5}$$

for all $x, y \in X$ and all $t \geq t_0$. Subsituting $y = 0$ in (2.5) we obtain

$$\nu\left(f(2x) - 4f(x)\right)(t\varphi(x,0)) > 1 - \epsilon \tag{2.6}$$

and replacing x by $2^n x$, we get

$$\nu\left(4^{-n-1}f(2^{n+1}x) - 4^{-n}f(2^n x)\right)\left(t4^{-n-1}\varphi(2^n x, 0)\right) > 1 - \epsilon. \tag{2.7}$$

Allowing to a nonincreasing subequence, if necessary, we assume that

$$\left\{4^{-n-1}\varphi(2^n x,0)\right\}$$

is nonincreasing.

Thus for each $n > m$ we have

$$\begin{aligned}
&\nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(t4^{-m-1}\varphi(2^m x,0)) \\
&\quad = \nu\sum_{k=m}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x))(t4^{-m-1}\varphi(2^m x,0)) \\
&\quad \geq \prod_M(\nu(4^{-m}f(2^m x) - 4^{-m-1}f(2^{m+1}x)), \\
&\qquad \nu(\sum_{k=m+1}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x)))(t4^{-m-1}\varphi(2^m x,0)) \\
&\quad \geq \prod_M(1-\epsilon; \prod_M((\nu(4^{-m-1}f(2^{m+1}x) - 4^{-m-2}f(2^{m+2}x)), \\
&\qquad \nu(\sum_{k=m+2}^{n-1}(4^{-k}f(2^k x) - 4^{-k-1}f(2^{k+1}x)))(t4^{-m-2}\varphi(2^{m+1}x,0))) \\
&\quad \geq 1-\epsilon. \qquad (2.8)
\end{aligned}$$

The convergence of (2.1) implies that for given $\delta > 0$ there is $n_0 \in N$ such that

$$t_0 4^{-n-1}\varphi(2^n x,0) < \delta \quad \forall n \geq n_0. \qquad (2.9)$$

Thus by (2.8) we deduce that

$$\begin{aligned}
&\nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(\delta) \\
&\quad \geq \nu(4^{-m}f(2^m x) - 4^{-n}f(2^n x))(t_0 4^{-m-1}\varphi(2^m x,0)) \geq 1-\epsilon \qquad (2.10)
\end{aligned}$$

for each $n \geq n_0$. Hence $4^{-n}f(2^n x)$ is a Cauchy sequence in γ. Since $(\gamma, \nu, \prod_M)$ is complete,this sequence converges to some $T(x) \in \gamma$. Therefore we can define a mapping $T : X \to \gamma$ by $T(x) := \lim_{n\to\infty} 4^{-n}f(2^n x)$, namely,for each $t > 0$, and $x \in X$,

$$\nu(T(x) - 4^{-n}f(2^n x))(t) = 1. \qquad (2.11)$$

Next, let $x, y \in X$. Temporarily fix $t > 0$ and $0 < \epsilon < 1$. Since $4^{-n-1}\varphi\left(2^n x, 0\right)$ converges to zero, there is some $n_1 > n_0$ such that $t_0\varphi\left(2^n x, 0\right) < t2^{n+1}$ for all $n > n_1$, we have

$$\nu\left(T(2x+y) - 4T(x) - T(y) - T(x+y) + T(x-y)\right)(t)$$

$$
\begin{aligned}
&\geq \prod_M(\prod_M(\nu(T(2x+y)-4^{-n-1}f(2^{n+1}(2x+y)))(t), \nu 4(T(x) \\
&-4^{n-1}f(2^{n+1}x))(t)), \\
&\prod_M(\nu(T(y)-4^{n-1}f(2^{n+1}y))(t), \nu(T(x+y)-4^{-n-1}f(2^{n+1}(x+y)))(t)), \\
&\prod_M(\nu(T(x-y)-4^{-n-1}f(2^{n+1}(x-y)))(t), \\
&\nu(f(2^{n+1}(2x+y))-4f(2^{n+1}x)-f(2^{n+1}y)-f(2^{n+1}(x+y)) \\
&+(2^{n+1}(x-y)))(4^{n+1}t))), \qquad (2.12)
\end{aligned}
$$

we have

$$
\begin{aligned}
\lim_{n\to\infty} \nu\left(T(2x+y)-4^{-n-1}f(2^{n+1}(2x+y))\right)(t)=1, \\
\lim_{n\to\infty} \nu\left(T(x)-4^{-n-1}f(2^{n+1}x)\right)(t)=1, \\
\lim_{n\to\infty} \nu\left(T(y)-4^{-n-1}f(2^{n+1}y)\right)(t)=1, \qquad (2.13) \\
\lim_{n\to\infty} \nu\left(T(x+y)-4^{-n-1}f(2^{n+1}(x+y))\right)(t)=1, \\
\lim_{n\to\infty} \nu\left(T(x-y)-4^{-n-1}f(2^{n+1}(x-y))\right)(t)=1,
\end{aligned}
$$

and by (2.5) and large enough n, we have

$$
\begin{aligned}
&\nu(f(2^{n+1}(2x+y))-4f(2^{n+1}x)-f(2^{n+1}y)-f(2^{n+1}(x+y)) \\
&\quad+f(2^{n+1}(x-y)))(4^{n+1}t) \\
&\geq \nu(f(2^{n+1}(2x+y))-4f(2^{n+1}x)-f(2^{n+1}y) \\
&\quad-f(2^{n+1}(x+y)t)+f(2^{n+1}(x-y))(t_0\varphi(2^nx,0))\geq 1-\epsilon. \qquad (2.14)
\end{aligned}
$$

Thus

$$
\nu\left(T(2x+y)-4T(x)-T(y)-T(x+y)+T(x-y)\right)(t)\geq 1-\epsilon, \quad \forall t>0,\ 0<\epsilon<1. \qquad (2.15)
$$

To end the proof, let us take for some positive δ and α, (2.3) holds. Let $x\in X$. Setting $m=0$ and $\alpha=1-\epsilon$ in (2.10), we get

$$
\nu(f(2^nx)-4^nf(x))(\delta)\geq\alpha \qquad (2.16)
$$

for all positive integers $n\geq n_0$. For large enough n, we have

$\nu(f(x)-T(x))(\delta 4^{-n-1}\varphi(2^nx,0)$

$$\geq \prod_M \left\{\nu(f(x) - 4^{-n}f(2^n x)), \nu(4^{-n}f(2^n x) - T(x))\right\} (\delta 4^{-n-1}\varphi(2^n x, 0)) \geq \alpha \tag{2.17}$$

which implies

$$\nu(T(x) - f(x))(\delta\widetilde{\varphi_n}(x, 0)) > \alpha. \tag{2.18}$$

□

Corollary 2.1. *Let X be a linear space defined by* (2.1) *and* $(\gamma, \nu, \prod_M)$ *a Serstnev PB space Let $\varphi : X \times X \to [0, \infty)$ be a control function satisfying* (2.2). *Let $f : X \to \gamma$ be a uniformly approximately quadratic function with respect to φ.Then there is a unique quadratic mapping $T : X \to \gamma$ such that*

$$\lim_{n\to\infty} \nu(f(x) - T(x))(t\widetilde{\varphi_n}(x, 0)) \tag{2.19}$$

uniformly on X.

Proof. The existence of uniform limit (2.19) immediately follows from Theorem (2.1). It remans to prove the uniqueness assertion.

Let S be another quadratic mapping satisfying (2.19). Fix $c > 0$. Given $\epsilon > 0$, by (2.19) for T and S,we can find some $t_0 > 0$ such that

$$\begin{aligned} &\nu(f(x) - T(x))(t\widetilde{\varphi_n}(x, 0)) > 1 - \epsilon, \\ &\nu(f(x) - S(x))(t\widetilde{\varphi_n}(x, 0)) > 1 - \epsilon. \end{aligned} \tag{2.20}$$

for all $x \in X$ and $t \geq t_0$. Fix for some $x \in X$ and find some integer n_0 such that

$$t_0 4^{-n}\varphi(2^{n+1}x, 0) > c \forall n \geq n_0. \tag{2.21}$$

Then we have

$$\begin{aligned} &\nu\left(S(x) - T(x)\right)(c) \\ &\quad\geq \prod_M \left\{\nu\left(4^{-n}f(2^n x) - T(x)\right), \nu\left(S(x) - 4^{-n}f(2^n x)\right)\right\}(c) \\ &\quad= \prod_M \left\{\nu\left(f(2^n x) - T(2^n x)\right), \nu\left(S(2^n x) - f(2^n x)\right)\right\}(2^n c) \\ &\quad\geq \prod_M \left\{\nu\left(f(2^n x) - T(2^n x)\right), \nu\left(S(2^n x) - f(2^n x)\right)\right\}(2^n c)\left(t_0 4^{-n}\varphi\left(2^{n+1}x, 0\right)\right) \\ &\quad\geq 1 - \epsilon. \end{aligned} \tag{2.22}$$

It follows that $\nu\left(S(x) - T(x)\right)(c) = 1$ for all $c > 0$. Thus $T(x) = S(x)$ for all $x \in X$. □

Theorem 2.2. *Let X be a linear space. Let $(Z, \omega, \prod_M)$ be a Serstnev MPN space. Let $\psi : X \times X \to Z$ be a function such that for all $0 < \alpha < 4$,*

$$\omega(\psi(2x, 0))(t) \geq \omega(\psi(x, 0))(t) \tag{2.23}$$

for all $x, y \in X$ and $t > 0$. Let $(\gamma, \nu, \prod_M)$ be a Serstnev PB space and let $f : X \to \gamma$ be a ψ approximately quadratic mapping in the sense that

$$\nu(f(2x + y) - 4f(x) - f(y) - f(x + y) + f(x - y))(t) \geq \omega(\psi(x, 0))(t) \tag{2.24}$$

for eqch $t > 0$ and $x, y \in X$. Then there exists unique quadratic mapping $T : X \to \gamma$ such that

$$\nu(f(x) - T(x))(t) \geq \omega\left(\frac{1}{4}\psi(x, 0)(t)\right) \tag{2.25}$$

where $x \in X$ and $t > 0$.

Proof. Put $y = 0$ in (2.24) we get

$$\nu(f(2x) - 4f(x))(t) \geq \omega(\psi(x, 0))(t) \quad (x \in X, t > 0). \tag{2.26}$$

Using (2.23) and using induction on n, we obtain

$$\omega(\psi(2^n x, 0))(t) \geq \omega(\alpha^n (x, 0))(t) \tag{2.27}$$

for all $x \in X$ and $t > 0$. Replacing x by $4^{n-1}x$ in (2.26) and using (2.27) we get

$$\nu(f(2^n x) - 4f(2^{n-1}x))(t) \geq \omega\left((\alpha^{n-1}\psi(x, 0)\right)(t) \tag{2.28}$$

for all $x \in X$ and $t > 0$. It follows from (2.28), we get

$$\nu(4^{-n}f(2^n x) - 4^{-n+1}f(2^{n-1}x))(4^{-n}t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x, 0)\right)(\alpha^{-n}t) \tag{2.29}$$

whence

$$\nu\left(4^{-n}f(2^n x) - 4^{-n+1}f(2^{n-1}x)\right)\left(\left(\frac{\alpha^n}{4^n}\right)t\right) \quad \geq \quad \omega\left(\frac{1}{\alpha}\psi(x, 0)\right)(t), \tag{2.30}$$

for all $n > m \geq 0, x \in X$ and $t > 0$. So

$$\nu(4^{-n}f(2^n x) - 4^{-m}f(2^m x))\left(\left(\frac{\alpha^{m+1}}{4^{m+1}}\right)t\right)$$

$$= \nu\left(\sum_{n}^{k=m+1} 4^{-k}f(2^kx) - 4^{-k+1}f(2^{k-1}x)\right)\left(\left(\frac{\alpha^{m+1}}{4^{m+1}}\right)t\right)$$
$$\geq \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(t) \tag{2.31}$$

whence

$$\nu(4^{-n}f(2^nx) - 4^{-m}f(2^mx))(t) \geq \omega\left(\left(\frac{1}{\alpha}\right)\psi(x,0)\right)\left(\left(\frac{\alpha^{m+1}}{4^{m+1}}\right)t\right) \tag{2.32}$$

for all $n > m \geq 0, x \in X$ and $t > 0$.Fix $x \in X$. By

$$\lim_{s\to\infty} \omega\left(\frac{1}{\alpha}\psi(x,0)\right)(s) = 1 \tag{2.33}$$

we reduce that $4^{-n}f(2^nx)$ is a Cauchy sequence in $(\gamma, \nu, \prod_M)$. Since $(\gamma, \nu, \prod_M)$ is complete, this sequence converges to some point $T(x) \in \gamma$. It follows from (2.24) that

$$\begin{aligned}\nu(f(2^n(2x+y)) - 4f(2^nx) - f(2^ny) - f(2^n(x+y)) + f(2^n(x-y))(t)\\ \geq \omega(\psi(2^nx, 2^ny))(t)\\ \geq \omega(\alpha^n\psi(x,y))(t)\\ \geq \omega(\psi(x,y))(\alpha^{-n}t)\end{aligned} \tag{2.34}$$

whence

$$\begin{aligned}\nu(4^{-n}f(2^n(2x+y)) - 4^{-n}4f(2^nx) - 4^{-n}f(2^ny) - 4^{-n}f(2^n(x+y))\\ + 4^{-n}f(2^n(x-y)))(t) \geq \omega(\psi(x,y))\left(\left(\frac{4}{\alpha}\right)^n t\right),\end{aligned} \tag{2.35}$$

we have

$$\begin{aligned}&\nu(T(2x+y) - 4T(x) - T(y) - T(x+y) + T(x-y))(t)\\ &\geq \prod_M(\prod_M\{\nu(T(2x+y) - 4^{-n}f(2^n(2x+y))), \nu(4(T(x) - 4^{-n}f(2^nx)))(t),\\ &\quad \prod_M(\nu(T(y) - 4^{-n}f(2^ny)), \nu(T(x+y) - 4^{-n}f(2^n(x+y)))(t)),\\ &\quad \prod_M(\nu(T(x-y) - 4^{-n}f(2^n(x-y))), \nu(4^{-n}f(2^n(2x+y)) -\\ &\qquad 4^{-n}f(2^nx) - 4^{-n}f(2^ny) - 4^{-n}f(2^n(x+y)) + 4^{-n}(2^n(x-y))))(t).\end{aligned} \tag{2.36}$$

By (2.35) and the fact that

$$\lim_{n\to\infty} \nu(T(z) - 4^{-n}f(2^n z)) = 1 \tag{2.37}$$

for all $z \in X$ and $r > 0$, each term on the rignt-hand side tends to 1 as $n \to \infty$. Hence

$$\nu(T(2x+y) - 4T(x) - T(y) - T(x+y) + T(x-y)) = 1. \tag{2.38}$$

By (N1), it means that

$$T(2x+y) - 4T(x) - T(y) = T(x+y) - T(x-y) \tag{2.39}$$

Furthermore, let $x \in X$ and $t > 0$. Using (2.32) with $m = 0$ we get

$$\begin{aligned} &\nu(T(x) - f(x))(t) \\ &\quad \geq \prod_M \left\{\nu(T(x) - 4^{-n}f(2^n x), \nu(4^{-n}f(2^n x) - f(x))\right\}(t) \\ &\quad \geq \prod_M \left\{\nu(T(x) - 4^{-n}f(2^n x), \omega\left(\frac{1}{4}\psi(x,0)\right)\right\}(t). \end{aligned} \tag{2.40}$$

Hence

$$\begin{aligned} &\nu(T(x) - f(x))(t) \\ &\quad \geq \prod_M \left\{\nu(T(x) - 4^{-n}f(2^n x), \omega\left(\frac{1}{4}\psi(x,0)\right)\right\}(t) \\ &\quad \geq \omega\left(\frac{1}{4}\psi(x,0)\right)(t). \end{aligned} \tag{2.41}$$

The uniqueness of T can be proved in a similar manner as in the proof of corollary (2.1). □

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