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SEMIGROUP OF INJECTIVE PARTIAL TRANSFORMATIONS WITH BOUNDED GAP AND FIXED DEFECT

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Abstract: Let X be an infinite set and suppose that $\aleph_0 \leq q \leq |X|$. In 2004, Pinto and Sullivan considered algebraic properties of $PS(q)$, the partial Baer-Levi semigroup consisting of all injective partial transformations α of X such that $|X \setminus X\alpha| = q$. They also determined its subsemigroup $S(q,r) = \{\alpha \in PS(q) : |X \setminus \mathrm{dom}\alpha| \leq r\}$ where $\aleph_0 \leq r \leq |X|$. Recently, Singha and Sanwong showed that, when $q < |X|$, almost every maximal subsemigroup of $PS(q)$ is induced by a maximal subsemigroup of $S(q,r)$. Here, we use their work to describe some algebraic properties of $S(q,r)$ including its Green's relation and its ideal structure.

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1. Introduction

Suppose that X is a non-empty set, and let $I(X)$ denote the *symmetric inverse semigroup* on X (see [1] vol 1, p 29): that is, the set of all injective mappings from X into X. For any $\alpha \in I(X)$, we let $\mathrm{dom}\alpha$ and $\mathrm{ran}\alpha$ (or $X\alpha$) denote the

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domain of α and *range* of α, respectively. We also write

$$g(\alpha) = |X \setminus \mathrm{dom}\alpha|, \quad d(\alpha) = |X \setminus \mathrm{ran}\alpha|, \quad r(\alpha) = |\mathrm{ran}\alpha|,$$

and refer to these cardinals as the *gap*, the *defect* and the *rank* of α, respectively. For an infinite cardinal q such that $\aleph_0 \leq q \leq |X|$, we write

$$PS(q) = \{\alpha \in I(X) : d(\alpha) = q\}$$

where $PS(q)$ is the *partial Baer-Levi semigroup* on X. In [3] Theorem 1, Pinto and Sullivan showed that $PS(q)$ is *right reductive* in the sense that, if $\alpha\gamma = \beta\gamma$ for all $\gamma \in PS(q)$ implies $\alpha = \beta$ $(\alpha, \beta \in PS(q))$, and *left reductive* which defined dually (see [1] vol 1, p 9), and they characterised the Green's relations and ideals of $PS(q)$. They also determined its subsemigroups

$$S_r = \{\alpha \in I(X) : d(\alpha) = q \text{ and } \ g(\alpha) \leq r\}$$

where $\aleph_0 \leq r \leq |X|$. In fact, when $r = |X|$ we have $S_r = PS(q)$, then we may regard S_r as a generalization of $PS(q)$. More recently, in [4], Singha and Sanwong studied maximal subsemigroups of $PS(q)$. In particular, when $q < |X|$ they found that almost every maximal subsemigroup of $PS(q)$ is induced by a maximal subsemigroup of S_r for some cardinal r such that $q \leq r < |X|$ (see [4] Theorem 3.5).

In this paper, we examine a subsemigroup S_r of $PS(q)$ and we write $S(q, r)$ in place of S_r to highlight the defect q and the maximum gap r of elements in this semigroup.

2. Basic Properties

We modify the convention introduced in [1] vol 2, p 241: namely, if $\alpha \in I(X)$ is non-zero, then we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $X\alpha = \mathrm{ran}\alpha = \{x_i\}$, $x_i\alpha^{-1} = \{a_i\}$ and $\mathrm{dom}\alpha = \{a_i : i \in I\}$. For simplicity, if $A \subseteq X$, we sometimes write $A\alpha$ in place of $(A \cap \mathrm{dom}\alpha)\alpha$, and if $A \subseteq \mathrm{dom}\alpha$, then we write $\alpha|A$ to mean the restriction of α on A.

In this paper, $Y = A \dot{\cup} B$ means Y is a *disjoint* union of A and B. Also, for each non-empty $A \subseteq X$, we write id_A for the identity transformation on A:

these mappings constitute all the idempotents in $I(X)$ and it is known from [3] that $\mathrm{id}_A \in PS(q)$ precisely when $|X \setminus A| = q$. As usual, $\emptyset$ denotes the empty (one-to-one) mapping which acts as a zero for $I(X)$ and in [3] Theorem 1, the authors showed that it belongs to $PS(q)$ precisely when $q = |X|$.

Theorem 1. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. Then $S(q,r)$ contains a zero precisely when $|X| = q = r$. Moreover, $S(q,r)$ is a right and left reductive semigroup without identity.*

Proof. Suppose that γ is a zero for $S(q,r)$. For each $x \in X$, if $x \in X\gamma$, then we write $X = P \dot\cup Q \dot\cup \{x\}$ where $|P| = |X|, |Q| = q$. Then a bijection $\beta : X \to P$ belongs to $S(q,r)$. Since γ is a zero for $S(q,r)$, we have $\gamma = \gamma\beta$. Then $X\gamma \subseteq X\beta = P$. This follows that $Q \dot\cup \{x\} = X \setminus P \subseteq X \setminus X\gamma$, a contradiction. This means that $\gamma = \emptyset$. Since $d(\emptyset) = |X| = g(\emptyset)$, then $\emptyset$ belongs to $S(q,r)$ precisely when $|X| = q = r$.

To show that $S(q,r)$ is right reductive, suppose that $\alpha, \beta \in S(q,r)$ and $\alpha\gamma = \beta\gamma$ for all $\gamma \in S(q,r)$. If (say) $\alpha = \emptyset$, then $|X| = q = r$. In this case $\mathrm{id}_{X\beta} \in S(q,r)$ and thus $\beta = \beta.\mathrm{id}_{X\beta} = \alpha.\mathrm{id}_{X\beta} = \emptyset$. Now suppose $\alpha, \beta \neq \emptyset$ and we will consider in two cases. First, if $q \leq r$, then $\mathrm{id}_{X\alpha} \in S(q,r)$ and thus $\alpha = \alpha.\mathrm{id}_{X\alpha} = \beta.\mathrm{id}_{X\alpha}$. This implies $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$ and $\beta|\mathrm{dom}\alpha = \alpha$. Similarly, we have $\beta = \beta.\mathrm{id}_{X\beta} = \alpha.\mathrm{id}_{X\beta}$ since $\mathrm{id}_{X\beta} \in S(q,r)$, and hence $\mathrm{dom}\beta \subseteq \mathrm{dom}\alpha$. It follows that $\alpha = \beta$. If $r < q$, then we write $X \setminus X\alpha = Q_1 \dot\cup Q_2 \dot\cup R$ where $|Q_1| = q = |Q_2|, |R| = r$. Define

$$\lambda = \begin{pmatrix} X\alpha & Q_1 \cup Q_2 \\ X\alpha & Q_1 \end{pmatrix} \in S(q,r)$$

where $\lambda|X\alpha = \mathrm{id}_{X\alpha}$ and $\lambda : Q_1 \cup Q_2 \to Q_1$ is a bijection. Then $\alpha = \alpha\lambda = \beta\lambda$. This implies that $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$ and $\beta|\mathrm{dom}\alpha = \alpha$. Similarly, by using the same arguments we can show that $\mathrm{dom}\beta \subseteq \mathrm{dom}\alpha$ and this implies $\alpha = \beta$. Therefore, $S(q,r)$ is right reductive.

Now suppose that $\gamma\alpha = \gamma\beta$ for all $\gamma \in S(q,r)$. As before, if (say) $\alpha = \emptyset$, then $|X| = q = r$. In this case, $\mathrm{id}_x \in S(q,r)$ for all $x \in X$. Thus, $\emptyset = \mathrm{id}_x.\alpha = \mathrm{id}_x.\beta$ for all $x \in X$. This implies that $\beta = \emptyset$. Now we suppose that $\alpha, \beta \neq \emptyset$, then, for each $t \in \mathrm{dom}\alpha$, we write $X = P \dot\cup Q \dot\cup \{t\}$ where $|P| = |X|, |Q| = q$. We define

$$\mu = \begin{pmatrix} X \setminus \{t\} & t \\ P & t \end{pmatrix}$$

where $\mu : X \setminus \{t\} \to P$ is a bijection. Then $\mu \in S(q,r)$ and $\mu\alpha = \mu\beta$. Therefore $t\mu\alpha = t\mu\beta$, and thus $t\alpha = t\beta$. Therefore $t \in \mathrm{dom}\beta$, that is, $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$ and

$\beta|\text{dom}\alpha = \alpha$. Similarly, we can show that $\text{dom}\beta \subseteq \text{dom}\alpha$ and therefore $\alpha = \beta$, that is, $S(q, r)$ is left reductive.

Finally, to see that $S(q, r)$ has no identity. We observe that, for each $x \in X$, $x \in X\alpha$ for some $\alpha \in S(q, r)$. Suppose that γ is the identity in $S(q, r)$. Then $\alpha\gamma = \alpha$. Thus, if $a\alpha = x$, then $x = a\alpha = a\alpha\gamma = x\gamma$. That is, $\gamma = \text{id}_X$, and so $d(\gamma) = 0$, contrary to $\gamma \in S(q, r)$. Therefore $S(q, r)$ has no identity. □

Theorem 2. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. Then $S(q, r)$ contains idempotents precisely when $q \leq r$. In this case, the set of all idempotents in $S(q, r)$ is*

$$E(S(q, r)) = \{\text{id}_A : A \subseteq X,\ |X \setminus A| = q\}.$$

Proof. Suppose that α is an idempotent in $S(q, r)$. If $\alpha = \emptyset$, then $|X| = q = r$ and α is a one-to-one mapping on the empty set, and we have that $|X \setminus \emptyset| = q$. Suppose that $\alpha \neq \emptyset$ and let $x \in \text{dom}\alpha$. Then $x\alpha = x\alpha^2$, and thus $x\alpha = x$ since α is injective. Therefore $\alpha = \text{id}_A$ for some subset A of X. Moreover, since $g(\alpha) \leq r$ and $d(\alpha) = q$, we have $q = |X \setminus A| \leq r$. Conversely, suppose $q \leq r$. Then for any subset Q of X with $|Q| = q$, we see that $\alpha = \text{id}_{X\setminus Q}$ is an idempotent, and it belongs to $S(q, r)$ since $d(\alpha) = g(\alpha) = |Q| = q \leq r$. □

3. Regular Elements

In [3] Corollary 1, the authors showed that

$$R(q) = \{\alpha \in I(X) : g(\alpha) = q = d(\alpha)\}$$

is the largest regular subsemigroup of $PS(q)$. Moreover, $R(q)$ is an inverse semigroup. Now we characterise all regular elements in $S(q, r)$.

Theorem 3. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. Then $S(q, r)$ contains regular elements precisely when $q \leq r$. In this case, α is regular in $S(q, r)$ if and only if $g(\alpha) = q$.*

Proof. Let $\alpha \in S(q, r)$. If α is a regular element in $S(q, r)$, then $\alpha = \alpha\beta\alpha$ for some $\beta \in S(q, r)$. For convenience, we say

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix},$$

then $a_i\alpha\beta\alpha = a_i\alpha$, thus $a_i\alpha\beta = a_i$ (since α is injective) and it follows that $x_i\beta = a_i$. That is, $X\alpha \subseteq \text{dom}\beta$ and

$$X\alpha\beta = (X\alpha \cap \text{dom}\beta)\beta = \{x_i\}\beta = \{a_i\} = \text{dom}\alpha.$$

Therefore, since $\alpha\beta \in S(q,r)$, we have

$$q = |X \setminus X\alpha\beta| = |X \setminus \mathrm{dom}\alpha| = g(\alpha) \leq r.$$

Now we suppose that $q \leq r$ and $g(\alpha) = q$. From the above notation of α, since $q = d(\alpha) = |X \setminus \{x_i\}|$ and $q = g(\alpha) = |X \setminus \{a_i\}|$, we have

$$\beta = \begin{pmatrix} x_i \\ a_i \end{pmatrix} \in S(q,r)$$

and $\alpha = \alpha\beta\alpha$. Therefore α is regular in $S(q,r)$. □

In view of [3] Corollary 1 and Theorem 3, we have proved the following result.

Corollary 1. *Suppose that* $|X| \geq q \geq \aleph_0$, $|X| \geq r \geq \aleph_0$ *and* $q \leq r$. *Then*

$$R(q) = \{\alpha \in I(X) : d(\alpha) = q = g(\alpha)\}$$

is the largest regular subsemigroup of $S(q,r)$ *and it is also an inverse semigroup.*

4. Green's Relations

The concept of Green's relations on a semigroup was first studied by J.A. Green in 1951 and they have played a fundamental role in a development of semigroup theory. In this paper, the knowledge and the notations about Green's relations, is based on [2] Chapter II.

In [3] Section 4, the authors described the Green's relations on $PS(q)$. Here, we follow some of their arguments to describe the Green's relations on $S(q,r)$. And, as might be expected, our results are similar to those obtained in [3] Section 4.

Theorem 4. *Suppose that* $|X| \geq q \geq \aleph_0$ *and* $|X| \geq r \geq \aleph_0$. *If* $\alpha, \beta \in S(q,r)$, *then* $\alpha = \beta\mu$ *for some* $\mu \in S(q,r)$ *if and only if* $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$. *Hence* $\alpha \,\mathcal{R}\, \beta$ *in* $S(q,r)$ *if and only if* $\mathrm{dom}\alpha = \mathrm{dom}\beta$.

Proof. It is clear that, if $\alpha = \beta\mu$ for some $\mu \in S(q,r)$, then $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$. For the converse, if $\mathrm{dom}\alpha \subseteq \mathrm{dom}\beta$, we write

$$\alpha = \begin{pmatrix} a_i \\ x_i \end{pmatrix}, \ \beta = \begin{pmatrix} a_i & b_j \\ y_i & z_j \end{pmatrix},$$

where $\{b_j\} = \mathrm{dom}\beta \setminus \mathrm{dom}\alpha$ and $|J| = |\mathrm{dom}\beta \setminus \mathrm{dom}\alpha| \leq |X \setminus \mathrm{dom}\alpha| \leq r$. Since $d(\alpha) = q$, we write $X \setminus \{x_i\} = A \dot{\cup} B$ where $|A| = q = |B|$, and define

$$\mu = \begin{pmatrix} y_i & X \setminus X\beta \\ x_i & A \end{pmatrix},$$

where $\mu : X \setminus X\beta \to A$ is a bijection. Then $\alpha = \beta\mu$, and $\mu \in S(q, r)$ since $d(\mu) = |B| = q$ and $g(\mu) = |J| \leq r$. □

To describe the $\mathcal{L}$ relation on $S(q, r)$, we recall from [3] Theorem 8 that, $\alpha \mathrel{\mathcal{L}} \beta$ in $PS(q)$ if and only if

$$(X\alpha = X\beta \text{ and } g(\alpha) = g(\beta) \geq q) \text{ or } (\alpha = \beta \text{ and } g(\alpha) < q).$$

By modifying some arguments in the proof of [3] Theorem 8, we obtain the following result.

Theorem 5. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. Let $\alpha, \beta \in S(q, r)$. Then $\alpha = \gamma\beta$ for some $\gamma \in S(q, r)$ if and only if $X\alpha \subseteq X\beta$ and*

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q). \tag{1}$$

Hence:

(a) if $q \leq r$, then $\alpha \mathrel{\mathcal{L}} \beta$ in $S(q, r)$ if and only if

$$(X\alpha = X\beta \text{ and } g(\alpha) = g(\beta) \geq q) \text{ or } (\alpha = \beta \text{ and } g(\alpha) < q);$$

(b) if $q > r$, then $\alpha \mathrel{\mathcal{L}} \beta$ in $S(q, r)$ if and only if $\alpha = \beta$.

Proof. Suppose that $\alpha, \beta \in S(q, r)$ and $\alpha = \gamma\beta$ for some $\gamma \in S(q, r)$. Then $X\alpha \subseteq X\beta$ and

$$|\mathrm{dom}\beta \setminus X\gamma| = |X\beta \setminus X\gamma\beta| = |X\beta \setminus X\alpha| \leq |X \setminus X\alpha| = q.$$

Then

$$\begin{aligned} q = |X \setminus X\gamma| &= |(X \setminus X\gamma) \cap (X \setminus \mathrm{dom}\beta)| + |\mathrm{dom}\beta \setminus X\gamma| \\ &\leq g(\beta) + |X\beta \setminus X\alpha| = \max(g(\beta), |X\beta \setminus X\alpha|) \end{aligned}$$

It remains to show $\max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q)$. Since $\alpha = \gamma\beta$, we have

$$\mathrm{dom}\alpha = \mathrm{dom}\gamma\beta = (X\gamma \cap \mathrm{dom}\beta)\gamma^{-1},$$

and this follows that $(X\gamma \setminus \mathrm{dom}\beta)\gamma^{-1} \subseteq X \setminus \mathrm{dom}\alpha$, and so $|X\gamma \setminus \mathrm{dom}\beta| \leq |X \setminus \mathrm{dom}\alpha|$. Therefore,

$$\begin{aligned} g(\beta) = |X \setminus \mathrm{dom}\beta| &= |X\gamma \setminus \mathrm{dom}\beta| + |(X \setminus X\gamma) \cap (X \setminus \mathrm{dom}\beta)| \\ &\leq |X \setminus \mathrm{dom}\alpha| + q = \max(g(\alpha), q). \end{aligned}$$

We also see that $|X\beta \setminus X\alpha| \leq |X \setminus X\alpha| = q$. Hence, $\max(g(\beta), |X\beta \setminus X\alpha|) \leq \max(g(\alpha), q)$ as required.

For the converse, we suppose the conditions hold. Since $X\alpha \subseteq X\beta$, we may write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} c_i & c_j \\ b_i & b_j \end{pmatrix}$$

where $|\{c_j\}| = |X\beta \setminus X\alpha| \leq q$. We also define

$$\gamma = \begin{pmatrix} a_i \\ c_i \end{pmatrix},$$

then $g(\gamma) = g(\alpha) \leq r$ and $\alpha = \gamma\beta$. It remains to show $d(\gamma) = q$. If $g(\alpha) < q$, then $\max(g(\beta), |X\beta \setminus X\alpha|) = q$ by (1). Also, if $\max(g(\beta), |X\beta \setminus X\alpha|) = |X\beta \setminus X\alpha|$, then

$$q \leq \max(g(\beta), |X\beta \setminus X\alpha|) = |X\beta \setminus X\alpha| \leq q$$

by (1) again, that is, $\max(g(\beta), |X\beta \setminus X\alpha|) = q$. In both cases, we have $d(\gamma) = |J| + g(\beta) = q$, and this follows that $\gamma \in S(q, r)$. Now we suppose that $g(\alpha) \geq q$ and $\max(g(\beta), |X\beta \setminus X\alpha|) = g(\beta)$. Then (1) implies that $q \leq g(\beta) \leq g(\alpha)$. We write $X \setminus \mathrm{dom}\beta = \{c_k\} \dot{\cup} \{c_l\}$ where $|K| = g(\beta), |L| = q$, and choose $\{z_k\} \subseteq X \setminus \mathrm{dom}\alpha$ (possible since $g(\beta) \leq g(\alpha)$). Then re-define

$$\gamma = \begin{pmatrix} a_i & z_k \\ c_i & c_k \end{pmatrix},$$

we have $\alpha = \gamma\beta$ and $\gamma \in S(q, r)$ since $d(\gamma) = |\{c_j\} \cup \{c_l\}| = q$ and $g(\gamma) \leq g(\alpha) \leq r$.

Now, to see (a), we suppose that $q \leq r$ and let $\alpha, \beta \in S(q, r)$ be such that $\alpha \neq \beta$ and $\alpha \mathcal{L} \beta$. Thus, $\alpha = \gamma\beta$ and $\beta = \mu\alpha$ for some $\gamma, \mu \in S(q, r)$. Then $X\alpha = X\beta$ and so $|X\beta \setminus X\alpha| = 0 = |X\alpha \setminus X\beta|$. Therefore, $q \leq g(\alpha) = g(\beta)$ by (1). Also, the converse is quite obvious. In particular, when $r < q$, $S(q, r)$ contains only mappings with gap less than q, therefore $\alpha \mathcal{L} \beta$ if and only if $\alpha = \beta$, that is (b) holds. □

Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, Theorem 4 and Theorem 5 lead to our next result.

Corollary 2. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. If $\alpha, \beta \in S(q,r)$, then $\alpha\ \mathcal{H}\ \beta$ in $S(q,r)$ if and only if*

$$(\mathrm{dom}\alpha = \mathrm{dom}\beta, X\alpha = X\beta \ \ \textit{and} \ \ g(\alpha) = g(\beta) \geq q) \ \ \textit{or} \ \ (\alpha = \beta \ \ \textit{and} \ \ g(\alpha) < q).$$

It quite difficult to describe the $\mathcal{J}$ relation on $S(q,r)$, so we will consider in the following three cases.

Lemma 1. *Suppose that $|X| \geq q \geq \aleph_0$, $|X| \geq r \geq \aleph_0$ and $q < r$. If $\alpha, \beta \in S(q,r)$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S(q,r)$ if and only if $g(\alpha) \leq q$ or $g(\beta) \geq g(\alpha) > q$.*

Hence, $\alpha\ \mathcal{J}\ \beta$ in $S(q,r)$ if and only if

$$\max(g(\alpha), g(\beta)) \leq q \ \ \textit{or} \ \ g(\alpha) = g(\beta) > q. \tag{2}$$

Proof. Suppose that the conditions hold and let $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S(q,r)$. We assume $q < g(\alpha)$. Since

$$g(\alpha) = |(X \setminus X\lambda) \cap (X \setminus \mathrm{dom}\alpha)| + |X\lambda \setminus \mathrm{dom}\alpha|$$

where $|(X \setminus X\lambda) \cap (X \setminus \mathrm{dom}\alpha)| \leq q$, we have $|X\lambda \setminus \mathrm{dom}\alpha| = g(\alpha) > q$. Also, since $\mathrm{dom}\beta = (X\lambda \cap \mathrm{dom}\alpha\mu)\lambda^{-1}$, we have $(X\lambda \setminus \mathrm{dom}\alpha\mu)\lambda^{-1} \subseteq X \setminus \mathrm{dom}\beta$. It follows that

$$\begin{aligned} q < g(\alpha) &= |X\lambda \setminus \mathrm{dom}\alpha| \leq |X\lambda \setminus \mathrm{dom}\alpha\mu| \\ &= |(X\lambda \setminus \mathrm{dom}\alpha\mu)\lambda^{-1}| \leq |X \setminus \mathrm{dom}\beta| = g(\beta). \end{aligned}$$

For the converse, since $d(\alpha) = q < r \leq |X|$, we have $r(\alpha) = |X|$. Similarly, $r(\beta) = |X|$. Then we can write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix},$$

where $|I| = |X|$. If $g(\alpha) \leq q$, then we write $\{a_i\} = \{x_i\} \dot\cup \{x_j\}$ where $|J| = q$. Then $X\alpha = \{x_i\alpha\} \dot\cup \{x_j\alpha\}$. Next, we define

$$\lambda = \begin{pmatrix} c_i \\ x_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} x_i\alpha \\ d_i \end{pmatrix},$$

then $\beta = \lambda\alpha\mu$. We can see that $g(\lambda) = g(\beta) \leq r$ and $d(\lambda) = |\{x_j\}| + g(\alpha) = q$, that is $\lambda \in S(q,r)$. Moreover, $g(\mu) = |\{x_j\alpha\}| + d(\alpha) = q < r$ and $d(\mu) = d(\beta) =$

q, this means that $\mu \in S(q,r)$. On the other hand, if $q < g(\alpha) \le g(\beta)$, then we write

$$X \setminus \mathrm{dom}\alpha = \{m_k\} \dot\cup A \quad \text{and} \quad X \setminus \mathrm{dom}\beta = \{n_k\} \dot\cup B$$

where $|K| = g(\alpha)$, $|A| = q$ and $|B| = g(\beta)$. Then re-define

$$\lambda = \begin{pmatrix} c_i & n_k \\ a_i & m_k \end{pmatrix} \text{ and } \mu = \begin{pmatrix} b_i \\ d_i \end{pmatrix},$$

we have $\beta = \lambda\alpha\mu$ and $g(\lambda) = g(\beta) \le r$, $d(\lambda) = |A| = q$, $g(\mu) = d(\alpha) = q < r$ and $d(\mu) = d(\beta) = q$, that is, $\lambda, \mu \in S(q,r)$. □

Lemma 2. *Suppose that $|X| \ge q \ge \aleph_0$, $|X| \ge r \ge \aleph_0$ and $q = r$. If $\alpha, \beta \in S(q,r)$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S(q,r)$ if and only if $r(\beta) \le r(\alpha)$.*

Hence,

$$\alpha \,\mathcal{J}\, \beta \;\; \textit{in} \;\; S(q,r) \;\; \textit{if and only if} \;\; r(\alpha) = r(\beta). \tag{3}$$

Proof. Suppose that the conditions hold. If $\beta = \lambda\alpha\mu$, then it is clear that $r(\beta) \le r(\alpha)$. Conversely, suppose that $r(\beta) \le r(\alpha)$. Then we write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_j \\ d_j \end{pmatrix} \text{ where } |J| \le |I|,$$

and write $\{a_i\} = \{x_j\} \dot\cup \{x_k\}$. Define

$$\lambda = \begin{pmatrix} c_j \\ x_j \end{pmatrix} \text{ and } \mu = \begin{pmatrix} x_j\alpha \\ d_j \end{pmatrix},$$

then $\beta = \lambda\alpha\mu$ and it is clear that $g(\lambda) = g(\beta) \le r$, $d(\mu) = d(\beta) = q$. We also see that

$$d(\lambda) = |\{x_k\}| + g(\alpha) \quad \text{and} \quad g(\mu) = |\{x_k\alpha\}| + d(\alpha).$$

If $|X| > q$, then $r(\alpha) = r(\beta) = |X|$, that is, $|I| = |J| = |X|$. Thus, when writing $\{a_i\} = \{x_j\} \dot\cup \{x_k\}$, we can assume $|\{x_k\}| = q$. It follows that $d(\lambda) = q$ and $g(\mu) = q = r$, that is, $\lambda, \mu \in S(q,r)$. On the other hand, if $|X| = q$, then $|X| = q = r$ and thus $g(\mu) = r$. In this case, if $g(\alpha) = |X|$, then $d(\lambda) = |X| = q$, otherwise, if $g(\alpha) < |X|$, then $|I| = |X|$. It follows that, when writing $\{a_i\} = \{x_j\} \dot\cup \{x_k\}$, we can assume $|\{x_k\}| = |X|$. Thus $d(\lambda) = |X| = q$ and therefore $\lambda, \mu \in S(q,r)$. □

Lemma 3. *Suppose that $|X| \ge q \ge \aleph_0$, $|X| \ge r \ge \aleph_0$ and $q > r$. Then the $\mathcal{J}$ relation is the universal relation on $S(q,r)$.*

Proof. Suppose that the conditions hold and let $\alpha, \beta \in S(q,r)$. Since $g(\alpha) \leq r < q \leq |X|$, we have $r(\alpha) = |X|$, similarly, $r(\beta) = |X|$. We write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix}, \text{ where } |I| = |X|.$$

We also write

$$\{a_i\} = \{x_i\} \dot{\cup} \{x_j\}, \ X \setminus X\alpha = A \dot{\cup} B \text{ and } X \setminus X\beta = C \dot{\cup} D,$$

where $|J| = q, |A| = |C| = |D| = q$ and $|B| = r$. Then $X\alpha = \{x_i\alpha\} \dot{\cup} \{x_j\alpha\}$. Define

$$\lambda = \begin{pmatrix} c_i \\ x_i \end{pmatrix}, \text{ and } \mu = \begin{pmatrix} x_i\alpha & \{x_j\alpha\} \cup A \\ d_i & C \end{pmatrix},$$

where $\mu : \{x_j\alpha\} \cup A \to C$ is a bijection. Then $\beta = \lambda\alpha\mu$, and $g(\lambda) = g(\beta) \leq r$, $d(\lambda) = |\{x_j\}| + g(\alpha) = q$, $g(\mu) = |B| = r$ and $d(\mu) = |D| = q$, that is $\lambda, \mu \in S(q,r)$. Similarly, we can define $\lambda', \mu' \in S(q,r)$ such that $\alpha = \lambda'\beta\mu'$. Therefore, $\alpha \ \mathcal{J} \ \beta$ and hence $\mathcal{J} = S(q,r) \times S(q,r)$. □

Recall from [3] Theorem 9 that, if $\alpha, \beta \in PS(q)$, then $\alpha \ \mathcal{J} \ \beta$ in $PS(q)$ if and only if

$$[\max(g(\alpha), g(\beta)) \leq q \text{ and } r(\alpha) = r(\beta)] \text{ or } [g(\alpha) = g(\beta) > q].$$

In what follows, by using Lemma 1, Lemma 2 and Lemma 3, we reach the same result for $\alpha, \beta \in S(q,r)$.

Theorem 6. *Suppose that $|X| \geq q \geq \aleph_0$ and $|X| \geq r \geq \aleph_0$. If $\alpha, \beta \in S(q,r)$, then $\alpha \ \mathcal{J} \ \beta$ in $S(q,r)$ if and only if*

$$[\max(g(\alpha), g(\beta)) \leq q \ \textit{ and } \ r(\alpha) = r(\beta)] \ \textit{ or } \ [g(\alpha) = g(\beta) > q]. \tag{4}$$

Proof. Suppose that the conditions hold and let $\alpha, \beta \in S(q,r)$. If $q < r$, then $r(\alpha) = |X| = r(\beta)$ since $d(\alpha) = d(\beta) = q < r \leq |X|$. Thus (2) and (4) are equivalent. Next, if $q = r$, then $g(\alpha) \leq r = q$ and similarly, $g(\beta) \leq q$. Therefore $\max(g(\alpha), g(\beta)) \leq q$ and so (3) and (4) are equivalent. Finally, if $q > r$, then $|\text{dom}\alpha| = |X|$ since $g(\alpha) \leq r < q \leq |X|$. Then $r(\alpha) = |X|$ since α is injective. Similarly, $r(\beta) = |X|$ and this follows that $r(\alpha) = r(\beta)$. Thus, the condition (4) is simply the universal relation on $S(q,r)$. Therefore, in each case $\alpha \ \mathcal{J} \ \beta$ in $S(q,r)$ if and only if α and β satisfy (4). □

Theorem 7. *Suppose that* $|X| \geq q \geq \aleph_0$ *and* $|X| \geq r \geq \aleph_0$*. Let* $\alpha, \beta \in S(q,r)$*. Then* $\alpha \ \mathcal{D} \ \beta$ *in* $S(q,r)$ *if and only if*

$$(r(\alpha) = r(\beta) \text{ and } g(\alpha) = g(\beta) \geq q) \quad or \quad (\mathrm{dom}\alpha = \mathrm{dom}\beta \text{ and } g(\alpha) < q). \quad (5)$$

In particular, if $q > r$*, then* $\alpha \ \mathcal{D} \ \beta$ *in* $S(q,r)$ *if and only if* $\mathrm{dom}\alpha = \mathrm{dom}\beta$.

Proof. Suppose that the conditions hold. If $\alpha \ \mathcal{D} \ \beta$ in $S(q,r)$, then $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ for some $\gamma \in S(q,r)$. If $\alpha = \gamma$, then $\alpha \ \mathcal{R} \ \beta$, that is, $\mathrm{dom}\alpha = \mathrm{dom}\beta$ by Theorem 4, and this leads us to (5). Otherwise, if $\alpha \neq \gamma$, then Theorem 4 and Theorem 5 imply that $X\alpha = X\gamma$, $g(\alpha) = g(\gamma) \geq q$ and $\mathrm{dom}\gamma = \mathrm{dom}\beta$. Therefore $q \leq g(\alpha) = g(\beta)$ and $r(\alpha) = r(\beta)$, that is, (5) holds. Conversely, since $\mathcal{R} \subseteq \mathcal{D}$, if $\mathrm{dom}\alpha = \mathrm{dom}\beta$, then $\alpha \ \mathcal{R} \ \beta$ by Theorem 4 and so $\alpha \ \mathcal{D} \ \beta$. For the case that $r(\alpha) = r(\beta)$ and $g(\alpha) = g(\beta) \geq q$, we write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix} \text{ and let } \gamma = \begin{pmatrix} c_i \\ b_i \end{pmatrix}.$$

Then $g(\gamma) = g(\beta) \leq r$ and $d(\gamma) = d(\alpha) = q$, that is, $\gamma \in S(q,r)$. Moreover, $\alpha \ \mathcal{L} \ \gamma \ \mathcal{R} \ \beta$ by Theorem 4 and Theorem 5, therefore $\alpha \ \mathcal{D} \ \beta$ as required. In particular, if $q > r$, then $g(\alpha) \leq r < q$ for all $\alpha \in S(q,r)$. Thus, by (5), $\alpha \ \mathcal{D} \ \beta$ if and only if $\mathrm{dom}\alpha = \mathrm{dom}\beta$. □

5. Two-Sided Ideals

In what follows, we first recall that a semigroup S without zero is called simple if it has no proper ideals and it is known that S is simple if and only if $\mathcal{J} = S \times S$ (see [2] Chapter III). Also, let u be a cardinal number, the successor of u is denoted by u', where

$$u' = min\{v : v > u\}.$$

Note that u' always exists since the cardinals are well-ordered, and when u is finite we have $u' = u + 1$.

Theorem 8. *Suppose that* $|X| \geq q \geq \aleph_0$ *and* $|X| \geq r \geq \aleph_0$*. Then the following statements hold:*

(*a*) *if* $q > r$*, then* $S(q,r)$ *has no proper ideals;*

(*b*) *if* $q = r$*, then* $S(q,r)$ *has a proper ideal precisely when* $|X| = q$ *and, in this case, all ideals of* $S(q,r)$ *are of the form*

$$V_t = \{\alpha \in S(q,r) : r(\alpha) < t\}$$

where $1 \leq t \leq |X|'$. *Moreover,* V_t *is principal precisely when* $t = s'$ *where* $0 \leq s \leq |X|$;

(c) *if* $q < r$, *then the proper ideals of* $S(q, r)$ *are precisely the sets:*

$$U_t = \{\alpha \in S(q, r) : g(\alpha) \geq t\}$$

where $q < t \leq r$. *Moreover, each* U_t *is a principal ideal.*

Proof. Suppose that the conditions hold. Here, for convenience we write S in place of $S(q, r)$. If $q > r$, then Lemma 3 implies that $\mathcal{J} = S \times S$ and so $S(q, r)$ is simple. It follows that (a) holds.

For (b), we suppose that $q = r$. In this case, if $|X| > q$, then every element in S has the same rank $|X|$. Then, by Lemma 2, $\mathcal{J} = S \times S$ and thus S has no proper ideals. We now suppose $|X| = q$. It follows that $|X| = q = r$ and thus $S = PS(q)$. Hence, all ideals of S are exactly the same as those ideals of $PS(q)$. Therefore, we have completed this case by [ps] Theorem 14.

Finally, to prove (c), suppose that $q < r$ and let t be a cardinal such that $q < t \leq r$. To show U_t is an ideal. Let $\alpha \in U_t$ and $\beta \in S$. Since $\mathrm{dom}\alpha\beta \subseteq \mathrm{dom}\alpha$, we have $t \leq g(\alpha) \leq g(\alpha\beta)$. Therefore U_t is a right ideal. Also, since

$$X \setminus \mathrm{dom}\alpha = [X\beta \cap (X \setminus \mathrm{dom}\alpha)] \,\dot{\cup}\, [(X \setminus X\beta) \cap (X \setminus \mathrm{dom}\alpha)]$$

where $g(\alpha) \geq t > q$ and the last term has cardinality at most q (since $|X \setminus X\beta| = q$), we have $|X\beta \cap (X \setminus \mathrm{dom}\alpha)| \geq t$. This means that

$$t \leq |[X\beta \cap (X \setminus \mathrm{dom}\alpha)]\beta^{-1}| = |\mathrm{dom}\beta \setminus \mathrm{dom}\beta\alpha|.$$

Thus, the equation

$$X \setminus \mathrm{dom}\beta\alpha = (X \setminus \mathrm{dom}\beta) \,\dot{\cup}\, (\mathrm{dom}\beta \setminus \mathrm{dom}\beta\alpha)$$

implies that $g(\beta\alpha) \geq t$, that is, U_t is a left ideal. For the converse, let I be a proper ideal of S and choose α with the least gap in I, say $g(\alpha) = t$, then $t \leq r$ and $I \subseteq U_t$. We claim that $q < t \leq r$. Suppose not, that is $t \leq q$ and we also note that, since $q < r$, we have that every element in S has the same rank $|X|$. Then, let $\beta \in S$ and write

$$\alpha = \begin{pmatrix} a_i \\ b_i \end{pmatrix}, \text{ then } \beta = \begin{pmatrix} c_i \\ d_i \end{pmatrix},$$

where $|I| = |X|$. We also write

$$\{a_i\} = \{x_i\} \,\dot{\cup}\, \{x_j\}, \;\; X \setminus X\alpha = A \,\dot{\cup}\, B, \;\; X \setminus X\beta = C \,\dot{\cup}\, D,$$

where $|J| = q, |A| = |C| = |D| = q$ and $|B| = t$. Then $X\alpha = \{x_i\alpha\} \dot{\cup} \{x_j\alpha\}$. Define

$$\lambda = \begin{pmatrix} c_i \\ x_i \end{pmatrix} \text{ and } \mu = \begin{pmatrix} x_i\alpha & \{x_j\alpha\} \cup A \\ d_i & C \end{pmatrix}$$

where $\mu : \{x_j\alpha\} \cup A \to C$ is a bijection. Then $\beta = \lambda\alpha\mu$, and $g(\lambda) = g(\beta) \leq r$, $d(\lambda) = |\{x_j\}| + g(\alpha) = q + t = q$, $g(\mu) = |B| = t \leq r$ and $d(\mu) = |D| = q$, that is $\lambda, \mu \in S$. Therefore $S = S.\alpha.S \subseteq I$ (since I is an ideal), a contradiction. Hence $q < t \leq r$. Also, for each $\gamma \in U_t$, we have $g(\gamma) \geq t = g(\alpha) > q$, thus, Lemma 1 implies that $\gamma = \delta\alpha\epsilon \in I$ for some $\delta, \epsilon \in S$. Therefore $U_t = I$. Finally, for each $q < t \leq r$, choose $\alpha \in U_t$ with gap t. Then, for each $\beta \in U_t$, $g(\beta) \geq g(\alpha) = t > q$. Again, by Lemma 1, we have that $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in S$. Therefore $U_t \subseteq S^1.\alpha.S^1 \subseteq U_t$ since U_t is an ideal, and so $U_t = S^1.\alpha.S^1$, that is, U_t is principal. □

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