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EXISTENCE OF PERIODIC WAVE SOLUTION IN SPRING-BLOCK MODEL WITH SLIP-DEPENDENT FRICTION

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Abstract: The global conditions under which a spring-block model coupled with non smooth slip-dependent friction force depicts a seismic fault are studied. Existence and uniqueness conditions of the periodic orbits solutions were proved using Poincare-Bendixsom theorem and Lyapounov-Schmidt reduction technique. The insight gained has important implications for the understanding of earthquakes and other dissipative driven systems.

AMS Subject Classification: 86A17
Key Words: spring-block, earthquakes, friction, slip-dependent, existence

1. Introduction

Earthquakes display a rich set of complex behaviors, and a wealth of quantitative data exists describing these behaviors. Yet the origin of these behaviors remains largely unexplained, and finding a set of well behaved mathematical equations which can reproduce all the behaviors is still an as yet unachieved goal. However, simple spatially extended dynamical models have shown that a remarkably rich set of earthquake-like behaviors can arise from slip-dependent

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non-smooth friction applied to the spring-block models [1]. It is generally believed that one of the basic mechanisms of earthquakes is frictional instability arising at the interface of tectonic plates [2]. This viewpoint is supported by the manifestation of static-dynamic friction duality causing the jerking motion that often occurs when two surfaces are sliding over each other (also known as the stick-slip phenomenon [3]).

A mathematical description of the stick-slip motion is a long-standing problem in dynamical systems, since it requires a description of the dynamics at very different spatial and temporal scales, ranging from microscopic interactions between two surfaces to the macroscopic nonlinear dynamics of the moving bodies [4-8]. Attempts to reduce the complexity in the theory of frictional dynamics resulted in the introduction of the spring-block models, which reproduce well the basic features of the frictional phenomena and do not contradict observational results. The basic idea [4] underlying this class of models includes the mechanism of contacts to the constitutive "friction law", while considering the effects of inertia and deformations through the introduction of masses and springs [5-8]. The aforementioned models are further complicated by some variants of non-smooth friction laws and model parameters that make it highly difficult to obtain solution [9].

Since the introduction of the spring-block model for fault mechanisms many similar models have been proposed which purportedly generate earthquake moments [10-13]. Carlson and Langer [12], for example, demonstrated that block arrays subjected to a simple velocity weakening friction reproduced the Gutenberg-Richter scaling law between the magnitude of earthquakes and their occurrence rate [13]. The periodic solution generated from their model was spatially homogeneous with initial conditions. However, any spatial in-homogeneity fault model caused by variable non-smooth friction during the slip may affect the characteristics of the traveling waves.

In this paper, we apply the velocity-weakening friction law introduced by Carlson and Langer to prove the existence of a periodic orbit and solution generated by spring-block model. The proof for the non-smooth uncoupled model was based on Poincare-Bendixsom theorem and is similar to the one presented in [14]. Since the non-smoothness in the slip-dependent friction added more coupled differential equation to the model system, we deployed Lyapounov-Schmidt reduction technique to show that there exists a periodic solution for the weakly coupled spring-block model system.

The paper is presented as follows: We state the model problem, notation and preliminaries in Section 2. We then prove the existence of a periodic solution for non-smooth uncoupled and coupled versions of the model in Section 3.

Conclusion is given in Section 4.

2. Problem, Notation and Preliminaries

2.1. Problem Statement

We state here the dimensionless non-smooth equation for a spring-block periodic wave solution x_i in the moving blocks as [14]

$$\frac{\ddot{x}_i}{\tau^2} = \ell^2(x_{i+1} - 2x_i + x_{i-1}) - x_i - F(V + \frac{\dot{x}_i}{\tau}). \tag{1}$$

Here the spring stiffness ratio ℓ, the non-linear velocity-dependent friction force F, and the weakening velocity strength V are all dimensionless parameters and defined in [14, 15]. Equation (1) reveals that the model characteristics is controlled by those three dimensionless parameters with the traveling wave velocity of τ^{-1}. The existence and uniqueness of the periodic solution for (1) was proved for uncoupled and coupled problems with smoothed friction force [14]. We now prove the existence and uniqueness of (1) with non-smooth friction force for both uncoupled ($\ell = 0$) and coupled ($\ell \neq 0$) at any value of $\tau = \tau_0 \in \mathbb{R}^+$.

2.2. Notation and Preliminaries

A function $x : [0, T] \to \mathbb{R}^n$ is absolutely continuous if given $\varepsilon > 0$, there exists $\delta > 0$ such that for any countable collection of disjoint subintervals $[t_n, t'_n]$ of $[0, T]$ satisfying $\sum(t'_n - t_n) < \delta$ whenever $\sum |x(t'_n) - x(t_n)| < \varepsilon$.

Definition 2.1. Consider a differential inclusion of order one

$$\dot{x}(t) \in F(x(t)),\ t \in [0, T], \tag{2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a set-valued map. The solution to (2) is continuous absolutely and satisfies (2) almost everywhere in the time interval $[0, T]$.

Definition 2.2. Let $x \in C^1(\mathbb{R})$ be a T-periodic function and piecewise in $C^2(\mathbb{R})$. Then x is a periodic solution of (1) if it satisfies almost everywhere in $[0, T]$.

Since $x \in C^1$, it is Lischitz continuous and easy to use the mean value theorem to show that it is continuous absolutely. Also, the derivative $\dot{x}$ is piecewise in C^2. Suppose that there exists a point of discontinuity $t_0 \in [t_n, t'_n]$, then $f_- = \dot{x}_{|[t_n,t_0)|} \in C^1$ which can be extended to $[t_n, t_0]$ by the C^1 function

$\tilde{f}_-$. Similarly, $f_+ = \dot{x}_{|]t_0,t'_n]|}$ in $]t_0, t'_n]$ is a C^1 function ($\tilde{f}_+ \in [t_0, t'_n]$) that can be extended. Thus,

$$\begin{aligned} f(t_n) - f(t_0) &= \tilde{f}_-(t_n) - \tilde{f}_-(t_0) = \int_{t_n}^{t_0} \tilde{f}'_-(t)dt = \int_{|t_n,t_0|} f_-, \\ f(t_0) - f(t'_n) &= \tilde{f}_+(t_0) - \tilde{f}_+(t'_n) = \int_{t_0}^{t'_n} \tilde{f}'_+(t)dt = \int_{|t_0,t'_n|} f'_+. \end{aligned} \tag{3}$$

Defining $g_n = f'_- \in [t_n, t_0[$, $g_n = f'_+ \in]t_0, t'_n]$ and, $g_n(t_0) = 0$, in a C^0_m function, we have

$$\begin{aligned} f(t_n) - f(t'_n) =& f(t_n) - f(t_0) + f(t_0) - f(t'_n) \\ =& \int_{[t_n,t_0[} g_n + \int_{]t_0,t'_n]} g_n = \int_{[t_n,t'_n]} g_n \end{aligned} \tag{4}$$

$$\Rightarrow \ \left| f(t_n) - f(t'_n) \right| \leq \|g_n\|_{\infty,[t_n,t'_n]} \left| t_n - t'_n \right| \leq \sup_k \|g_n\|_{\infty,[t_n,t'_n]} \left| t_n - t'_n \right| .$$

Thus, $f = \dot{x}$ is absolutely continuous.

Theorem 2.3. *For all $V > 0$, there exists a periodic orbit to equation (1).*

Proof. The proof is based on Poincare-Bendixson theorem which is proved in [14]. □

However, we assume that there exists a periodic orbit given by theorem 2.3 of the form:

$$\dot{x}(t) = -\tau V \ \text{ in } \ [0, t_g] \ \text{ and } \dot{x}(t) \neq -\tau V \ \text{ in } \ [t_g, T_0] \, , \tag{5}$$

This means that in an interval $[0, t_g]$, the masses stick to the lower surface of the spring-block model while at $t = t_g$, when the strain threshold is exceeded, they begin to move.

3. Global Existence of Periodic Solutions

3.1. Existence of Uncoupled Non-Smooth Problem

We now prove the existence of the periodic solution for the uncoupled non-smooth version of equation (1) for any value of $\tau = \tau_0 \in \mathbb{R}^+$ by setting $\ell = 0$, scale time as $t\tau$, and define a non-smooth friction force F. Thus (1) becomes

$$\ddot{x} + x + F\left(V + \dot{x}\right) = 0. \tag{6}$$

Proposition 3.1. *The inclusion equation (6) has at least one solution on* $\mathbb{R}^+$ *for all initial conditions.*

Proof. The differential $\dot{X}(t) \in G(X)$, where $G : \mathbb{R}^2 \to P(\mathbb{R}^2)\backslash\Phi(P(\mathbb{R}^2))$. We have existence of an absolute continuous solution on $\mathbb{R}^+$. If the multivalued function, G, is a closed convex, upper semi-continuous and does not grow too fast then there exists a constant $c > 0$ such that for all $X \in \mathbb{R}^2$ and $z \in G(X)$, we have $\|z\| \leq c(1+\|X\|)$. So, the second dimensional form of (6) in first order can be written as

$$\dot{X} \in G(X), \text{ where } X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}; \; G(X) = \begin{pmatrix} \dot{x} \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ -F(V+\dot{x}) \end{pmatrix}. \tag{7}$$

Clearly, F is a closed convex in $\mathbb{R}^2$ and so G is also a closed convex in $\mathbb{R}^2$. The function G is upper semi-continuous if for all A closed in $\mathbb{R}^2$, we have $G^{-1}(A)$ also closed in $\mathbb{R}^2$. This is correspondent to showing that F is an upper semi-continuous. If we define a segment $[a, b]$in $\mathbb{R}$, then $F^{-1}([a, b])$ is of the form Φ, $\{0\} \cup [\alpha, \beta]$, $\{0\} \cup [\alpha, +\infty[$, $\{0\} \cup]-\infty, \beta]$ or $\{0\} \cup]-\infty, \beta] \cup [\alpha, +\infty[$. Hence, $F^{-1}([a, b])$ is closed and F is upper semi-continuous. In addition we can write

$$\begin{aligned} \|G(X)\|_2 &\leq \left\| \begin{pmatrix} \dot{x} \\ -x \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} 0 \\ -F(V+\dot{x}) \end{pmatrix} \right\|_2 \\ &\leq \|X\|_2 + \sup_y \left\| \begin{pmatrix} 0 \\ F(y) \end{pmatrix} \right\|_2 \leq \|X\|_2 + M, \end{aligned} \tag{8}$$

Since F is bounded, the not too fast growing condition is satisfied. Thus, we have existence of a solution for all initial conditions □

Proposition 3.2. *There exists a unique solution for the inclusion equation (6) in* C^1 *and piecewise in* C^2.

Proof. By phase plot analysis illustrated in Figure 1, clearly, as long as x is a solution of $\dot{x}(t) = -V$, the right-hand-side of (6) is perfectly smooth. Thus, by Cauchy-Lipschitz theorem the solution is unique. To verify that the two different solutions cannot pass through a point $(x_0, -V)$, we define a vector field $\nu(x, \dot{x}) = \vec{\nu} = \begin{pmatrix} \dot{x} \\ -x - F(V+\dot{x}) \end{pmatrix}$. Then

$$\lim_{(x,\dot{x}) \to (x_0, -V^+)} \vec{\nu} = \begin{pmatrix} -V \\ -x_0 - F_0 \end{pmatrix} \text{ and } \lim_{(x,\dot{x}) \to (x_0, -V^-)} \vec{\nu} = \begin{pmatrix} -V \\ -x_0 + F_0 \end{pmatrix}. \tag{9}$$

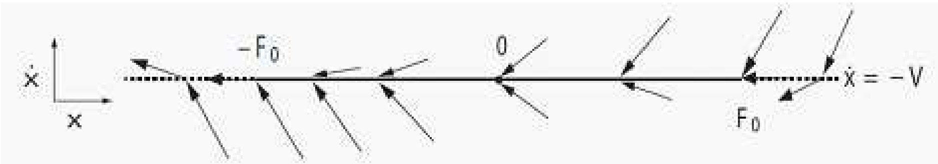

Figure 1: Phase space plot for $\dot{x} = -V$.

Therefore, as $x \to (x_0, -V)$ with $|x_0| > F_0$, we have the two solutions crossing the line $\dot{x} = -V$. However, if $x \to (x_0, -V)$ with $|x_0| \leq F_0$, the solution x cannot cross the segment $[-F_0, F_0] \times \{-V\}$ since it is contractive (see phase space below). In addition, since x and $\dot{x}$ are continuous, $\ddot{x}$ is also continuous except when $\dot{x} \to -V$(i.e. at the end of the sliding period). Thus, $\ddot{x}$ is piecewise in class C^2. □

Lemma 3.3. *Let $L(t) = (x(t), \dot{x}(t))$ be a trajectory of (6) and X_{eq} defines a unique equilibrium point. Then the distance $D(L(t), X_{eq})$ is increasing in the half plane $y > -V$.*

Proof. We define $X_{eq} = (-F(V), 0)$ so that $D(L(t), X_{eq})^2 = (x(t)+F(V))^2 + \dot{x}^2$. This implies

$$\frac{d}{dx}[D(L, X_{eq})^2](t) = 2\dot{x}[\ddot{x} + x + F(V)]. \tag{10}$$

Implying

$$\frac{d}{dt}[D(L, X_{eq})^2](t) > 0 \Leftrightarrow 2\dot{x}[-x - F(V + \dot{x}) + x + F(V)] > 0,$$
$$\Leftrightarrow \dot{x}[F(V) - F(V + \dot{x})] > 0. \tag{11}$$

Assume that $\dot{x} > 0$,then $F(y) = \frac{F_0}{1+y}$ for all $y > 0$. Thus, F is decreasing in $\mathbb{R}^+$ and $F(V) > F(V + \dot{x})$. Hence $D(L, X_{eq})$ is increasing. If $-V < \dot{x} < 0$, then since $V + \dot{x} > 0$, it implies $F(V + \dot{x}) > F(V)$ and consequently $\dot{x}[F(V) - F(V + \dot{x})] > 0$. Therefore, $D(L(t), X_{eq})$ is also increasing. □

Proposition 3.4. *Suppose that x_0 is a T-periodic solution of (6) of the form*

$$\begin{cases} \dot{x}_0(t) = -V \text{ for } [0, t_{g0}], \\ \dot{x}_0(t) \neq -V \text{ for } [t_{g0}, T]. \end{cases} \tag{12}$$

Then any trajectory of (6) with initial values into domain delimited by $X_0 = (x_0, \dot{x}_0)$ reaches the periodic orbit in finite time.

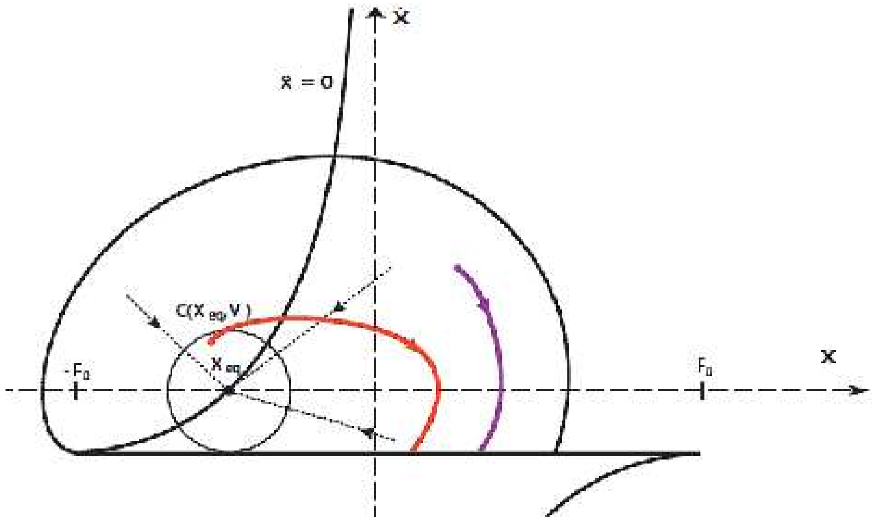

Figure 2: Semi-attractive plot of the periodic solution.

Proof. We consider initial conditions in the domain delimited by X_0 which is outside of the disk of radius V and center X_{eq}, as shown in Figure 2. Since the disk is tangent to the segment$[-F_0, F_0] \times \{-V\}$, it follows from Lemma 3.3 that the trajectory must cross the segment in finite time. If we locate an initial point in the disk of radius V and center X_{eq}, we finds that the distance to X_{eq} is strictly increasing. Suppose then that the trajectory does not travel outside the disk, then there is no attractive equilibrium point into the disk and so the trajectory converges to a limit cycle. Since the trajectory distance is increasing, the limit cycle is a circle. However, the distance to X_{eq} is constant in the case of a circle trajectory which contradicts the lemma. Thus, every trajectory with initial point into the disk must go out of the disk. □

Proposition 3.5. *If* $X_0 = (x_0(t), \dot{x}_0(t))$ *is a periodic orbit passing through the point* $(-F_0, -V)$*, then there is no other periodic orbit in the domain delimited by the orbit* X_0.

Proof. Suppose there exists a periodic orbit with initial condition into the domain delimited by the orbit X_0. Then from Proposition 3.4 it reached $(x_0(t), \dot{x}_0(t))$ in finite time. Therefore, it would not be a periodic orbit unless it is X_0. □

Remark 3.6. Therefore, the existence of the periodic orbit for the uncoupled non-smooth model follow the Poincare-Bendixson property proved in [14]. Now, it is necessary to determine the conditions under which it is unique by considering two periodic orbits as follow:

- Case 1: When $V$is smaller than F_0, the periodic trajectories start at the point $(-F_0, -V)$ and then crosses again the line $\dot{x} = -V$ at the point $(y, -V)$ such that $-F_0 < y < F_0$.

- Case 2: When for large V, the periodic trajectories crosses the line $\dot{x} = -V$ at the point $(y, -V)$ such that $y > F_0$.

For both cases, the trajectory orbits are in C^1 and piecewise in C^2. Therefore, from propositions 3.4 and 3.5 the periodic orbits generated by cases 1 and 2 are semi-attractive with no other periodic orbits inside them.

3.2. Existence of Coupled Non-Smooth Problem

Again assuming that x_0 defined in (13) is a T_0-periodic solution for $\tau = \tau_0$ that describes a stick-slip characteristic for $t \in [0, t_{g0}]$.

$$\begin{cases} \dot{x}_0(t) = -\tau_0 V \text{ for } [0, t_{g0}], \\ \dot{x}_0(t) \neq -\tau_0 V \text{ for } [t_{g0}, T_0]. \end{cases} \tag{13}$$

Thus, equation (13) indicates that at time $t \in [0, t_g]$, the masses stick to the lower surface while at time $t \approx t_g = t_{g0}$ the masses begin to move. We now show the existence of periodic solution x of (1) traveling at speed $\tau^{-1} > 0$ when ℓ is close to zero by first stating the following hypotheses and theorem.

Hypotheses 14.

$$\text{(i) } T_0 \gg 1; \quad \text{(ii) } T_0 - t_{g0} \ll 1; \quad \text{(iii) } x_0 \text{ is Case 1 Type.} \tag{14}$$

Theorem 3.7. *Assume that x_0 is a T_0-periodic solution of the form (13) for $\tau = \tau_0 \in \mathbb{R}^+$, $x_0(0) = x_0(T_0) < F_0$ of (6) and satisfying the hypotheses (14). Let denote t_{g0} as time at which the mass begins to slide and x_{0i} for $i \in \{1, 2\}$ as two linearly independent solutions on the time interval $]t_{g0}, T_0[$ with initial conditions $x_{01}(t_{g0}) = 1,\ \dot{x}_{01}(t_{g0}) = 0,\ x_{02}(t_{g0}) = 0,\ \dot{x}_{02}(t_{g0}) = 1$, for the equation*

$$\tau_0^{-2}\ddot{x}_{0i} + x_{0i} + F'(V + \tau_0^{-1}\dot{x}_0)\tau_0^{-1}\dot{x}_{0i} = 0; \quad i \in \{1, 2\}.$$

If we represent (15) by the solution $x_{p,\hbar}$

$$\tau_0^{-2}\ddot{x} + x + F'(V + \tau_0^{-1}\dot{x}_0)\tau_0^{-1}\dot{x} = \hbar, \quad t \in \left]t_{g0}, T_0\right[, \tag{15}$$

where $\hbar = 2\ddot{x}_0 + \tau_0 F'(V + \tau_0^{-1}\dot{x}_0)\dot{x}_0$ *and* $x_{p,\hbar}(t_{g0}) = \dot{x}_{p,\hbar}(t_{g0}) = 0$ *and we further assume that*

$$-\tau_0 V\dot{x}_{02}(T_0) + \tau_0 V + \tau_{p,\hbar}^{-2}(T_0) \neq 0; \text{ for } x_{02}(t_{g0}) = 0;\ \dot{x}_{02}(t_{g0}) = 1. \tag{16}$$

Then, there exists a neighborhood ψ *of* $0 \in \mathbb{R}$, *and a neighborhood* Ω *of* (x_0, τ_0, t_{g0}) *in* $H^2(0, t_{g0}) \times (\mathbb{R}^+)^2$ *such that* $\forall\ \ell \in \psi$, *the solution*

$$(x(\ell), \tau(\ell), t_g(\ell)) \in \Omega,$$

satisfies (1) with $\tau = \tau(\ell)$.

Proof. For $t \in \left]t_{g0}, T_0\right[$, we are looking for periodic orbits of (1) in the form of (13) within (t_g, T_0) corresponding to when the masses slide. Thus, (t_g, T_0) for (1) is no more inclusion and t_g is unknown. Defining x_1 as a small perturbation depending on parameter ℓ, equation (1) is a differential equation with $x \in C^1(\mathbb{R})$ and piecewise in $C^2(\mathbb{R})$.Therefore, in order to use the Lyapounov-Schmidt reduction to find $x = x_0 + x_1$ on $t \in \left]t_{g0}, T_0\right[$, we need to satisfy the compatibility condition (16) in Theorem (3.7) and show that the sum of all external forces equals the threshold $(-F_0)$ when the system begins to slip. That is,

$$\begin{aligned}
-F_0 &= x(t_g) - \ell^2(x(t_g - 1) - 2x(t_g) + x(t_g + 1)),\\
&= (1 + 2\ell^2)x(t_g) - \ell^2(-2\tau V t_g + \tau V T_0 + 2x(0)),\\
&= (1 + 2\ell^2)(-\tau V t_g + x(0)) - \ell^2(-2\tau V t_g + \tau V T_0 + 2x(0)),\\
&= -\tau V t_g - \ell^2\tau V T_0 + x(0)\\
\Rightarrow x(0) &= -F_0 + \tau V t_g + \ell^2\tau V T_0
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
x(t) &= -\tau V t - F_0 + \tau V t_g + \ell^2\tau V T_0\\
&= \tau V(t_g - t) - F_0 + \ell^2\tau V T_0 \quad \text{for } t \in [0, t_g].
\end{aligned} \tag{17}$$

So far we have shown that we can perform a Lyapounov-Schmidt reduction to determine the unknowns τ, t_g and x in $\left]t_g, T_0\right[$ provided $(i.)$ x is a T_0-periodic solution of (1), $(ii.)$ there exists τ and t_g such that x is of the form (13), $(iii.)$ $t_g \approx t_{g0}$ and $\tau \approx \tau_0$, $(iv.)$ equation (17) is true and $(v.)$ $\|x\|_\infty \leq 2\|x_0\|_\infty$.

Lemma 3.8. *The periodic solution,* x, *given by (17) is the solution of (1) in the time interval* $[t_g, T_0]$ *under the hypotheses (14).*

Proof. We first prove that x is a solution of (1) in $[0, t_g]$. When$t \in]0, t_g]$, we have $\ddot{x} = 0$, thus, equation (1) becomes

$$x(t) - \ell^2 \underbrace{(x(t+1) - 2x(t) + x(t-1))}_{W} \in -F_0. \tag{18}$$

Given that x is continuous, (18) holds for all $t \in [0, t_g]$. To compute the term W in $t \in]0, t_g]$, we already know that $x(t)$ is given in (17). However, $t-1$ and $t+1$ may not be in $t \in]0, t_g]$. So, we split the interval for $t \in]0, t_g]$ as $t+1 \in]1, t_g+1]$ into three subintervals as

- **SUBINTERVAL 1:** $t+1 \in]1, t_g] \Rightarrow t \in]0, t_g - 1]$,
- **SUBINTERVAL 2:** $t+1 \in [t_g, T_0] \Rightarrow t \in [t_g - 1, T_0 - 1]$,
- **SUBINTERVAL 3:** $t+1 \in [T_0, t_g + 1] \Rightarrow t \in [T_0 - 1, t_g]$.

  Similarly, $t - 1 \in]-1, t_g - 1]$ **gives**

- **SUBINTERVAL 4:** $t - 1 \in]-1, -(T_0 - t_{g0})] \Rightarrow t \in]0, 1 - (T_0 - t_{g0})]$,
- **SUBINTERVAL 5:** $t - 1 \in [-(T_0 - t_{g0}), 0] \Rightarrow t \in [1 - (T_0 - t_{g0}), 1]$,
- **SUBINTERVAL 6:** $t - 1 \in [0, t_g - 1] \Rightarrow t \in [1, t_g]$.

Only subintervals 2 and 5 remain unknown. The subintervals 1, 3, 4 and 6 can be given by explicit formula. Regrouping the common subintervals for both $t+1 \in]1, t_g+1]$ and $t-1 \in]-1, t_g-1]$ produces five distinct subintervals (DS) which we will analyze separately below.
DS 1 and 6: $t \in [1, t_g - 1] \;\Rightarrow\; t-1,\ t,\ t+1 \in [0, t_g]$.
Since (17) is satisfied for $x(t-1)$, $x(t)$, and $x(t+1)$, (18) becomes

$$\begin{aligned} & x(t) = -\tau V t + x(0) \in [-F_0, F], \quad \forall t \in [1, t_g - 1] \\ \Leftrightarrow\ & \tau V(t_g - t) - F_0 + \ell^2 \tau V T_0 \in [-F_0, F], \quad \forall t \in [1, t_g - 1] \\ \Leftrightarrow\ & 0 \le \tau V(t_g - t) - F_0 + \ell^2 \tau V T_0 \in [-F_0, F], \quad \forall t \in [1, t_g - 1] \end{aligned} \tag{19}$$

Since $\tau, V, T_0 \ge 0$ and $t_g - t \ge 0$ in $[1, t_g - 1]$, it follows that the lower inequality of (19) is satisfied. Furthermore for small ℓ, since $\tau V t_g$ is very close to $\tau_0 V t_{g0}$ the upper inequality of (19) is satisfied. Hence, the last equation of (19) is satisfied.
DS 2 and 6: $t \in [t_g - 1, T_0 - 1[\;\Rightarrow\; t-1,\ t \in]0, t_g]$, and $t+1 \in [t_g, T_0[$.

From (17), we have $x(t) = -\tau V t + x(0)$ and $x(t-1) = -\tau V t(t-1) + x(0)$ explicitly. However, $x(t+1)$ is unknown. Thus,

$$\begin{aligned}
&x(t+1) - 2x(t) + x(t-1) = x(t+1) + \tau V t + \tau V - x(0) \\
&\Rightarrow x(t) - \ell^2(x(t+1) - 2x(t) + x(t-1)) \in [-F_0, F_0] \\
&\Leftrightarrow \left|-\tau V t + x(0) - \ell^2(x(t+1) + \tau V t + \tau V - x(0))\right| \leq F_0
\end{aligned}$$

Therefore,

$$0 \leq \tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2(x(t+1) + \tau V t + \tau V - x(0)) \leq 2F_0. \qquad (20)$$

Since $\|x\|_\infty \leq 2\|x_0\|_\infty$, it follows that $|x(t+1) + \tau V t + \tau V - x(0)| \leq 4\|x_0\|_\infty + \tau V T_0 + \tau V = \mathrm{H}$. We need to show that the following inequalities are satisfied in order to conclude (20).

$$\begin{aligned}
&0 \leq \tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2 \mathrm{H}, \quad \text{at } t = T_0 - 1, \\
&\tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2 \mathrm{H} \leq 2F_0, \quad \text{at } t = t_g - 1.
\end{aligned}$$

From $t_g \approx t_{g0}$ and $\tau \approx \tau_0$ and using the hypotheses (14), we have for $t = T_0 - 1$ and $t = t_g - 1$ respectively,

$$\begin{aligned}
&\tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2 \mathrm{H} = \tau V(1 - (T_0 - T_g) + \ell^2 T_0) - \ell^2 \mathrm{H} \geq 0 \\
&\tau V(t_g - t) + \ell^2 \tau V T_0 - \ell^2 \mathrm{H} = \tau V + \ell^2 \tau V T_0 + \ell^2 \mathrm{H}.
\end{aligned}$$

Since $t_g - T_0 \ll 1$, we deduce that $1 - (T_0 - t_g) + \ell^2 T_0 > 0$ when ℓ is close to zero. Also for $\ell \approx 0$ and $|\tau - \tau_0| \to 0$, we have $\tau V + \ell^2 \tau V T_0 + \ell^2 \mathrm{H} \leq 2F_0$. Thus, (20) is satisfied for small ℓ.

DS 3 and 6: $t \in [T_0 - 1, t_g] \Rightarrow\ t-1,\ t \in]0, t_g]$, and $t+1 \in [T_0, T_0 + t_g]$. In this case $x(t-1)$, $x(t)$, and $x(t+1)$ are given explicitly as $x(t) = -\tau V t + x(0)$, $x(t-1) = -\tau V(t-1) + x(0)$ and $x(t+1) = -\tau V(t+1-T_0) + x(0)$ which deduce to

$$\begin{aligned}
&x(t+1) - 2x(t) + x(t-1) = \tau V T_0, \\
&x(t) - \ell^2(x(t+1) - 2x(t) + x(t-1)) = \tau V(t_g - t) - F_0.
\end{aligned}$$

Hence x is a solution of (1) if and only if

$$0 \leq \tau V(t_g - t) \leq 2F_0, \qquad \forall t \in [T_0 - 1, t_g]. \qquad (21)$$

Clearly $\tau V(t_g - t) \geq 0$. Moreover, since $\tau V(T_0 - t_g) \geq 0$ and $t_{g0} \gg 1$ (based on hypotheses (14)) we have $\tau_0 V \ll 2F_0$. Thus, $\tau V \ll 2F_0$ for small ℓ and $|\tau - \tau_0| \to 0$. So (21) is satisfied.

DS 1 and 4: $t \in]0, 1-(T_0 - t_{g0})] \Rightarrow\ t,\ t+1 \in]0, t_g]$, and $t-1 \in [-1, -T_0 + t_g] \subset [-T_0, -T_0 + t_g]$.
This means, $x(t) = -\tau V t + x(0)$, $x(t-1) = -\tau V(t-1+T_0) + x(0)$ and $x(t+1) = -\tau V(t+1) + x(0)$, and so

$$\begin{aligned} &x(t+1) - 2x(t) + x(t-1) = \tau V T_0, \\ &x(t) - \ell^2(x(t+1) - 2x(t) + x(t-1)) = \tau V(t_g - t) - F_0 + 2\ell^2\tau V T_0. \end{aligned}$$

Implying

$$\begin{aligned} &x(t) + \ell^2\tau V T_0 \in [-F_0, F_0] \\ &\quad \Leftrightarrow \tau V(t_g - t) + 2\ell^2\tau V T_0 \le 2F_0 \ \ \forall t \in]0, 1-(T_0 - t_{g0})]\,. \end{aligned} \tag{22}$$

Clearly, for ℓ small the inequalities are satisfied at $t = 0$ under hypotheses (14).
DS 1 and 5:$t \in [1-(T_0 - t_{g0}, 1] \Rightarrow\ t-1 \in [-T_0 + t_g, 0]$. Thus, $t, t+1 \in]0, t_g]$ and $t-1 \in [-T_0 + t_g, 0]$.
Similar to DS 2 and 6 we have $x(t) + \ell^2\tau V T_0 \in [-F_0, F_0]\ \Leftrightarrow$

$$0 \le \tau V(t_g - t) + \ell^2\tau V T_0 - \ell^2(\tau V(t-1) - x(0) + x(t-1) \le 2F_0, \tag{23}$$

which satisfies in $[1-(T_0 - t_{g0}), 1]$ for small ℓ and concludes the prove for the lemma. □

We now construct the solution of (1) in the time interval $x \in [t_g, T_0]$ and prove that it is a T_0-periodic solution when $x \in C^1$ and piecewise in C^2.
Let define a function $\bar{x}(k) = x(t)$ for $k \in [t_{g0}, T_0]$ and rescale time by stating that $k(t) = mt + b$ then for $k(t_g) = t_{g0}$ and $k(T_0) = T_0$ so that we can compare x_0 and x in the same interval, we have

$$\begin{aligned} &m^2(t_g)\tau^{-2}\ddot{\bar{x}}(k) + \bar{x}(k) + F(V + m(t_g)\tau^{-1}\dot{\bar{x}}) \\ &\qquad = \ell^2(-2m^{-1}\tau V k - 2\bar{x}(k) + C(\tau, t_g, \ell), \end{aligned} \tag{24}$$

where $m = m(t_g) = \frac{t_{g0} - T_0}{t_g - T_0}$, $b = b(t_g) = T_0\frac{t_{g0} - t_g}{T_0 - t_g}$ and

$$C(\tau, t_g, \ell) = \ell^2(2m^{-1}Vb + \tau V T_0 + 2x(0)).$$

In addition $\bar{x}$ must be in C^1 such that the following four conditions are satisfied.

$$\bar{x}(t_{g0}) = \ell^2\tau V T_0, \tag{25}$$

$$\dot{\bar{x}}(t_{g0}) = \tau_0 V - m^{-1}\tau V, \tag{26}$$

$$\bar{x}(T_0) = -\tau_0 V t_{g0} + \tau V t_g + \ell^2 \tau V T_0, \tag{27}$$

$$\dot{\bar{x}}(T_0) = \tau_0 V - m^{-1}\tau V. \tag{28}$$

Thus, (24) reduces to

$$\begin{aligned} m^2\tau^{-2}(\ddot{\bar{x}} + \ddot{x}_0) + \bar{x} + x_0 + F(V + m\tau^{-1}(\dot{\bar{x}} + \dot{x}_0)) \\ = \ell^2(-2m^{-1}\tau V k - 2\bar{x} - 2x_0) + C(\tau, t_g, \ell). \end{aligned} \tag{29}$$

Lemma 3.9. *(Applying Lyapounov-Schmidt Reduction)*
*For all $\ell \approx 0$, equation (29) has T_0-*periodic *solution $\bar{x}$ satisfying conditions (25)-(28) for τ and t_g.*

Proof. Let represent (29) and its reliability conditions by $g : C^2(]t_{g0}, T_0[) \times \mathbb{R}^3 \to C^0(]t_{g0}, T_0[) \times \mathbb{R}^4$ written as $q(\bar{x}, \tau, t_g, \ell)$

$$= \begin{cases} m^2\tau^{-2}(\ddot{\bar{x}} + \ddot{x}_0) + \bar{x} + x_0 + F(V + m\tau^{-1}(\dot{\bar{x}} + \dot{x}_0)) \\ \quad -\ell^2(-2m^{-1}\tau V k - 2\bar{x} - 2x_0) + C(\tau, t_g, \ell); \\ \text{for } \bar{x}(t_{g0}^+) - \ell^2\tau V T_0, \qquad \dot{\bar{x}}(t_{g0}^+) - m^{-1}\tau V - \tau_0 V, \\ \quad \bar{x}(T_0^-) + \tau_0 V t_{g0} - \tau V t_g - \ell^2\tau V T_0, \ \dot{\bar{x}}(T_0^-) + m^{-1}\tau V - \tau_0 V, \end{cases} .$$

Then, we have to solve

$$q(\bar{x}, \tau, t_{g0}, \ell) = 0 \quad \text{for} \quad q(0, \tau_0, t_{g0}, 0) = 0. \tag{30}$$

Proposition 3.10. *Let define a linear operator, L, by the linear part of q as $L(\bar{x}) = D_{\bar{x}}q(0, \tau_0, t_{g0}, 0) \cdot \bar{x}$ at $m(t_{g0}) = 1$, where $\bar{x} \in C^2([t_{g0}, T_0[$, then the kernel of L is zero, i.e.* $\ker L = \{0\}$.

Proof. Since $m(t_{g0}) = 1$ for $\bar{x} \in C^2([t_{g0}, T_0[$, we have

$$L(\bar{x}) = \begin{cases} \tau_0^{-2}\ddot{\bar{x}} + \bar{x} + F'(V + \tau_0^{-1}\dot{x}_0) \cdot \tau_0^{-1}\dot{\bar{x}} \\ \bar{x}(t_{g0}^+), \quad \dot{\bar{x}}(t_{g0}^+), \quad \bar{x}(T_0^-), \quad \dot{\bar{x}}(T_0^-) \end{cases} \tag{31}$$

Suppose that $\bar{x}(t_{g0}^+) = \lim_{t \to t_{g0}^+} \bar{x}(t) = 0$ and $\bar{x}(T_0^-) = \lim_{t \to T_0^-} \bar{x}(t) = 0$, then $\bar{x} \equiv 0$ since $F'(V + \tau_0^{-1}\dot{x}_0)$ results to finite limits when $t \to t_{g0}^-$ and $t \to T_0^+$. We now extend $F'(V + \tau_0^{-1}\dot{x}_0)$ under $[t_{g0}, T_0]$ and claim that $\hat{x}$ is a unique solution of (31) with initial condition $\hat{x}(t) = A$ and $\dot{\hat{x}}(t) = B$. Thus, $\hat{x}_{|]t_{g0}, T_0[|} = \bar{x}$. Since $\hat{x}$ is C^1 under $[t_{g0}, T_0]$ and both $\bar{x}$ and $\dot{\hat{x}}$ identical to zero at t_{g0} and T_0, we have $\hat{x}(t_{g0}) = \hat{x}(T_0) = \dot{\hat{x}}(t_{g0}) = \dot{\hat{x}}(T_0) = 0$. Thus $\hat{x} = \bar{x} \equiv 0$. ∎

Now, representing the non-linear form of (30) by $L(\bar{x}) = \lambdabar(\bar{x}, \tau, t_g, \ell)$, we have $\lambdabar(\bar{x}, \tau, t_g, \ell)$

$$= \begin{cases} (\tau_0^{-2} - m^2\tau^{-2})\ddot{\bar{x}} - m^2\tau^{-2}\ddot{x}_0 - x_0 + \ell^2(-2m^{-1}\tau V k - 2\bar{x} - 2x_0) \\ \quad +C(\tau, t_g, \ell) - F(V + m\tau^{-1}(\dot{\bar{x}} + \dot{x}_0)) + F'(V + \tau_0^{-1}\dot{x}_0) \cdot \tau_0^{-1}\dot{\bar{x}}; \\ \ell^2\tau V T_0,\ V(\tau_0 - m^{-1}\tau),\ V(\tau t_g - \tau_0 t_{g0}) + \ell^2\tau V T_0,\ V(\tau_0 - m^{-1}\tau) \end{cases}.$$

Projecting Π onto the range of L gives

$$L(\bar{x}) = \Pi\lambdabar(\bar{x}, \tau, t_g, \ell), \tag{32}$$

$$\Rightarrow\ (Id - \Pi)\lambdabar(\bar{x}, \tau, t_g, \ell) = 0. \tag{33}$$

Since L is bijective from $C^2([t_{g0}, T_0]) \to \mathbb{R}(L)$, we can solve (32) using the Implicit Function Theorem. However, (33) is not easy to solve for τ and t_g as a function of ℓ. Let define an equivalent of (33) as

$$\varphi_1(\lambdabar^*(\tau, t_g, \ell)) = \varphi_2(\lambdabar^*(\tau, t_g, \ell)) = 0 \tag{34}$$

where $\lambdabar^*(\tau, t_g, \ell) = \lambdabar(\bar{x}^*(\tau, t_g, \ell), \tau, t_g, \ell)$ and φ_1, φ_2 are given by

$$\begin{cases} \varphi_1(\eta, \bar{\alpha}) = \alpha_2 x_{02}(T_0) - \alpha_3 + x_{p,\eta}(T_0) + \alpha_1 \\ \varphi_2(\eta, \bar{\alpha}) = \alpha_2 \dot{x}_{02}(T_0) - \alpha_4 + \dot{x}_{p,\eta}(T_0) + \alpha_1 \dot{x}_{01}(T_0), \\ \text{for } (\eta, \bar{\alpha}) = (\eta, (\alpha_1, \alpha_2, \alpha_3, \alpha_4)) \in R(L), \\ \text{and } x_{p,\eta}(t) = c_1(t)x_{01}(t) + c_2(t)x_{02}(t);\ \ \dot{c}_1 x_{01} + \dot{c}_2 x_{02} = 0. \end{cases}, \tag{35}$$

and define $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ given by $\Phi(\tau, t_g, \ell) = \varphi_1(\lambdabar^*(\tau, t_g, \ell)) = \varphi_2(\lambdabar^*(\tau, t_g, \ell)) = 0$. Then, solving $\Phi(\tau, t_g, \ell) = 0$ with respect to ℓ in the neighborhood of $(\tau_0, t_{g0}, 0)$ gives the homogeneous solution of (34). For the particular solution, let $\lambdabar^*(\tau, t_g, \ell) = \begin{pmatrix} \eta^* \\ \bar{\alpha} \end{pmatrix}$, where

$$\eta^* = (\tau_0^{-2} - m^2\tau^{-2})\ddot{\bar{x}} - m^2\tau^{-2}\ddot{x}_0 - x_0 + \ell^2(-2m^{-1}\tau V k - 2\bar{x} - 2x_0) \\ +C(\tau, t_g, \ell) - F(V + m\tau^{-1}(\dot{\bar{x}} + \dot{x}_0)) + F'(V + \tau_0^{-1}\dot{x}_0) \cdot \tau_0^{-1}\dot{\bar{x}},$$

$$\bar{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} \ell^2\tau V T_0 \\ V(\tau_0 - m^{-1}\tau) \\ V(\tau t_g - \tau_0 t_{g0}) + \ell^2\tau V T_0 \\ V(\tau_0 - m^{-1}\tau) \end{pmatrix},$$

It follows from the computation of $D_{(\tau, t_g)}\Phi(\tau_0, t_{g0}, 0)$ that

$$\varphi_1(\lambdabar^*) = V x_{02}(T_0)(\tau_0 - m^{-1}\tau) + V(\tau_0 t_{g0} - \tau t_g) + x_{p,\eta^*}(T_0), \tag{36}$$

$$\varphi_2(\lambda^*) = V\dot{x}_{02}(T_0)(\tau_0 - m^{-1}\tau) + V(m^{-1}\tau - \tau_0) + \dot{x}_{p,\eta^*}(T_0) \\ + \ell^2\tau V T_0\dot{x}_{01}(T_0). \quad (37)$$

Since $m(t_g) = \frac{t_{g0}-T_0}{t_g-T_0} \Rightarrow m'(t_g) = -(t_{g0} - T)^{-1}$ and for $\ell = 0$, we have $C(\tau, t_g, \ell) = 0$,

$$\Rightarrow \partial_\tau\eta^*(\tau_0, t_g, 0) = F'(V+\tau_0^{-1}\dot{x}_0)(-\tau_0^{-2}(\dot{\bar{x}}(\tau_0, t_{g0}, 0)+\dot{x}_0)+\tau_0^{-1}\partial_\tau\dot{\bar{x}}(\tau_0, t_{g0}, 0)) \\ - 2\tau_0^{-3}(\ddot{\bar{x}}(\tau_0, t_{g0}, 0) + \ddot{x}_0) + F'(V + \tau_0^{-1}\dot{x}_0)\cdot\tau_0^{-1}\partial_\tau\dot{\bar{x}}(\tau_0, t_{g0}, 0).$$

Furthermore, since $\bar{x}(\tau_0, t_{g0}, 0) = \dot{\bar{x}}(\tau_0, t_{g0}, 0) = \ddot{\bar{x}}(\tau_0, t_{g0}, 0) = 0$, it follows that

$$\partial_\tau\eta^*(\tau_0, t_g, 0) = -2\tau_0^{-3}\ddot{x}_0 + \tau_0^{-2}F'(V + \tau_0^{-1}\dot{x}_0)\dot{x}_0.$$

Similarly,

$$\begin{aligned}
&\partial_{t_g}\eta^*(\tau_0, t_{g0}, 0) \\
&\quad = F'(V + \tau_0^{-1}(\dot{\bar{x}}(\tau_0, t_{g0}, 0) + \dot{x}_0))(-\tau_0^{-1}(t_{g0} - T_0)^{-1}(\dot{\bar{x}}(\tau_0, t_{g0}, 0) + \dot{x}_0) \\
&\qquad -\tau_0^{-1}\partial_{t_g}\dot{\bar{x}}(\tau_0, t_{g0}, 0)) + F'(V + \tau_0^{-1}\dot{x}_0)\cdot\tau_0^{-1}\partial_{t_g}\dot{\bar{x}}(\tau_0, t_{g0}, 0) \\
&\qquad + 2\tau_0^{-2}(t_{g0} - T_0)^{-1}(\ddot{\bar{x}}(\tau_0, t_{g0}, 0) + \ddot{x}_0), \\
&\quad = 2\tau_0^{-2}(t_{g0} - T_0)^{-1}\ddot{x}_0 + F'(V + \tau_0^{-1}\dot{x}_0)\tau_0^{-1}(t_{g0} - T_0)^{-1}\dot{x}_0.
\end{aligned}$$

Assuming the wronskian of the form $\omega = x_{01}\dot{x}_{02} - \dot{x}_{01}x_{02}$ and the particular solution

$$x_{p,\eta}(T_0) = -\tau_0^2 x_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{02}\eta}{\omega}dk + \tau_0^2 x_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{01}\eta}{\omega}dk,$$

we have

$$\begin{aligned}
&\partial_\tau x_{p,\eta^*}(T_0)(\tau_0, t_{g0}, 0) \\
&\quad = -\tau_0^2 x_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{02}}{\omega}\partial_\tau\eta^* dk + \tau_0^2 x_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{01}}{\omega}\partial_\tau\eta^* dk, \\
&\quad = -\tau_0^2 x_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{02}}{\omega}\left[-2\tau_0^{-3}\ddot{x}_0 - \tau_0^{-2}F'(V + \tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk \\
&\qquad + \tau_0^2 x_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{01}}{\omega}\left[-2\tau_0^{-3}\ddot{x}_0 - \tau_0^{-2}F'(V + \tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk \\
&\quad = \tau_0^{-1}x_{01}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{02}}{\omega}\left[-2\ddot{x}_0 - \tau_0 F'(V + \tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk
\end{aligned}$$

$$-\tau_0^{-1}x_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{01}}{\omega}\left[2\ddot{x}_0+\tau_0F'(V+\tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk,$$

$$\Rightarrow \partial_\tau x_{p,\eta^*}(T_0)(\tau_0,t_{g0},0)=\tau_0^{-1}x_{01}(T_0)I_1-\tau_0^{-1}x_{02}(T_0)I_2,$$

where

$$\begin{aligned} I_1&=\int_{t_{g0}}^{T_0}\frac{x_{01}}{\omega}\left[2\ddot{x}_0+\tau_0F'(V+\tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk,\\ I_2&=\int_{t_{g0}}^{T_0}\frac{x_{02}}{\omega}\left[-2\ddot{x}_0-\tau_0F'(V+\tau_0^{-1}\dot{x}_0)\dot{x}_0\right]dk. \end{aligned} \tag{38}$$

Following the same procedure,

$$\begin{aligned} \partial_{t_g}x_{p,\eta^*}(T_0)(\tau_0,t_{g0},0)=-\tau_0^2x_{01}(T_0)&\int_{t_{g0}}^{T_0}\frac{x_{02}}{\omega}\left[\frac{2\ddot{x}_0}{\tau_0^2(t_{g0}-T_0)}\right.\\ &\left.+F'(V+\frac{\dot{x}_0}{\tau_0})\frac{\dot{x}_0}{\tau_0(t_{g0}-T_0)}\right]dk\\ &+\tau_0^2x_{02}(T_0)\int_{t_{g0}}^{T_0}\frac{x_{01}}{\omega}\left[\frac{-2\ddot{x}_0}{\tau_0^3}-\frac{1}{\tau_0^2}F'(V+\frac{\dot{x}_0}{\tau_0})\dot{x}_0\right]dk. \end{aligned}$$

$$\Rightarrow \partial_{t_g}x_{p,\eta^*}(T_0)(\tau_0,t_{g0},0)=-\frac{x_{01}(T_0)}{(t_{g0}-T_0)}I_2+\frac{x_{02}(T_0)}{(t_{g0}-T_0)}I_1,$$

Therefore, the partial derivatives of (36) and (37), respectively, becomes

$$\begin{aligned} \partial_\tau\varphi_1(\tau_0,t_{g0},0)&=x_{02}(T_0)V+Vt_{g0}+\partial_\tau x_{p,\eta^*}(T_0)(\tau_0,t_{g0},0),\\ &=x_{02}(T_0)V+Vt_{g0}+\tau_0^{-1}x_{01}(T_0)I_2-\tau_0^{-1}x_{02}(T_0)I_1, \end{aligned}$$

$$\begin{aligned} \partial_\tau\varphi_2(\tau_0,t_{g0},0)&=\dot{x}_{02}(T_0)V+V+\partial_\tau\dot{x}_{p,\eta^*}(T_0)(\tau_0,t_{g0},0),\\ &=\dot{x}_{02}(T_0)V-V+\tau_0^{-1}x_{01}(T_0)I_2-\tau_0^{-1}x_{02}(T_0)I_1, \end{aligned}$$

$$\begin{aligned} \partial_{t_g}\varphi_1(\tau_0,t_{g0},0)&=-\frac{\tau_0V}{t_{g0}-T_0}x_{02}(T_0)+V\tau_0+\partial_{t_g}x_{p,\eta^*}(T_0)(\tau_0,t_{g0},0)\\ &=-\frac{\tau_0V}{t_{g0}-T_0}x_{02}(T_0)-V\tau_0-\frac{x_{01}(T_0)I_2}{t_{g0}-T_0}+\frac{x_{02}(T_0)I_1}{t_{g0}-T_0}, \end{aligned}$$

$$\partial_{t_g}\varphi_2(\tau_0,t_{g0},0)=-\frac{\tau_0V}{t_{g0}-T_0}\dot{x}_{02}(T_0)+\frac{\tau_0V}{t_{g0}-T_0}+\partial_{t_g}\dot{x}_{p,\eta^*}(T_0)(\tau_0,t_{g0},0),$$

$$= -\frac{\tau_0 V}{t_{g0}-T_0}\dot{x}_{02}(T_0) - \frac{\tau_0 V}{t_{g0}-T_0} - \frac{\dot{x}_{01}(T_0)I_2}{t_{g0}-T_0} + \frac{\dot{x}_{02}(T_0)I_1}{t_{g0}-T_0}.$$

Thus, the determinant of $D_{(\tau,t_g)}\Phi(\tau_0,t_g,0)$ becomes

$$\det D_{(\tau,t_g)}\Phi(\tau_0,t_g,0) = (\partial_\tau\varphi_1\cdot\partial_{t_g}\varphi_2 - \partial_\tau\varphi_2\cdot\partial_{t_g}\varphi_1)(\tau_0,t_g,0) =$$

$$\big(x_{02}(T_0)V + Vt_{g0} + \tau_0^{-1}x_{01}(T_0)I_2 - \tau_0^{-1}x_{02}(T_0)I_1\big)\begin{bmatrix} -\frac{\tau_0 V}{t_{g0}-T_0}\dot{x}_{02}(T_0) + \frac{\tau_0 V}{t_{g0}-T_0} \\ -\frac{\dot{x}_{01}(T_0)I_2}{t_{g0}-T_0} + \frac{\dot{x}_{02}(T_0)I_1}{t_{g0}-T_0}\end{bmatrix}$$

$$-\big(\dot{x}_{02}(T_0)V - V + \tau_0^{-1}x_{01}(T_0)I_2 - \tau_0^{-1}x_{02}(T_0)I_1\big)\begin{bmatrix} -\frac{\tau_0 V}{t_{g0}-T_0}x_{02}(T_0) - \tau_0 V \\ -\frac{x_{01}(T_0)I_2}{t_{g0}-T_0} + \frac{x_{02}(T_0)I_1}{t_{g0}-T_0}\end{bmatrix}$$

$$= -\frac{\tau_0 V^2 t_{g0}}{t_{g0}-T_0}\dot{x}_{02}(T_0) + \tau_0 V^2\dot{x}_{02}(T_0) - \tau_0 V^2 + \frac{\tau_0 V^2 t_{g0}}{t_{g0}-T_0} - \frac{Vt_{g0}}{t_{g0}-T_0}\dot{x}_{01}(T_0)I_2$$
$$+V\dot{x}_{01}(T_0)I_2 + \frac{Vt_{g0}}{t_{g0}-T_0}\dot{x}_{02}(T_0)I_1 - \dot{x}_{02}(T_0)VI_1,$$

$$= -\frac{\tau_0 V^2 T_0}{t_{g0}-T_0}\dot{x}_{02}(T_0) + \frac{\tau_0 V^2 T_0}{t_{g0}-T_0} - \frac{VT_0}{t_{g0}-T_0}\dot{x}_{01}(T_0)I_2 + \frac{VT_0}{t_{g0}-T_0}\dot{x}_{02}(T_0)I_1.$$

Thus, $\det D_{(\tau,t_g)}\Phi(\tau_0,t_{g0},0) \neq 0$

$$\Leftrightarrow\ -\tau_0 V^2\dot{x}_{02}(T_0) + \tau_0 V + \dot{x}_{02}(T_0)I_1 - \dot{x}_{01}(T_0)I_2 \neq 0 \tag{39}$$

Therefore, if (39) is satisfied, equation (33) can be solved using Implicit Function Theorem. □

That means we can obtain (τ,t_g) in the neighborhood of

$$(\tau,t_g,\ell) = (\tau_0,t_{g0},0) \in \mathbb{R}^3.$$

Hence, $x(t,\ell) = x_0(t)+\bar{x}(\tau(\ell),t_g(\ell),\ell)$ is the solution of (1) in the neighborhood of $(x_0,\tau_0,0)$. This concludes the proof of Theorem 3.7 and hence the existence of the solution for (1). □

4. Conclusion

The global existence and uniqueness of wave solutions under spring-block model with non-smooth slip-dependent friction were proved using Poincare-Bendixson theroem and Lyapounov-Schmidt reduction technique. The non-smooth uncoupled version of the model was inclusion and the existence of its periodic solution was proved using Poincare-Bendixson theroem. However, the non-smooth coupled model was not an inclusion but a nonlinear differential equation which we used Lyapounov-Schmidt reduction technique to solve for both the periodic solution and the small perturbation solution that depended on parameter ℓ. To derive the perturbation solution and hence use the Lyapounov-Schmidt reduction, the compatibility condition (39) needed to be satisfied. Thus, analytical solutions for non-smooth slide-dependent friction laws provided more insight to periodic waves and may help provide more understanding of earthquakes and other dissipative driven systems.

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