ON SUBSPACE-TRANSITIVE OPERATORS

Sorayya Talebi\textsuperscript{1}§, Meysam Asadipour\textsuperscript{2}

\textsuperscript{1,2}Payame Noor University
P.O. Box 19395-4697, Tehran, IRAN

Abstract: The purpose of the present paper is to treat a notion, which can be viewed as a localization of the recent notion of subspace-transitivity. We conclude this paper to answer in affirmative one question asked by Madore and Martinez-Avandano with an additional assumption.

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1. Introduction

Let $X$ be a Banach space. In what follows, the symbol $T$ stands for a bounded linear operator acting on $T$ and $M$ will be a nonzero closed subspace of $X$. Consider any subset $C$ of $X$. The symbol $\text{Orb}(T, C)$ denotes the orbit of $C$ under $T$, i.e. $\text{Orb}(T, C) = \{T^n x : x \in C, n = 0, 1, 2, \ldots\}$. If $C = \{x\}$ is a singleton and the orbit $\text{Orb}(T, x)$ is dense in $X$, then the operator $T$ is called hypercyclic and the vector $x$ is a hypercyclic vector for $T$. Observe that in case the operator is hypercyclic the underlying Banach space $X$ should be separable. Then it is well known and easy to show that an operator $T$ is hypercyclic if and only if $T$ is topologically transitive, to be precise, for every pair of nonempty open sets $U, V$ of $X$ there exists a non-negative integer $n$ such that $T^n(U) \cap V \neq \emptyset$. The study of hypercyclicity goes back a long way, and has

\textsuperscript{§}Correspondence author
been investigated in more general settings, for example in topological vector spaces. A nice source of examples and properties of hypercyclic operators is the survey article [5], and see also recent books [1], [4].

Recently, B. F. Madore and R. A. Martinez-Avendano in [7] introduced the concept of subspace-hypercyclicity. An operator $T$ is subspace-hypercyclic (or $M$-hypercyclic) for a subspace $M$ of $X$ if there exists a vector $x \in X$ such that the intersection of its orbit and $M$ is dense in $M$. Also authors introduced the notion of subspace-transitivity (or $M$-transitivity) and show that $M$-transitivity implies $M$-hypercyclicity, and C. M. Le in [6] construct an operator $T$ such that it is $M$-hypercyclic but it is not $M$-transitive. The authors in [7] prove several results analogous to hypercyclicity case. Other sources of examples and some properties of notions relating subspace-hypercyclicity are [8], [9].

The purpose of this paper is twofold. Firstly, we somehow localize the notion of subspace-transitivity by introducing certain set, which we called $M$-extended limit set of $x$ under $T$, $J(T, M, x)$, for an operator $T$ and a given vector $x$. It is worthwhile to mention that the notion of $J$-class operators was introduced by G. Costakis and A. Manoussos in [3], [2], and with it, the authors localized the notion of hypercyclicity.

In [7] authors raised five questions relating subspace-hypercyclic-ity. We are interested in the first one. The second purpose of this paper is an application of the localization notion of subspace-transitivity in order to answer in the affirmative question:” let $T$ be an invertible operator. If $T$ is subspace-transitive for some $M$, is $T^{-1}$ subspace-hypercyclic for some space? If so, for which space?”

2. Preliminaries and Some Results

**Definition 1.** Let $T \in B(X)$. We say that $T$ is $M$-hypercyclic if there exists $x \in X$ such that $\text{Orb}(T, x) \cap M$ is dense in $M$. Such a vector $x$ is called an $M$-hypercyclic vector for $T$.

**Definition 2.** Let $T \in B(X)$. We say that $T$ is $M$-transitive if for any nonempty open subsets $U, V$ of $M$ there exists a non-negative integer $n$ such that $T^{-n} \cap V$ contains a relatively open nonempty subset of $M$.

The proof of the following proposition can be found in [7].

**Proposition 3.** Let $T \in B(X)$. Then the following conditions are equivalent:
(i) The operator $T$ is $M$-transitive.

(ii) For any nonempty sets $U$ and $V$, both relatively open, there exists $n \geq 0$ such that $T^{-n} \cap V$ is a relatively open nonempty subset of $M$.

(iii) For any nonempty sets $U$ and $V$, both relatively open, there exists $n \geq 0$ such that $T^{-n} \cap V$ is nonempty and $T^n(M) \subseteq M$.

In [6], [7] the authors prove that $M$-transitivity implies $M$-hypercyclicity and the converse is not always correct.

**Theorem 4.** Let $T$ is $M$-transitive. Then for any nonempty open subset $U$ of $M$, $\bigcap_{n=0}^{\infty} T^n(U) \cap M$ is dense in $M$.

**Proof.** Let $U$ be a nonempty subset of $M$, by previous proposition there exists some $k \geq 0$ such that

$$T^{-k}U \cap V \neq \emptyset \quad \text{and} \quad T^k(M) \subseteq M$$

hence

$$\emptyset \neq T^k(T^{-k}(V) \cap U) \subseteq V \cap T^k(U).$$

Therefore $\bigcap_{n=0}^{\infty} T^n(U) \cap M$ is dense in $M$. \hfill \Box

**Definition 5.** Let $T \in B(X)$. Then for any subsets $A, B \subseteq M$, the return set from $A$ to $B$ defined as

$$N_{(T,M)}(A,B) = \{n \geq 0 : T^{-n}(A) \cap B \text{ is nonempty open subset of } M\}$$

In this notation, $T$ is $M$-transitive if and only if, for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, the return set $N_{(T,M)}(U,V)$ is nonempty.

**Remark 6.** When $T$ is $M$-transitive, the proposition3 rearrange the return set $N_{(T,M)}(U,V)$ for any nonempty sets $U \subseteq M$ and $V \subseteq M$, both relatively open, as below:

$$N_{(T,M)}(U,V) = \{n \geq 0 : T^{-n}(U) \cap V \neq \emptyset \quad \text{and} \quad T^n(M) \subseteq M\}.$$

**Theorem 7.** Let $T$ is an $M$-transitive operator. Then for any pair $U,V$, both nonempty relatively open subsets of $M$, the return set $N_{(T,M)}(U,V)$ is infinite.
Proof. Since \( T \) is \( M \)-transitive, there exists \( n \geq 0 \) such that \( W = T^{-n}(U) \cap V \) is nonempty relatively open subset of \( M \). Consider two distinct points \( x, y \) in \( W \) and two relatively open subsets \( W_1 \) and \( W_2 \) of \( M \) such that

\[
x \in W_1, \quad y \in W_2, \quad W_1 \subseteq W, \quad W_2 \subseteq W, \quad W_1 \cap W_2 = \emptyset,
\]

consequently there exists \( k \geq 1 \) such that

\[
T^{-k}(W_1) \cap W_2 \neq \emptyset, \quad T^k(M) \subseteq M
\]

hence

\[
\emptyset \neq T^{-k}(W_1) \cap W_2 \subseteq T^{-k}(W) \cap W \subseteq T^{-(k+n)}(U) \cap V.
\]

(1)

Since \( T^n(M) \subseteq M \), so \( T^{(k+n)}(M) \subseteq M \). Therefore (1) implies that the intersection of \( N_{(T,M)}(U,V) \) and Natural numbers is nonempty. Proceeding inductively we find infinite integers \( n \in N_{(T,M)}(U,V) \).

\[ \square \]

Remark 8. An equivalent definition of an \( M \)-transitive operator is the following: for any nonempty sets \( U \subseteq M \) and \( V \subseteq M \), both relatively open, and for any \( N \geq 1 \), there exists a positive integer \( n > N \) such that \( T^{-n}(U) \cap V \) is a relatively open nonempty subset of \( M \).

Definition 9. Let \( T \) be an operator. For every \( x \in M \) the set

\[
J(T, M, x) = \{ y \in M : \text{for every relatively open neighborhoods } U, V \text{ of } x, y \text{ in } M \text{ respectively, and every positive integer } N,
\]

\[
\text{there exists } n > N \text{ such that } T^n(U) \cap V \neq \emptyset \text{ and } T^n(M) \subseteq M \}
\]

denote the \( M \)-extended limit set of \( x \) under \( T \).

Proposition 10. An equivalent definition of \( J(T, M, x) \) is the following.

\[
J(T, M, x) = \{ y \in M : \text{there exists a strictly increasing sequence of positive integers } \{ k_n \} \text{ and a sequence } \{ x_n \} \subseteq M
\]

\[
\text{such that } x_n \rightarrow x \text{ and } T^{k_n}x_n \rightarrow y \text{ and for every } n, \ T^{k_n}(M) \subseteq M \}.
\]

Proof. Let us prove that

\[
J(T, M, x) \subseteq \{ y \in M : \text{there exists a strictly increasing sequence}
\]
of positive integers \( \{k_n\} \) and a sequence \( \{x_n\} \subset M \)
such that \( x_n \rightarrow x \) and \( T^{k_n}x_n \rightarrow y \) and for every
\( n, \ T^{k_n}(M) \subseteq M \). since the converse inclusion is obvious. Let \( y \in J(T, M, x) \) and consider the open balls
\( U_n = B(x, \frac{1}{n}) \cap M, \ V_n = B(y, \frac{1}{n}) \cap M, \) for \( n = 1, 2, ... \)
and \( N = k_{n-1}, \ k_0 = 1 \). Then there exists \( k_n > N = k_{n-1} \) such that
\( T^{k_n}(U_n) \cap V_n \neq \emptyset \) and \( T^{k_n}(M) \subseteq M \).

Hence there exists \( x_n \in U_n \) such that \( T^{k_n} \in V_n \) and \( T^{k_n}(M) \subseteq M \). Therefore \( \{k_n\} \) is an strictly increasing sequence of positive integers and \( \{x_n\} \) is a sequence in \( X \) such that \( x_n \rightarrow x \) and \( T^{k_n}x_n \rightarrow y \) and for every \( n, \ T^{k_n}(M) \subseteq M \). \( \square \)

3. Main Results

The following characterization of \( M \)-transitive operators help us to answer in the affirmative question:” let \( T \) be an invertible operator. If \( T \) is an \( T \)-transitive for some \( M \), is \( T^{-1} \) subspace-hypercyclic for some space? If so, for which space?”.

**Theorem 11.** Let \( T \) be an operator on \( X \). Then the following conditions are equivalent:

(i) \( T \) is an \( M \)-transitive.

(ii) For every \( x \in M, \ J(T, M, x) = M \).

**Proof.** We first prove that (i) implies (ii). Let \( x \in U, \ y \in V \) and \( U, V \) be relatively open subsets of \( M \) and \( N \geq 1 \). There exists \( n > N \) such that \( U \cap T^{-n}(V) \) is nonempty and \( T^n(M) \subseteq M \). Thus \( y \in J(T, M, x) \), and consequently \( J(T, M, x) = M \).

We will show that (ii) \( \Rightarrow \) (i). Let \( U \subseteq M, \ V \subseteq M \), both nonempty and relatively open. Consider \( x_0 \in U, \ y_0 \in V \). Since \( J(T, M, x_0) = M \), there exists \( n \geq 1 \) such that \( T^n(V) \cap U \neq \emptyset \) and \( T^n(M) \subseteq M \). Proposition3 implies that \( T \) is an \( M \)-transitive operator. \( \square \)
The next example will show that subspace-hypercyclicity does not imply subspace-transitivity with respect to $M$.

**Example 12.** Let $\lambda \in \mathbb{C}$ be of modulus greater than 1 and let $B$ be the backward shift on $l^2$. Let $m$ be a positive integer and $M$ be the subspace of $l^2$ consisting of all sequences with zero on the first $m$ entries, that is:

$$M = \{\{a_n\}_{n=0}^{\infty} \in l^2 : a_n = 0 \text{ for } n \leq m\}$$

then $T = \lambda B$ is $M$-hypercyclic, see [7]. Now consider

$$V = \{\{a_n\}_{n=0}^{\infty} \in l^2 : a_n = 0 \text{ for } n \leq m \text{ and } |a_n| > 0 \text{ for } n > m\}$$

so $V$ is relatively open subset of $M$. If $N = m + 1$, then for every $n > N$, $T^n(V) \cap M = \emptyset$. Thus for every $x \in M$, $J(T, M, x) \neq M$.

**Theorem 13.** Let $T$ be an invertible operator and $M$-transitive. Then $T^{-1}$ is $M$-hypercyclic.

**Proof.** Let $x, y \in M$. Since $T$ is $M$-transitive, so $J(T, M, x) = M$. If $U, V$ are relatively open subsets of $M$ such that contain $x, y$ respectively, then there exists $n > 1$,

$$T^n(U) \cap V \neq \emptyset \quad \text{and} \quad T^n(M) \subseteq M$$

thus invertibility of $T$ implies that

$$T^{-n}(M) \subseteq M \quad \text{and} \quad U \cap T^{-n}(V) \neq \emptyset.$$ 

Hence for every $x \in M$, $x \in J(T^{-1}, M, y)$. This means for every $y \in M$,

$$M = J(T^{-1}, M, y)$$

or equivalently $T^{-1}$ is $M$-transitive.

**References**


