

FINITE SOLVABLE GROUPS HAVING A UNIQUE IRREDUCIBLE CHARACTER OF A GIVEN DEGREE

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Abstract: It has been conjectured that $\text{PSL}(2, q)$, the projective special linear group of 2×2 matrices over a field of order q , is the only non-solvable group satisfying the property that it has a unique irreducible complex character χ of degree $m > 1$ and every other irreducible complex character is such that its degree is relatively prime to m . (Such a χ is a particular case of the Steinberg character of finite Chevalley groups.) In this paper, we consider finite solvable groups satisfying the above property and obtain a complete classification.

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1. Introduction

If G is a finite simple Chevalley group, then G has an irreducible complex character χ whose degree is the order of a p -Sylow subgroup of G , and all other irreducible complex characters are in the principal p -block. Such a χ is called the Steinberg character. See [1] and [3]. For simple groups G , this property seems to hold only if G is a finite simple Chevalley group. The conjecture, then, is that the above property about the irreducible complex characters of G

forces G to be a finite simple Chevalley group. This conjecture seems to be too difficult at present. However, $\text{PSL}(2, q)$ is the only known non-solvable group which satisfies the following stronger property:

Property 1.1. There is a unique irreducible complex character χ of degree $m > 1$ and every other irreducible complex character is such that its degree is relatively prime to m .

So the problem of classifying all finite groups G satisfying Property 1.1 can be viewed, at least in the simple case, as a first step towards classifying finite simple Chevalley groups by their Steinberg character.

A natural and more general question that arises is “What are *all* the finite groups that satisfy Property 1.1?” In this paper we obtain a complete classification of all finite solvable groups satisfying Property 1.1 (see Theorem 3.1).

2. Notations and Preliminary Results

In this section we set the needed notations and state without proof some well known results in character theory which are needed in the proof of the main theorem. A reference is given in each case. All groups considered in this paper are finite. For a group G :

- (i) $Z(G)$ is the center of G .
- (ii) G' is the derived group of G .
- (iii) $[G : H]$ is the index of a subgroup H in G . $|H|$ and $|x|$ are orders of a subgroup H and an element x in G , respectively.
- (iv) If $H \leq G$ and $x, y \in G$, then $H^x = x^{-1}Hx$ and $y^x = x^{-1}yx$.
- (v) If ψ is a character of a subgroup H , and $x \in G$, then ψ^x , defined on H^x by $\psi^x(h^x) = \psi(h)$, $h \in H$, is a character of H^x . If H is normal in G , the stabilizer of $\psi = \{x \in G \mid \psi^x = \psi\}$ is denoted by $T_{\psi, H}$ or simply by T_{ψ} if the domain of ψ is clear from the context.
- (vi) ψ^G is the character of G induced from the character ψ of the subgroup H . $\chi|_H$ is the restriction to H of a character χ of G .
- (vii) $\ker \chi$ is the kernel of a representation corresponding to the character of G .

- (viii) The inner product of two class functions α and β on G is defined by $(\alpha, \beta)_G = |G|^{-1} \sum_{g \in G} \alpha(g) \overline{\beta(g)}$. In particular, if α is a character and β an irreducible character of G , then $(\alpha, \beta)_G$ is the multiplicity of β in α .

We assume all characters to be complex characters. Throughout, C_n denotes the cyclic group of order n .

We now present the needed results without proofs. A reference is provided in each case.

Lemma 2.1 (Frobenius reciprocity law, [5] Theorem 9.4). *If H is a subgroup of G and θ and η are complex valued class functions of H and G respectively, then $(\eta, \theta^G)_G = (\eta|_H, \theta)_H$.*

Lemma 2.2 ([5] Theorem 9.10). *Let H be a normal subgroup of G . If η is an irreducible character of G , then there exists an irreducible character θ of H and a positive integer a such that $\eta|_H = a \sum_{i=1}^t \theta^{g_i}$ where $t = [G : T_\theta]$ and $\{g_i\}$ is a complete system of coset representatives of T_θ in G .*

Lemma 2.3 ([2] Corollary 45.5). *Let θ be an irreducible character of a normal subgroup H of G . Then θ^G is irreducible if and only if $T_\theta = H$.*

Lemma 2.4 ([12] Theorem 1.9). *Suppose G is a solvable group, p a prime such that $(p, \psi(1)) = 1$ for each irreducible character ψ of G . Then G has a normal abelian p -Sylow subgroup.*

Lemma 2.5 (Thompson [13] Theorem 1). *Suppose G is any finite group, p a prime such that $p \mid \psi(1)$ for each non-linear irreducible character ψ of G . Then G has a normal p -complement.*

Lemma 2.6 ([5] Theorem 9.13). *If N is a normal and abelian subgroup of G and ψ is any irreducible character of G , then $\psi(1) \mid [G : N]$.*

Lemma 2.7 ([5] Theorem 12.1). *Let $A = (a_{ij})$ be a nonsingular complex matrix of degree k . If σ is a permutation of the k^2 ordered pairs (i, j) with $i, j = 1, 2, \dots, k$, define $A^\sigma = (a_{\sigma(i,j)})$. Suppose G is a group of permutations on the k^2 ordered pairs (i, j) , such that for each σ in G , A^σ can be obtained from A by permuting the rows of A and A^σ can also be obtained from A by permuting the columns of A . Let C be the columns of A and R the rows of A . Then G acts on C and R as a permutation group, and the number of orbits is the same in both representations. Moreover, if G is cyclic, then the number of rows fixed by G equals the number of columns fixed by G .*

In all applications of the above lemma in this paper, A will be the character table of a normal subgroup.

Lemma 2.8 ([12] Proposition 1.5). *Let G be a group and $N \triangleleft G$. Then G is Frobenius with kernel N if and only if each non-principle irreducible character of N induces an irreducible character of G .*

Lemma 2.9 (Gallagher [6] Theorem 2). *Let $N \triangleleft G$, τ be an irreducible character of N . Suppose τ has an extension $\hat{\tau}$ to T_τ . Then $\tau^G = \sum_{\omega} \omega(1)(\omega\hat{\tau})^G$ where the sum runs over the irreducible characters ω of T_τ/N . Each $(\omega\hat{\tau})^G$ is irreducible and $(\omega_1\hat{\tau})^G = (\omega_2\hat{\tau})^G$ implies $\omega_1 = \omega_2$.*

Lemma 2.10 (Gallagher [6] Theorem 1). *Let $N \triangleleft G$ and τ be an irreducible character of N . Then for each irreducible constituent χ of τ^G , there is a unique irreducible character ζ of T_τ such that $(\tau^{T_\tau}, \zeta)_{T_\tau} \geq 1$ and $(\zeta^G, \chi)_G \geq 1$. For this ζ , $\zeta|_N = a\tau$ for some positive integer a and $\zeta^G = \chi$.*

Lemma 2.11 (Gallagher [6] Theorem 3). *Let N be a normal subgroup of prime index p in G . Then each invariant irreducible character of N extends to an irreducible character of G .*

Lemma 2.12 (Gallagher [6] Lemma 1 and Theorem 6). *Suppose $N \trianglelefteq G$. If N is a Hall subgroup of G , then each irreducible character of N extends to its stabilizer. If N is abelian and complemented in G , then each irreducible character of N extends to its stabilizer.*

Lemma 2.13 (Isaacs [10] Proposition 3). *Let $K \triangleleft H$ be such that H/K is an elementary abelian p -group and let A act on H and normalize K , with the induced action on H/K being non-trivial and irreducible. Then the following hold.*

(i) *If $\chi \neq 1_H$ is an irreducible character of H which is stabilized by A , we have that either:*

- (a) $\chi|_K$ is irreducible and $\neq 1_K$, or
- (b) $\chi|_K = a\theta$, θ irreducible, $\neq 1_K$ and $a^2 = [H : K]$, or
- (c) $\chi|_K = \sum_{i=1}^t \theta_i$ where $t = [H : K]$ and the θ_i are distinct irreducible characters of K , transitively permuted by H .

(ii) *If $\theta \neq 1_K$ is an irreducible character of K which is stabilized by A and θ^H is the induced character, we have that either:*

- (a) θ^H is irreducible, or
- (b) $\theta^H = a\chi$, χ irreducible and $a^2 = [H : K]$, or
- (c) $\theta^H = \sum_{i=1}^t \chi_i$ where $t = [H : K]$ and the χ_i are distinct irreducible characters of H which are transitively permuted by multiplication by the linear characters of H/K .

In all applications of the above lemma in this paper, we let H/K be a chief factor of a solvable group G and let $A = G$. We also note from the proof that the assumption that the action of A is non-trivial on H/K is used only to show $\theta \neq 1$ in (b) of (i).

Lemma 2.14 (Seitz [11]). *A group G has exactly one non-linear, irreducible character if and only if (i) G is an extra-special 2-group or (ii) G is isomorphic to the group of all transformations $x \mapsto \alpha x + \beta$, $\alpha \neq 0$ on a field of order $p^n \neq 2$.*

The following lemma is needed in the proof of the next lemma, which will play an important role in the proof of the main theorem in the next chapter.

Lemma 2.15. *Let $G = HA$ be a Frobenius group with kernel H and complement A . Suppose A is cyclic. Let V be a faithful, irreducible module of $\mathbb{C}G$, the group algebra over complex numbers. Then $V|_A = t \cdot \rho_A$ where ρ_A is the regular representation of A and t is some positive integer.*

Proof. By Lemma 2.2, $V|_{Z(H)} = V_1 + V_2 + \cdots + V_\ell$ where the V_i are homogeneous and transitively permuted by A .

Claim. *No non-identity element of A stabilizes V_i for any i .*

Suppose that the subgroup A_1 , the stabilizer of V_1 in A , is non-trivial. Then $G_1 = Z(H)A_1$ is a Frobenius group. Let N_1 be the kernel of the representation of G_1 on V_1 . Since G_1 is Frobenius, either $N_1 < Z(H)$ or $N_1 \geq Z(H)$. If $N_1 \geq Z(H)$, then, since the V_i 's are transitively permuted by A , $Z(H)$ is contained in the kernel of the representation of $Z(H)$ on every V_i , contradicting the hypothesis that V is a faithful module for G . So $N_1 < Z(H)$ and $G_1/N_1 = Z(H)/N_1 \cdot A_1N_1/N_1$ is a Frobenius group. Since G_1/N_1 acts faithfully on V_1 , $Z(H)/N_1$ is cyclic and each element of $Z(H)/N_1$ acts as a scalar transformation on V_1 . Therefore $[Z(H)/N_1, A_1N_1/N_1]$ is in the kernel of the representation of G_1/N_1 on V_1 , and hence, $[Z(H)/N_1, A_1N_1/N_1] = 1$. This is a contradiction since G_1 is Frobenius. This proves the claim.

If $v \neq 0 \in V_1$, the elements in the set $\{v^a \mid a \in A\}$ lie in distinct direct summands V_i by our claim and, hence, are linearly independent. Since $(v^a)^{a'} = v^{aa'}$, the space spanned by $\{v^a \mid a \in A\}$ gives a regular representation of A . Now let $\{v_1, v_2, \dots, v_t\}$ be a basis of V_1 . Then $\{v_i^a \mid 1 \leq i \leq t, a \in A\}$ is a basis of V and

$$V|_A = \langle \{v_1^a \mid a \in A\} \rangle \oplus \langle \{v_2^a \mid a \in A\} \rangle \oplus \cdots \oplus \langle \{v_t^a \mid a \in A\} \rangle$$

is a direct sum of t submodules, each giving a regular representation of A . Thus, $\theta|_A = t \cdot \rho_A$. \square

Lemma 2.16. *Let K be minimal normal in a solvable group G and let G/K be Frobenius with kernel M/K . Suppose that G/M is cyclic of order $q-1$ where $q = p^n$ for some prime p and M/K is elementary abelian of order q but M is not abelian. Then one of the following assertions holds:*

- (a) $q = 3$
- (b) K is a p -group (so that M is a p -group)
- (c) There exists an irreducible character $\varphi \neq 1_K$ of K and an irreducible character θ of G such that $(\varphi^G, \theta)_G \geq 1$ and $(\theta(1), q-1) \neq 1$.

Proof. Suppose that neither (b) nor (c) holds. We shall prove that $q = 3$. Let $|K| = r^m$, r a prime $\neq p$, and let Q be a p -Sylow subgroup of G . Then $Q \leq M$ and $M = Q \cdot K$. By the Frattini argument, $G = N_G(Q) \cdot M = N_G(Q) \cdot Q \cdot K = N_G(Q) \cdot K$. Noting that $N_G(Q) \cap K \triangleleft G$, K is minimal normal in G and that M is non-abelian, we get $N_G(Q) \cap K = 1$. So G is a semidirect product of $N_G(Q)$ and K and $N_G(Q)$ can be written as $Q \cdot T$ where T is a subgroup of order $q-1$ and is cyclic.

Let $\varphi \neq 1_K$ be a linear character of K . By Lemma 2.12 φ extends to a linear character of $\hat{\varphi}$ of the stabilizer T_φ of φ . Then $\hat{\varphi}^G$ is irreducible ([2] Corollary 45.4) of degree $\hat{\varphi}^G(1) = \hat{\varphi}(1)[G : T_\varphi] = [G : T_\varphi]$. Then $1 = (\hat{\varphi}^G, \hat{\varphi}^G)_G = (\hat{\varphi}^G, \hat{\varphi})_{T_\varphi}$ by Lemma 2.1 and this implies $(\hat{\varphi}^G|_K, \hat{\varphi}|_K)_K \geq 1$. Since $\hat{\varphi}|_K = \varphi$, we get $(\hat{\varphi}^G|_K, \varphi)_K \geq 1$, and using Lemma 2.1 again, we get $(\hat{\varphi}^G, \varphi^G)_G \geq 1$. Since we are assuming that the possibility (c) of the conclusion does not occur, we conclude that $\hat{\varphi}^G(1) = [G : T_\varphi]$ and $q-1$ are relatively prime. Hence $q-1 \mid |T_\varphi|$ and so by a theorem of Philip Hall ([8] Theorem 9.3.1), some conjugate of $TK \leq T_\varphi$. Let R denote the representation of G/K acting by conjugation on the irreducible module K . If the representation is not faithful, then the kernel, say N/K is either contained in M/K or contains M/K . Since M is not abelian, $N/K < M/K$. Therefore the character θ remains the same on TK/K whether K is considered as a G/K -module or as a G/N module.

Hence, by Lemma 2.15, $R|_{TK/K} = t \cdot \rho_{TK/K}$, where $\rho_{TK/K}$ is the regular representation of TK/K and t is some positive integer. In every regular representation module, the dimension of the subspace of fixed points is 1 and so TK/K fixes $r^t - 1$ non-trivial elements of K . Since the size of $\rho_{TK/K}(1) = |TK/K| = |T| = q-1$, we get $|K| = r^{t(q-1)}$ and so there are $r^{t(q-1)} - 1$ non-trivial elements in K . Since G/K is Frobenius and since $|TK/K| = |M/K| - 1$, TK/K acts transitively on the non-trivial elements of M/K . Therefore, two different TK/K 's generate all of G/K , and this implies two different TK/K 's cannot fix the same non-trivial element in K , since K

is minimal normal and M is non-abelian. There are q conjugates of TK/K , each fixing $r^t - 1$ non-trivial elements in K . Thus, equating the total number of fixed elements in K , we get

$$q(r^t - 1) = r^{t(q-1)} - 1$$

i.e.,

$$q = \frac{r^{t(q-1)} - 1}{r^t - 1} = r^{t(q-2)} + r^{t(q-3)} + \dots + 1 > 2^{q-2}.$$

This clearly implies that $q = 3$. The proof is complete. \square

3. Results

The main theorem of this paper is the following:

Theorem 3.1. *Suppose G is a finite solvable group having exactly one irreducible character χ of degree $\chi(1) = m > 1$ and every irreducible character η of G is such that $(\eta(1), m) = 1$. Then G is isomorphic to one of the following:*

- (a) *An extra-special 2-group*
- (b) *The group of all transformations $x \mapsto \alpha x + \beta$ ($\alpha \neq 0$) on a field of order $p^n \neq 2$*
- (c) *S_4 , the symmetric group on 4 letters*
- (d) *$SL(2, 3)$, the special linear group of 2×2 matrices over field of order 3*
- (e) *A semidirect product of $SL(2, 3)$ with $C_3 \times C_3$.*

The character χ is faithful in cases (a) and (b) and non-faithful in cases (c), (d), and (e).

The proof is carried out in a sequence of lemmas.

Lemma 3.2. *Suppose the hypotheses of the theorem hold and that χ is faithful. Then G is isomorphic to one of the groups listed in (a) and (b).*

Proof. Let $\chi(1) = m = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots \cdot p_k^{m_k}$, where each p_i is prime and let N be a minimal normal subgroup of G . Then, by hypothesis, B/N has no irreducible character ζ such that $p_i \mid \zeta(1)$ for any $i = 1, 2, \dots, k$. By Lemma 2.4, G/N has normal abelian p_i -Sylow subgroup for every i . Let P_i be a p_i -Sylow subgroup of G and H be the group $P_1 \cdot P_2 \cdot \dots \cdot P_k \cdot N$ so that H is normal in G .

Claim. $H = G$

Suppose $H \neq G$ and that M is a maximal normal subgroup of G containing H . Then G/M is cyclic of order p for some prime $p \neq p_i$ for any i . By Lemma 2.2,

$$\chi|_M = a \sum_{i=1}^t \theta^{g_i}$$

where θ^{g_i} are irreducible characters of M , a and t are positive integers and $t = [G : T_\theta]$. Equating the degrees, we get

$$\chi(1) = a \cdot t \cdot \theta(1).$$

If $\theta(1) = 1$, then $M' \subseteq \bigcap_{i=1}^t \ker \theta^{g_i} \subseteq \ker \chi = 1$, and so M is abelian. But then Lemma 2.6 implies $\chi(1) \mid [G : M]$, which is a contradiction. Therefore $\theta(1) > 1$. Also, since $[G : M]$ is a prime number, $t = 1$ or $[G : M]$. If $t = [G : M]$, then θ^G is irreducible (Lemma 2.3) of degree $\theta(1)[G : M]$, and since $\theta(1) \mid \chi(1)$, this is a contradiction to our hypothesis. Therefore $t = 1$. By Lemma 2.11, θ extends to an irreducible character $\hat{\theta}$ of G . By hypothesis, $\hat{\theta} = \chi$ and so $\chi|_M = \theta$. But then $\theta = \chi|_M$ has $[G : M]$ distinct extensions, once again contradicting our hypothesis. This proves our claim that $H = G$.

Since N is normal and abelian, Lemma 2.6 and our hypotheses imply that χ is the only non-linear irreducible character of G . Lemma 2.14 yields the required result. \square

Remark 3.3.

- (i) As seen in the above proof, the hypothesis of Lemma 3.2 implies that χ is the only non-linear irreducible character of G .
- (ii) If G is an extra-special 2-group of order 2^k , then $\chi(1) = 2^{(k-1)/2}$, so that the degree of χ determines the order of the group and conversely.
- (iii) If G is a group of all transformations $x \mapsto \alpha x + \beta$ ($\alpha \neq 0$) on a field of order $p^n \neq 2$, then G is a Frobenius group of order $p^n(p^n - 1)$. If M is the Frobenius kernel of G , then $|M| = p^n$, M is elementary abelian and G/M is cyclic. The degree of χ is $p^n - 1$ and so once again the order of G is determined by the degree of χ , and conversely.

We now set out to investigate $\ker \chi$.

Lemma 3.4. *Suppose that the hypothesis of Theorem 3.1 holds and that $\ker \chi = K > 1$. Then G/K is a Frobenius group of the type listed in (b) of Theorem 3.1. If M/K denotes the Frobenius kernel, then $M' = K$.*

Proof. By Lemma 3.2, G/K is either an extra special 2-group or a Frobenius group. We shall prove G/K cannot be a 2-group. To this end, assume G is a minimal counter-example so that G/K is a 2-group. By minimality of $|G|$, K is a minimal normal subgroup of G . Then Lemma 2.6 and the hypothesis of Theorem 3.1 together imply that χ is the only non-linear irreducible character of G . Therefore, by Lemma 2.14, G is either an extra-special 2-group or a Frobenius group, and in either case χ is faithful. This contradicts our assumption that $\ker \chi = K > 1$. Hence, G/K is a Frobenius group.

It remains to prove $M' = K$, where M/K is the Frobenius kernel of G/K . Clearly $M' \leq K$, since M/K is abelian (see Remark 3.3 (iii)). If $M' < K$, then M/M' is a normal, abelian subgroup of G/M' . Hence it follows once again by Lemma 2.6 and the hypotheses (note $\chi(1) = [G : M]$) that χ is the only non-linear character of G . In such a case χ is always faithful and again we have a contradiction. This proves Lemma 3.4. \square

As we will see later, the following lemma and Lemma 2.6 together reduce the proof to the case $\chi(1) = 2$ or 3.

Lemma 3.5. *Suppose the hypotheses of Theorem 3.1 hold and that $\ker \chi = K > 1$. Further, suppose that K is minimal normal and that M (same as in Lemma 3.4) is a p -group for some prime p . Then $G \cong SL(2, 3)$.*

Proof. The proof is carried out in 3 steps.

Step 1. $K \leq Z(G)$.

Suppose $K \not\leq Z(G)$. Let $|M/K| = p^n$ so that $[G : M] = \chi(1) = p^n - 1$. Since G is solvable, we can write $G = MT$ where $T \cong G/M$ is a cyclic group of order $p^n - 1$. Since K is minimal normal and M is a p -group, we have $K \leq Z(M)$. Then $G/C_G(K)$ is cyclic; say $G/C_G(K) = \langle \bar{g} \rangle$, where $\bar{g} = gC_G(K)$, $g \in T$. The group $\langle g \rangle$ acts on the irreducible characters of K and if g fixes every irreducible character of K , then by Lemma 2.7 (applied to the character table of K) $g \in C_G(K)$, contrary to our assumption that $G \neq C_G(K)$. So there exists $\phi \neq 1$, an irreducible character of K , such that $\phi \neq \phi^g$. Let η be an irreducible character of M such that $(\phi^M, \eta)_M \geq 1$. If $\eta(1) = 1$, then $\eta = 1$ on $M' = K$ and by Lemma 2.1, we get $(\phi^M, \eta)_M = (\phi, 1)_K \geq 1$. This is impossible since $\phi \neq 1$. Therefore $\eta(1) > 1$. Since $K \leq Z(M)$, $\eta|_K = a \cdot \phi$ for some positive integer a and so $\eta^g|_K = a \cdot \phi^g$. Hence, $\eta \neq \eta^g$ and so T_η , the stabilizer of η , is a proper subgroup of G . If $T_\eta = M$, then η^G would be an irreducible character (Lemma 2.3) of degree $\chi(1)\eta(1) > \chi(1)$ (since $\eta(1) > 1$). This is impossible by our hypotheses. Thus $G > T_\eta > M$. Let θ be an irreducible character of G

such that $(\eta^G, \theta)_G \geq 1$. By Lemma 2.10, there exists an irreducible character ζ of T_η such that

- (i) $\zeta^G = \theta$ and
- (ii) $\zeta|_M = r \cdot \eta$ for some positive integer r .

Then the equation (i) implies $(\theta(1), \chi(1)) \neq 1$. Therefore $\theta = \chi$, by hypothesis, and so $\chi(1) = p^n - 1 = \zeta^G(1) = \zeta(1)[G : T_\eta]$. Hence

$$\zeta(1) = \frac{p^n - 1}{[G : T_\eta]}. \tag{1}$$

On the other hand the equation (ii) implies $\zeta(1) = r \cdot \eta(1)$. Since $p|\eta(1)$ (η is an irreducible character of M , a p -group), we get $p|\zeta(1)$ and this is a contradiction to (1). Hence $K \leq Z(G)$.

Step 2. $Z(M) = K$.

If $Z(M) > K$, then $G/Z(M)$ is a Frobenius group with kernel $M/Z(M)$ and complement a cyclic group of order $p^n - 1$. But an element of order $p^n - 1$ cannot act fixed-point free on a group of order $< p^n$ and $|M/Z(M)| < p^n$. Therefore $Z(M) = K$.

Step 3. We now prove Lemma 3.5.

Since K is minimal normal and contained in $Z(G)$, $|K| = p$ for some prime p . Also M is a class 2, p -group with $Z(M) = K = M'$ (Lemma 3.4) and M/K is elementary abelian. Thus M/K is a vector space over K , considered as a field of order p . Then the map from $M/K \times M/K \rightarrow K$ defined by $[aK, bK] \mapsto a^{-1}b^{-1}ab$ is easily checked to be a well-defined alternating non-degenerate linear form on $M/K \times M/K$. Thus M/K is a symplectic space. The cyclic group T of order $p^n - 1$ can be considered a subgroup of $\text{Sp}(2 \cdot n/2, p)$ (the group of symmetries of the space M/K). Since T acts irreducibly on M/K , by a theorem on symplectic groups ([9] Satz 9.23, p.228) we get $|T| = p^n - 1 \mid p^{n/2} + 1$. This implies $p = n = 2$. Hence, $|K| = 2$, $|M/K| = 4$ and $|G/M| = 3$, i.e., G is a group of order 24 and $\chi(1) = 3$. It is now easy to check that $G \cong \text{SL}(2, 3)$. \square

The next two lemmas are particular cases of Theorem 3.1 and settle cases when $\chi(1) = 2$ or 3.

Lemma 3.6. *Suppose the hypotheses of Theorem 3.1 hold with $\chi(1) = 2$. If $\ker \chi = K > 1$, then $G \cong S_4$.*

Proof. From Lemma 3.4, we have the following picture:

$$\begin{array}{c} G \\ | \quad 2 \\ M \\ | \quad 3 \\ K \\ | \\ 1 \end{array}$$

where $M' = K$ and $G/K \cong S_3$.

The proof is carried out in a sequence of steps.

Step 1. M is a Frobenius group with kernel K .

By Lemma 2.8, it is enough to show that every non-trivial irreducible character φ of K is such that φ^M is an irreducible character of M . Let η be an irreducible character of M such that $(\eta, \varphi^M)_M \geq 1$. Lemma 2.1 and the fact that $M' = K$ imply that $\eta(1) > 1$. Then $T_\eta = G$, for if $T_\eta = M$, then η^G would be an irreducible character of G with $2 \mid \eta^G(1)$ and $2 < \eta^G(1)$, contrary to the hypothesis. Thus, Lemma 2.13 applies and we have the following possibilities.

- (i) $\eta|_K$ is irreducible,
- (ii) $\eta|_K = a\varphi$, $a^2 = [M : K]$,
- (iii) $\eta|_K = \sum_{i=1}^3 \varphi_i$, φ_i distinct irreducible characters of K transitively permuted by M ($\varphi_1 = \varphi$, say).

Possibility (ii) is ruled out since a^2 cannot be 3. Suppose possibility (i) occurs. Since $T_\eta = G$, by Lemma 2.12, η extends to an irreducible character $\hat{\eta}$ of G . Then the characters $\{\omega \cdot \hat{\eta} \mid \omega \text{ irreducible character of } G/K \cong S_3\}$ are all irreducible by Lemma 2.9. But one of the ω 's is χ and $\hat{\eta}(1) > 1$, and this is a contradiction to our hypothesis. Therefore, only the possibility (iii) can occur and this is equivalent to saying $\varphi^M = \eta$ (Lemma 2.3). The proof of Step 1 is complete.

Step 2. K is an abelian 2-group.

We first note that K is nilpotent since it is the kernel of a Frobenius group ([7] Theorem 10.3.1 (iii), p.339). Next we prove the following.

Claim. $|K|$ is even.

Proof. Suppose $|K|$ is odd and G is a minimal counterexample. If there is a normal subgroup L of G such that $1 < L < K$, then by minimality of G , $2 \mid [K : L]$, contrary to our supposition that $|K|$ is odd. Therefore we can assume K is minimal normal in G and $K \cong C_p \times C_p \times \cdots \times C_p$, with $p > 3$ ($p \neq 3$ since M is Frobenius with kernel K). Thus, we can regard K as an irreducible module for $G/K \cong S_3$. Since $p > 3$, the degree of an irreducible representation of S_3 over C_p is either 1 or 2. Also, $K \neq C_p$, since $\text{Aut}(C_p)$ is cyclic and G/K is a non-cyclic group acting faithfully on K . Therefore, we can assume $K \cong K_1 \times K_2$ with $K_1 \cong K_2 \cong C_p$. Let tK and xK be elements of order 2 and 3 respectively in G/K . The minimal polynomial of tK on K is $Y^2 - 1 = (Y + 1)(Y - 1)$, for if tK acts as either 1 or -1 on K , then tK and xK commute on K , so that $\langle [tK, xK] \rangle = \langle xK \rangle$ acts trivially on K and this is not the case. Therefore, we may suppose $tK|_{K_1} = -1$ and $tK|_{K_2} = 1$. Let $\zeta \neq 1_K$ be an irreducible character of K/K_2 . Then $\zeta \neq \zeta^t$ and we know $\zeta, \zeta^x, \zeta^{x^2}$ are distinct since M is Frobenius. Let $g \neq 1 \in K_2$ and let $g^{x^{-1}} = g_1 \cdot g_2$ for some $g_1 \neq 1 \in K_1$ and $g_2 \in K_2$. (If $g_1 = 1$, then K_2 would be normal in G whereas K is minimal normal in G). Then $\zeta^t(g) = \zeta(g^t) = 1$ whereas $\zeta^x(g) = \zeta(g^{x^{-1}}) = \zeta(g_1)\zeta(g_2) = \zeta(g_1) \neq 1$. Thus $\zeta^x \neq \zeta^t$. Similarly, $\zeta^{x^2} \neq \zeta^t$ and hence, there are more than 3 conjugates of ζ . It now follows from Lemma 2.3 that ζ^G is irreducible. This is a contradiction to our hypothesis on the degrees of the irreducible characters of G and the proof of the claim is complete. \square

Since K is nilpotent, we can write $K = P_1 \times P_2 \times \cdots \times P_r$ with P_1 a 2-Sylow subgroup of K . Considering the quotient group G/P_1 and using induction on $|G|$, we obtain $P_i = 1$ for $i > 1$. Thus, $K = P_1$, a 2-group. If K is not abelian, then there exists an irreducible character θ of K such that $\theta(1) = 2^\ell$, $\ell \geq 1$. Then θ^M is irreducible since M is Frobenius. If $T_{\theta^M} = M$, then $(\theta^M)^G = \theta^G$ is irreducible by Lemma 2.3, and if $T_{\theta^M} = G$, then θ^M extends to an irreducible character of G by Lemma 2.11. In either case, we have a contradiction to our hypothesis. Therefore K is an abelian 2-group. The proof of Step 2 is complete.

Step 3. $K \cong C_2 \times C_2$.

Let $\Omega_1(K) = \{g \in K \mid g^2 = 1\}$. Then, $1 < \Omega_1(K)$ is a characteristic subgroup of K , and so $\Omega_1(K) \triangleleft G$. Using induction on $|G|$, we conclude that

$$\text{either } K/\Omega_1(K) \cong C_2 \times C_2, \text{ or } K = \Omega_1(K) \tag{2}$$

Since x (same as in Step 2) does not fix anything in $\Omega_1(K)$, $\Omega_1(K)$ decomposes

into $\Omega_1(K) = L_1 \times L_2 \times \cdots \times L_r$ with each $L_i \cong C_2 \times C_2$ and is $\langle x \rangle$ -irreducible.

Claim. L_i is $\langle t \rangle$ -invariant for every i .

Proof. Fix i and suppose $L_i \neq L_i^t$. By the Frattini argument, $N_G(\langle x \rangle) \cdot M = G$. Since $M = \langle x \rangle K$, we obtain $N_G(\langle x \rangle) \cdot K = G$. Also, since $\langle x \rangle \cdot K$ is Frobenius, $N_G(\langle x \rangle) \cap K = 1$. Thus, we may suppose that $t \in N_G(\langle x \rangle)$, and so x leaves L_i^t invariant. Therefore, $L_i \cap L_i^t = 1$, for otherwise $L_i \cap L_i^t$ would be an $\langle x \rangle$ -invariant subgroup of order 2. But then, x fixes some non-trivial element of K . Hence we may suppose that $L_i^t = L_j$ for some j , i.e., $\Omega_1(K) = L_i \times L_j \times \prod_{s \neq i, j} L_s$ with $L_i^t = L_j$. Let $L_i = \langle a \rangle \times \langle b \rangle$, $a^2 = b^2 = 1$. Let $\varphi \neq 1$ be an irreducible character of $\Omega_1(K)$ having $\langle b \rangle \times L_j \times \prod_{s \neq i, j} L_s$ as its kernel. Then $\varphi, \varphi^x, \varphi^{x^2}$ are distinct since $\langle x \rangle \cdot \Omega_1(K)$ is Frobenius. Also, since L_j is $\langle x \rangle$ -invariant, L_j is contained in the kernel of each φ, φ^x and φ^{x^2} , whereas $\varphi^t(a^t) = \varphi(a) \neq 1$. Therefore $\varphi^t \neq \varphi, \varphi^x, \varphi^{x^2}$. Since K is abelian, φ extends to an irreducible character $\hat{\varphi}$ of K (Lemma 2.12) and the above argument implies $T_{\hat{\varphi}} = K$. Hence $\hat{\varphi}^G$ is irreducible (Lemma 2.3) of degree 6. This contradiction proves the claim. \square

Thus $L_i \triangleleft G$ for every i . Then, by induction on $|G|$, $G/L_1 \cong S_4$ or S_3 . This, together with (2), implies either (i) $K/\Omega_1(K) \cong C_2 \times C_2$ with $\Omega_1(K) \cong C_2 \times C_2$, or (ii) $K = \Omega_1(K) \cong C_2 \times C_2$, or (iii) $K = \Omega_1(K) \cong (C_2 \times C_2) \times (C_2 \times C_2)$ with each $(C_2 \times C_2) \triangleleft G$. Thus, we conclude that one of the following holds.

- (i) $K \cong C_2 \times C_2$
- (ii) $K \cong (C_2 \times C_2) \times (C_2 \times C_2)$ with each $(C_2 \times C_2) \triangleleft G$
- (iii) $K \cong C_4 \times C_4$

If (i) holds, we are done. We shall prove that neither (ii) nor (iii) can hold. To this end, first suppose (ii) holds and let $K = L_1 \times L_2$ where $L_i \cong C_2 \times C_2$ and $L_i \triangleleft G$ for $i = 1, 2$. As before, if t is an element of order 2 in $N_G(\langle x \rangle)$, then t cannot centralize either L_1 or L_2 ; for if it does, say L_1 , then $C_G(L_1)$ would be a normal subgroup of G containing K and $3 \mid [G : C_G(L_1)]$. Since $G/K \cong S_3$, $C_G(L_1) = G$, contradicting the fact that M is Frobenius. Therefore, there exists an irreducible character say $\zeta \neq 1$ of L_1 such that $\zeta^t \neq \zeta$ (Lemma 2.7). Actually we can produce an irreducible character θ of L_1 such that $\theta^t \neq \theta$ and also $\theta^t \neq \theta^x$. to get such a θ , first we get a ζ , as above. If $\zeta^t \neq \zeta^x$, then we set $\theta = \zeta$. If $\zeta^t = \zeta^x$, then we let $\theta = \zeta^t$. Then $\theta^t = (\zeta^t)^t = \zeta \neq \theta$ and $\theta^x = (\zeta^t)^x = (\zeta^x)^x = \zeta^{x^2} \neq \zeta = \theta^t$. Similarly, we can obtain an irreducible character φ of L_2 such that $\varphi^t \neq \varphi$ and also $\varphi^t \neq \varphi^{x^2}$. Then $\theta \cdot \varphi$ is an irreducible character of $K \cong L_1 \times L_2$ ([7] Theorem 3.7.1, p.100). We now check

that $\theta \cdot \varphi$ has 6 conjugates. Clearly, $\theta \cdot \varphi$, $(\theta \cdot \varphi)^x$ and $(\theta \cdot \varphi)^{x^2}$ are distinct since M is Frobenius. $(\theta \cdot \varphi)^t = \theta \cdot \varphi$ implies $\theta^t = \theta$ and $\varphi^t = \varphi$, a contradiction. $(\theta \cdot \varphi)^t = (\theta \cdot \varphi)^x$ implies $\theta^t = \theta^x$, a contradiction. $(\theta \cdot \varphi)^t = (\theta \cdot \varphi)^{x^2}$ implies $\varphi^t = \varphi^{x^2}$, a contradiction. Hence $(\theta \cdot \varphi)^G$ is an irreducible character of G (Lemma 2.3) of degree 6. This is a contradiction to our hypothesis. Therefore (ii) cannot hold.

Now suppose (iii) holds, i.e. $K \cong C_4 \times C_4$, and $\Omega_1(K) = C_2 \times C_2 \triangleleft G$. Then t does not centralize $\Omega_1(K)$ since G/K has no normal subgroup of index 3. Therefore, we may suppose $\Omega_1(K) = \langle a \rangle \times \langle b \rangle$, $a^2 = b^2 = 1$ and $a^t = b$, $b^t = a$. Let $h \in K$ be such that $h^2 = a$, $|h| = 4$. Then, $\langle h \rangle \cap \langle h^t \rangle = 1$, since $h^2 = a$ is not fixed by t . Let $K_\circ = \langle hh^t \rangle$. Then $|K_\circ| = 4$ and t centralizes K_\circ . Also $(hK_\circ)^t = h^t K_\circ = h^{-1} K_\circ$. If $\varphi \neq 1$ is a faithful irreducible character of K/K_\circ then $\varphi^t \neq \varphi$ and $\ker \varphi^t = \ker \varphi$. Moreover $|K_\circ \cdot \Omega_1(K)/\Omega_1(K)| = 2$ and so x cannot normalize $K_\circ \Omega_1(K)/\Omega_1(K)$, since $G/\Omega_1(K) \cong S_4$, and $M/\Omega_1(K)$ is Frobenius. Therefore, K_\circ is not normalized by x , and so $K_\circ, K_\circ^x, K_\circ^{x^2}$ are distinct. Hence, $\varphi, \varphi^x, \varphi^{x^2}$ are distinct and none of them equals φ^t since the kernels are different. This implies, again, that φ has 6 conjugates and so φ^G is an irreducible character of G of degree 6. This contradiction proves Step 3 and hence the lemma. □

Lemma 3.7. *Suppose the hypothesis of Theorem 3.1 holds with $\chi(1) = 3$. If $\ker \chi = K > 1$ then G is isomorphic to either $SL(2, 3)$ or to a semidirect product of $SL(2, 3)$ with $C_3 \times C_3$.*

Proof. From Lemma 3.4, we have the following picture:

$$\begin{array}{c}
 G \\
 | \quad 3 \\
 M \\
 | \quad 4 \\
 K \\
 | \\
 1
 \end{array}$$

with $M' = K$ and $G/K \cong A_4$.

Once again the proof is carried out in a sequence of steps.

Step 1. M has a normal 2-complement, say H .

By Lemma 2.5, it is enough to show that $2 \mid \theta(1)$ for every non-linear irreducible character θ of M . To see this, we first observe that $T_\theta = G$, for

if $T_\theta = M$, then by Lemma 2.3, θ^G would be an irreducible character of G of degree divisible by 3 and > 3 . This is a contradiction to the hypotheses. Lemma 2.13 now yields the following three possibilities:

- (i) $\theta|_K$ is irreducible
- (ii) $\theta|_K = a\varphi$ for some irreducible character φ of K and $a^2 = [M : K] = 4$
- (iii) $\theta|_K = \sum_{i=1}^4 \varphi_i$, φ_i distinct irreducible characters of K

If (i) occurs, then since $[G : M]$ is prime and θ is stabilized by G , θ extends to an irreducible character say $\hat{\theta}$ of G by Lemma 2.11. Thus $\theta|_K$ as an irreducible character of K extends to an irreducible character $\hat{\theta}$ of G and is stabilized by G . We can now apply Lemma 2.9 and get $(\theta|_K)^G = \sum_{\omega} \omega(1)\omega\hat{\theta}$ where ω ranges over all irreducible characters of G/K and the characters $\{\omega\hat{\theta}\}$ are distinct and irreducible. Since G/K is A_4 , and $\hat{\theta}$ is non-linear, one of $\omega\hat{\theta}$ is an irreducible character of degree > 3 and divisible by 3. This is a contradiction to the hypothesis. Thus, only either (ii) or (iii) can occur and in either case, clearly, $2 \mid \theta(1)$. This proves Step 1.

Step 2. If the normal 2-complement $H = 1$, so that M is a 2-group, then G is isomorphic to $\text{SL}(2, 3)$.

Let $K = K_0 \triangleright K_1 \triangleright K_2 \triangleright K_3 \cdots \triangleright K_n \triangleright 1$ be a part of a chief series for G . Then $K_n \cap Z(M) \neq 1$, since M is a 2-group, and so the minimality of K_n implies $K_n \leq Z(M)$. Therefore every subgroup of K_n is normal in M . Let g be an element of order 3 in G so that $G = \langle g \rangle \cdot M$. Since $|g| = 3$, under the action of g each irreducible constituent of K_n has order equal to either 2 or 4. But then, every such constituent of K_n is normal in G . Therefore $|K_n| = 2$ or 4. The same argument applied to the group G/K_n shows that $|K_{n-1}/K_n| = 2$ or 4, etc. Thus, $|K_i/K_{i+1}| = 2$ or 4 $\forall i$. Suppose $|K_n| = 4$, so that $K_n \cong C_2 \times C_2$. Then g does not centralize any non-trivial element of K_n and so there exists a linear character $\zeta \neq 1$ of K_n such that $\zeta \neq \zeta^g$ (Lemma 2.7). Let φ be an irreducible character of M such that $(\zeta^M, \varphi)_M \geq 1$. Then Frobenius reciprocity (Lemma 2.1) and the fact that $K_n \leq K = M'$ imply that $\varphi(1) > 1$. Since $K_n \leq Z(M)$, we conclude that $\varphi|_{K_n} = a \cdot \zeta$ for some positive integer a , and so $\varphi^g|_{K_n} = a \cdot \zeta^g \neq \varphi|_{K_n}$. Thus, $\varphi^g \neq \varphi$, and so $T_\varphi < G$. Therefore $T_\varphi = M$, since $[G : M]$ is prime. But then, Lemma 2.3 implies φ^G is irreducible. This is a contradiction since $\varphi^G(1) = 3 \cdot \varphi(1) > 3$. Hence $|K_n| = 2$. Applying the same argument to G/K_n , we obtain $|K_{n-1}/K_n| = 2$, etc. Thus $|K_i/K_{i+1}| = 2 \forall i$, and g centralizes K_i/K_{i+1} for every i . This implies that g centralizes K

since $(|g|, |K|) = 1$. Also, $G' = M$, for clearly, $G' \subseteq M$ and $G' \supseteq M' = K$ and G/K , being isomorphic to A_4 , does not have any proper normal subgroups between M and K . Since $g \in C_G(K)$, $G/C_G(K)$ is a 2-group. This implies that $G = C_G(K)$, as M is 2-Sylow and contained in G' . It now follows that $Z(M) = K$, since G/K is isomorphic to A_4 . Thus M is a class 2, 2-group. Also, M' is generated by commutators $[x, y]$, where x and y range over the coset representatives of $Z(M)$ in M . Let $M = K \cup Kx \cup Ky \cup Kxy$. Then, $M' = \langle [x, y], [x, xy], [y, xy] \rangle$. But, $[x, xy] = [x, x][x, y]$ and $[y, xy] = [y, x][y, y]$, since M is a class 2-group. Therefore $M' = \langle [x, y] \rangle$ is a cyclic 2-group. Also, $[x, y][x, y] = [x, y^2] = 1$, since $y^2 \in K = Z(M)$. Hence, $|K| = 2$ and so $|G| = 24$. This clearly implies that G is isomorphic to $SL(2, 3)$.

From now on, we suppose $H > 1$.

Step 3. We may assume $[K : H] = 2$.

Since H is the normal 2-complement in M and since K is a normal subgroup of M of index 4 in M , we get $H \subset K$. By using induction on $|G|$, we obtain either G/H is isomorphic to A_4 , so that $K = H$, or G/H is isomorphic to $SL(2, 3)$, so that $[K : H] = 2$, or G/H is isomorphic to $SL(2, 3) \cdot (C_3 \times C_3)$. The last possibility cannot occur since M/H is a 2-group. Thus it is enough to show that $K \neq H$. Suppose $K = H$. Then, H is minimal normal in G , for if $L < H$ and $L \triangleleft G$, then by induction G/L is isomorphic to either $SL(2, 3)$ or $SL(2, 3) \cdot (C_3 \times C_3)$ and so in either case $2 \mid |H|$, contrary to the fact that H is a $2'$ -group. Therefore, $H = C_p \times C_p \times \cdots \times C_p$ for some odd prime p . We now claim M is a Frobenius group with kernel H . Let $\varphi \neq 1$ be a linear character of H . By Lemma 2.8, it is enough to show φ^M is irreducible. To this end, first consider the case $T_\varphi = G$. In particular, M stabilizes φ and so by Lemma 2.12, φ extends to a linear character $\hat{\varphi}$ of M . But then, since $M' = K$ (Lemma 3.4) and $K = H$, we get $\hat{\varphi}|_H = 1_H$ whereas $\hat{\varphi}|_H = \varphi \neq 1_H$. Therefore $T_\varphi < G$. Suppose $3 \mid [G : T_\varphi]$ and let η be an irreducible character of G such that $(\varphi^G, \eta)_G \geq 1$. By Lemma 2.10, there exists an irreducible character ζ of T_φ such that $\zeta^G = \eta$. Since $3 \mid \zeta^G(1)$, we get $\eta = \chi$. So by Lemma 2.1, $(\varphi^G, \chi)_G = (\varphi, \chi|_H)_H \geq 1$. This is impossible since $\chi|_H = 3 \cdot 1_H$. Thus, we have $T_\varphi < G$ and $3 \nmid [G : T_\varphi]$. We can write $M = R \cdot H$, $R \cap H = 1$, where R is a 2-Sylow subgroup of M and R is isomorphic to $C_2 \times C_2$. Let gH be an element of order 3 in G/H . Then since G/H has no subgroup of index 2, and since $3 \nmid [G : T_\varphi]$, we conclude that T_φ is a conjugate of $\langle g \rangle H$. Thus $T_\varphi \cap R = 1$, and so by Lemma 2.3, φ^M is irreducible, proving our claim that M is Frobenius with kernel H . But then, since a 2-Sylow subgroup of a Frobenius complement

is either cyclic or generalized quaternion, we cannot have R being isomorphic to $C_2 \times C_2$. This contradiction proves that $K \neq H$ and Step 3 is complete.

Step 4. We can now assume $1 < H < K$ and $[K : H] = 2$, because of Step 2 and Step 3. Then G is isomorphic to $\text{SL}(2, 3) \cdot (C_3 \times C_3)$.

First we claim that H is an abelian p -group for some prime p . If H is minimal normal in G this is certainly true. So, suppose H is not minimal normal in G and let $L < H$ with $L \triangleleft G$. By induction on $|G|$, we may assume G/L is isomorphic to $\text{SL}(2, 3) \cdot (C_3 \times C_3)$ and that L is minimal normal. We have the following picture:

$$\begin{array}{c}
 G \\
 | \quad C_3 \\
 M \\
 | \quad C_2 \times C_2 \\
 K \\
 | \quad C_2 \\
 H \\
 | \quad C_3 \times C_3 \\
 L \\
 | \quad C_p \times C_p \times \cdots \times C_p \\
 1
 \end{array}$$

with $G' = M$ and $G/H \cong \text{SL}(2, 3)$.

Suppose that H is not abelian and let θ be a non-linear irreducible character of H . By Ito's Theorem (Lemma 2.6) $3 \mid \theta(1)$. Let φ be an irreducible character of G such that $(\theta^G, \varphi)_G \geq 1$. This implies by Lemma 2.1, $(\theta, \varphi|_H)_H \geq 1$. Hence by Lemma 2.2, $\varphi(1) = r \cdot \theta(1)$ for some positive integer r . Therefore by hypothesis, $\varphi = \chi$, $r = 1$ and $\chi|_H = \theta$. This is impossible since $\chi|_H = 3 \cdot 1_H$, whereas $\theta \neq 3 \cdot 1_H$ by our choice of θ . Therefore H is abelian. Let $L = C_p \times C_p \times \cdots \times C_p$. If $p \neq 3$, then $H = C_3 \times C_3 \times C_p \times C_p \times \cdots \times C_p$. By considering the smaller group $G/(C_3 \times C_3)$ and using induction hypothesis, we get $H/(C_3 \times C_3)$ is isomorphic to $C_3 \times C_3$. So we may assume $p = 3$ and that H is an abelian 3-group. This proves the claim that H is an abelian p group for some prime p . Next we claim that $3 \nmid [G : T_\zeta]$ for any irreducible character $\zeta \neq 1$ of H . To see this, let φ be any irreducible character of G such that $(\zeta^G, \varphi)_G \geq 1$. So $(\zeta, \varphi|_H)_H \geq 1$ by 2.1. This in turn implies by 2.2, $\varphi|_H = a \sum_{i=1}^t \zeta^{g_i}$ where $t = [G : T_\zeta]$. If $3 \mid t$, then $3 \mid \varphi(1)$, and so $\varphi = \chi$. But then, $3 \cdot 1_H = \chi|_H = \zeta^{g_1} + \zeta^{g_2} + \zeta^{g_3}$; a contradiction to the linear independence of irreducible characters.

Thus, given any irreducible character $\zeta \neq 1$ of H , there is a 3-Sylow subgroup of G/H ($\cong \text{SL}(2, 3)$) stabilizing ζ . There are 4 3-Sylow subgroups of G/H and any two of them generate G/H . So if two distinct 3-Sylow subgroups of G/H stabilize the same character $\zeta \neq 1$ of H , then G stabilizes ζ . In particular, M stabilizes ζ , and by Lemma 2.12, ζ extends to a linear character $\hat{\zeta}$ of M . But then, $M' = K \supset H$ and so $\hat{\zeta}|_H = \zeta = 1_H$, whereas $\zeta \neq 1_H$. Thus no two distinct 3-Sylow subgroups of G/H stabilize the same irreducible character of H . Moreover, since the 3-Sylow subgroups in G/H are conjugate, the number of irreducible characters of H fixed by any one 3-Sylow subgroup of G/H is the same as the number of irreducible characters of H fixed by any other 3-Sylow subgroup of G/H . By Brauers' Lemma (2.7), the number of irreducible characters of H fixed by a 3-Sylow subgroup of G/H is equal to the number of elements of H fixed by it. Also, the set of elements of H fixed by a 3-Sylow subgroup of G/H form a subgroup of H . Thus if $|H| = p^n$, and if the number of elements fixed by a 3-Sylow subgroup of G/H is p^{n_o} ($n_o \leq n$), then, noting that there are 4 3-Sylow subgroups of G/H , we get $p^n - 1 = 4(p^{n_o} - 1)$, or equivalently, $p^{n_o}(p^{n-n_o} - 4) = -3$. This clearly implies $p = 3$, $n_o = 1$, and $n - n_o = 1$. Therefore $|H| = 9$. But we know that either H is normal minimal or H/L is isomorphic to $C_3 \times C_3$. Hence, H is minimal normal and isomorphic to $C_3 \times C_3$. It is now easy to see by the Frattini argument, and noting that H is minimal normal, that G is isomorphic to a semidirect product of $\text{SL}(2, 3)$ and $C_3 \times C_3$. This proves Step 4 and the lemma. \square

Proof of Theorem 3.1. By Lemma 3.2, we only need to consider the case that $\ker \chi = K > 1$. So suppose $K > 1$ and that the theorem is false. Let G be a minimal counterexample. If K is not minimal normal in G , let $N < K$ with $N \triangleleft G$. Then considering χ as an irreducible character of G/N , we see that G/N satisfies the hypotheses of the theorem with $\ker \chi = K/N > 1$, and so by the minimality of G , G/N is isomorphic to one of the groups listed in (c), (d), and (e) of the conclusion of Theorem 3.1. This implies $\chi(1) = 2$ or 3 . But then, Lemma 3.6 and 3.7 imply that G itself is isomorphic to one of the groups listed in (c), (d), and (e), contrary to our assumption that G is a counter-example. Therefore we may assume K is minimal normal in G . By Lemma 3.4, G/K is a Frobenius group of the type listed in (b) and if M/K is the Frobenius kernel, then $M' = K$. We can now apply Lemma 2.16 (see Remark 3.3(iii)) and conclude that one of the following three holds.

- (i) $|M/K| = 3$,
- (ii) M is a p -group for some prime p ,

- (iii) There exists an irreducible character $\varphi \neq 1$ of K and θ and irreducible character of G such that $(\varphi^G, \theta)_G \geq 1$ and $(\theta(1), q-1) \neq 1$.

If $|M/K| = 3$, then $\chi(1) = |M/K| - 1 = 2$ and so by Lemma 3.6, G is isomorphic to S_4 , contrary to the assumption that G is a counterexample. If M is a p -group, then by Lemma 3.5, G is isomorphic to $\text{SL}(2, 3)$, again contradicting our assumption. So statement (3) above holds. Since $\chi(1) = q-1$, $\chi = \theta$ by hypotheses. Then $(\varphi^G, \chi)_G = (\varphi, \chi|_K)_K \geq 1$. But $\chi|_K = \chi(1) \cdot 1_K$ whereas $\varphi \neq 1_K$. This is once again a contradiction and the proof of Theorem 3.1 is complete. \square

Remark 3.8. We observe that there is no group satisfying the hypotheses of the Theorem 3.1 with $\ker \chi = K > 1$ and with $(\chi(1), 6) = 1$.

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