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ON 0-MINIMAL $(0,2)$-BI-Γ-IDEALS IN Γ-SEMIGROUPS

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Abstract: We describe $(0,2)$-Γ-ideals, $(1,2)$-Γ-ideals and 0-minimal $(0,2)$-Γ-ideals in Γ-semigroups, and then the notion of $(0,2)$-bi-Γ-ideal in Γ-semigroups will be introduced and studied.

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1. Preliminaries

Let S and Γ be two non-empty sets. Then S is called a Γ-*semigroup* if there exists a mapping $S \times \Gamma \times S \to S$; $(a, \gamma, b) \to a\gamma b$ such that

$$(a\alpha b)\beta c = a\alpha(b\beta c)$$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. This notion has been introduced by Sen in [4]. If A, B are non-empty subsets of S, we let

$$A\Gamma B = \{x\gamma y \mid x \in A, y \in B, \gamma \in \Gamma\}.$$

A non-empty subset T of S is called a *sub*-Γ-*semigroup* of S if $T\Gamma T \subseteq T$, that is $x\alpha y \in T$ for all $x, y \in T$ and for all $\alpha \in \Gamma$.

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Let S be a Γ-semigroup. A non-empty subset A of S is called a *left* Γ*-ideal* of S if $S\Gamma A \subseteq A$, a *right* Γ*-ideal* of S if $A\Gamma S \subseteq A$, and a (two-sided) Γ*-ideal* of S if A is both a left and a right Γ-ideal of S. It is clear that every left Γ-ideal, right Γ-ideal and (two-sided) Γ-ideal of S is a sub-Γ-semigroup of S.

Let S be a Γ-semigroup and let m be a positive integer. If $A \subseteq S$, we let

$$A^m = A\Gamma A \cdots \Gamma A \text{ where } A \text{ occurs } m \text{ times},$$

i.e.

$$A^m = \{x_1\gamma_1x_2\gamma_2x_3\gamma_3 \cdots \gamma_{m-1}x_m \mid x_i \in A, \gamma_i \in \Gamma\};$$

if $A = \{a\}$, we let $a^m = \{a\}^m$.

Definition 1. Let S be a Γ-semigroup, and let m, n be non-negative integers. A sub-Γ-semigroup A of S is called an (m,n)-Γ*-ideal* of S if

$$A^m\Gamma S\Gamma A^n \subseteq A.$$

Here, $A^0\Gamma S = S$ and $S\Gamma A^0 = S$.

Every $(1,1)$-Γ-ideal is called a *bi*-Γ*-ideal* [2]. If A is a non-empty subset of a Γ-semigroup S, it is known that

$$A \cup A\Gamma S\Gamma A$$

is a bi-Γ-ideal of S.

Now, we define $(0,2)$-bi-Γ-ideals in Γ-semigroups analogue to $(0,2)$-bi-ideals in semigroups introduced in [3].

Definition 2. A sub-Γ-semigroup A of a Γ-semigroup S is called a $(0,2)$-*bi*-Γ*-ideal* of S if A is both a bi-Γ-ideal and a $(0,2)$-Γ-ideal of S.

In [3], Krgović described $(0,2)$-ideals, $(1,2)$-ideals and 0-minimal $(0,2)$-ideals in semigroups. The author also introduced and studied $(0,2)$-bi-ideals in semigroups. The purpose of this paper is to extend the results proved by Krgović using Γ-semigroups.

2. Main Results

Lemma 3. *Let S be a Γ-semigroup and let $A \subseteq S$. Then A is a $(0,2)$-Γ-ideal of S if and only if A is a left Γ-ideal of some left Γ-ideal of S.*

Proof. If A is a $(0,2)$-Γ-ideal of S, then

$$(A \cup S\Gamma A)\Gamma A = A^2 \cup S\Gamma A^2 \subseteq A,$$

and so A is a left Γ-ideal of the left Γ-ideal $A \cup A\Gamma S$ of S.

Conversely, if A is a left Γ-ideal of a left Γ-ideal L of S then

$$S\Gamma A^2 = S\Gamma A\Gamma A \subseteq S\Gamma L\Gamma A \subseteq L\Gamma A \subseteq A.$$

Thus A is a $(0,2)$-Γ-ideal of S. □

Theorem 4. *Let A be a subset of a Γ-semigroup S. Then the following statements are equivalent:*

(i) *A is a $(1,2)$-Γ-ideal of S;*

(ii) *A is a left Γ-ideal of some bi-Γ-ideal of S;*

(iii) *A is a bi-Γ-ideal of some left Γ-ideal of S;*

(iv) *A is a $(0,2)$-Γ-ideal of some right Γ-ideal of S;*

(v) *A is a right Γ-ideal of some $(0,2)$-Γ-ideal of S.*

Proof. (i)⇒(ii). Since A is a $(1,2)$-Γ-ideal of S, we have

$$(A \cup A\Gamma S\Gamma A)\Gamma A \subseteq A^2 \cup A\Gamma S\Gamma A^2 \subseteq A,$$

and so A is a left Γ-ideal of the bi-Γ-ideal $A \cup A\Gamma S\Gamma A$ of S.

(ii)⇒(iii). Assume that A is a left Γ-ideal of a bi-Γ-ideal B of S. Then

$$A\Gamma(A \cup S\Gamma A)\Gamma A \subseteq A \cup B\Gamma S\Gamma B\Gamma A \subseteq B\Gamma A \subseteq A,$$

and hence A is a bi-Γ-ideal of the left Γ-ideal $A \cup S\Gamma A$ of S.

(iii)⇒(iv). Let A is a bi-Γ-ideal of a left Γ-ideal L of S. We have

$$(A \cup A\Gamma S)\Gamma A^2 \subseteq A \cup A\Gamma S\Gamma L\Gamma A \subseteq A \cup A\Gamma L\Gamma A = A.$$

Thus A is a $(0,2)$-Γ-ideal of the right Γ-ideal $A \cup A\Gamma S$ of S.

(iv)⇒(v). Assume that A is a $(0,2)$-Γ-ideal of a right Γ-ideal R of S. Then

$$A\Gamma(A \cup S\Gamma A^2) \subseteq A \cup R\Gamma S\Gamma A^2 \subseteq A \cup R\Gamma A^2 = A.$$

Therefor, A is a right Γ-ideal of a $(0,2)$-Γ-ideal $A \cup S\Gamma A^2$ of S.

(v)⇒(i). If A is a right Γ-ideal of a $(0,2)$-Γ-ideal R of S, then

$$A\Gamma S\Gamma A^2 = A\Gamma S\Gamma A^2 \subseteq A\Gamma S\Gamma R^2 \subseteq A\Gamma R \subseteq A,$$

and thus A is a $(1,2)$-Γ-ideal of S. □

Lemma 5. *Let S be a Γ-semigroup and let A be a sub-Γ-semigroup of S. Then A is a $(1,2)$-Γ-ideal of S if and only if there exist $(0,2)$-Γ-ideal L of S and right Γ-ideal R of S such that $R\Gamma L^2 \subseteq A \subseteq R \cap L$.*

Proof. Assume that A is a $(1,2)$-Γ-ideal of S. Let $L = A \cup S\Gamma A^2, R = A \cup A\Gamma S$, then

$$\begin{aligned} R\Gamma L^2 &= (A \cup A\Gamma S)\Gamma(A \cup S\Gamma A^2)\Gamma(A \cup S\Gamma A^2) \\ &= (A \cup A\Gamma S)\Gamma(A^2 \cup A\Gamma S\Gamma A^2 \cup S\Gamma A^3 \cup S\Gamma A^2\Gamma S\Gamma A^2) \\ &= A^3 \cup A\Gamma(A\Gamma S\Gamma A^2) \cup (A\Gamma S\Gamma A^2)\Gamma A \cup (A\Gamma S\Gamma A^2)\Gamma S\Gamma A^2 \\ &\quad \cup A\Gamma S\Gamma A^2 \cup A\Gamma(S\Gamma A\Gamma S)\Gamma A^2 \cup A\Gamma(S\Gamma S)\Gamma A^3 \\ &\quad \cup A\Gamma S^2\Gamma A\Gamma(A\Gamma S\Gamma A^2) \\ &\subseteq A \cup A^2 \cup A\Gamma S\Gamma A^2. \end{aligned}$$

Therefore $R\Gamma L^2 \subseteq A \subseteq L \cap R$.

Conversely, if $R\Gamma L^2 \subseteq A \subseteq L \cap R$ for some right Γ-ideal R and $(0,2)$-Γ-ideal L of S then

$$\begin{aligned} A\Gamma S\Gamma A^2 &\subseteq (R \cap L)\Gamma S\Gamma(R \cap L)\Gamma(R \cap L) \\ &\subseteq (R\Gamma S)\Gamma L^2 \\ &\subseteq R\Gamma L^2 \\ &\subseteq A. \end{aligned}$$

So A is a $(0,2)$-Γ-ideal of S. □

Let S be a Γ-semigroup. An element $0 \in S$ is called a *zero element* of S if $0\gamma a = a\gamma 0 = 0$ for all $a \in S$ and for all $\gamma \in \Gamma$.

A $(0,2)$-bi-Γ-ideal A of a Γ-semigroup S with a zero element 0 will be said to be 0-*minimal* if $A \neq \{0\}$ and $\{0\}$ is the only $(0,2)$-bi-Γ-ideal of S properly contained in A.

Assume that S is a Γ-semigroup with a zero element 0. It is easy to see that every left Γ-ideal of S is a $(0,2)$-Γ-ideal of S. Hence if L is a 0-minimal $(0,2)$-Γ-ideal of S and A is a left Γ-ideal of S contained in L then $A = \{0\}$ or $A = L$. One can ask: what can we say about $(0,2)$-Γ-ideals contained in some 0-minimal left Γ-ideal of S ?

Lemma 6. *Let S be a Γ-semigroup with a zero element 0. Let L be a 0-minimal left Γ-ideal of S and let A be a sub-Γ-semigroup of L. Then A is a $(0,2)$-Γ-ideal of S if and only if $A^2 = \{0\}$ or $A = L$.*

Proof. Assume that A is a $(0,2)$-Γ-ideal of S. Since $S\Gamma A^2$ is a left Γ-ideal of S and $S\Gamma A^2 \subseteq A \subseteq L$, it follows that $S\Gamma A^2 = \{0\}$ or $S\Gamma A^2 = L$. If $S\Gamma A^2 = L$, then $S\Gamma A^2 \subseteq A \subseteq L = S\Gamma A^2$, and so $A = L$. Assume that $S\Gamma A^2 = \{0\}$, then $0 \in A$. Since A^2 is a left Γ-ideal of S contained in L, we have $A^2 = \{0\}$ or $A^2 = L$. If $A^2 = L$, then $A = L$.

Conversely, if $A^2 = \{0\}$ then $S\Gamma A^2 \subseteq A$, and if $A = L$ then $S\Gamma A^2 = S\Gamma L\Gamma L \subseteq L = A$. Hence A is a $(0,2)$-Γ-ideal of S. □

Lemma 7. *Let S be a Γ-semigroup with a zero element 0. If L is a 0-minimal $(0,2)$-Γ-ideal of S, then $L^2 = \{0\}$ or L is a 0-minimal left Γ-ideal of S.*

Proof. Assume that L is a 0-minimal $(0,2)$-Γ-ideal of S. Then

$$S\Gamma(L^2)^2 = (S\Gamma(L^2))\Gamma L^2 \subseteq L\Gamma L^2 \subseteq L^2,$$

i.e. L^2 is $(0,2)$-Γ-ideal of S contained in L, and so $L^2 = \{0\}$ or $L^2 = L$. Assume that $L^2 = L$, then L is a left Γ-ideal of S. If B is a left Γ-ideal of S contained in L such that $B \neq \{0\}$, then B is a $(0,2)$-Γ-ideal of S. Hence $B = L$ since L is a 0-minimal $(0,2)$-Γ-ideal of S. Therefore, L is a 0-minimal left Γ-ideal of S. □

Corollary 8. *Let S be a Γ-semigroup without zero. Then L is a minimal $(0,2)$-Γ-ideal of S if and only if L is a minimal left Γ-ideal of S.*

Proof. This follows from Lemma 6 and Lemma 7. □

Lemma 9. *Let S be a Γ-semigroup without zero, and let A be a non-empty subset of S. Then A is a minimal $(2,1)$-Γ-ideal of S if and only if A is a minimal bi-Γ-ideal of S.*

Proof. Assume that A is a minimal $(2,1)$-Γ-ideal of S. Then $A^2\Gamma S\Gamma A$ is a $(2,1)$-Γ-ideal of S contained in A, hence $A^2\Gamma S\Gamma A = A$. This implies that A is a bi-Γ-ideal of S. If B is a bi-Γ-ideal of S contained in A, then B is a $(2,1)$-Γ-ideal of S contained in A, and so $B = A$. Hence A is a minimal bi-Γ-ideal of S.

Conversely, if A is a minimal bi-Γ-ideal of S, then A is a $(2,1)$-Γ-ideal of S. Let B be a $(2,1)$-Γ-ideal of S contained in A. Since $B^2\Gamma S\Gamma B$ is a bi-Γ-ideal of S contained in A, so $B^2\Gamma S\Gamma B = A$, and hence $A = B^2\Gamma S\Gamma B \subseteq B \subseteq A$. □

Lemma 10. *Let A be a subset of a Γ-semigroup S. Then A is a $(0,2)$-bi-Γ-ideal of S if and only if A is a Γ-ideal of some left Γ-ideal of S.*

Proof. Assume that A is a $(0,2)$-bi-Γ-ideal of S. Since

$$A\Gamma(A \cup S\Gamma A) = A^2 \cup A\Gamma S\Gamma A \subseteq A,$$

$$(A \cup S\Gamma A)\Gamma A = A^2 \cup S\Gamma A^2 \subseteq A,$$

we have A is a Γ-ideal of the left Γ-ideal $A \cup S\Gamma A$ of S.

Conversely, assume that A is a Γ-ideal of a left Γ-ideal L of S. Then

$$A\Gamma S\Gamma A \subseteq A\Gamma(S\Gamma L) \subseteq A\Gamma L \subseteq A,$$

and by Lemma 3 it follows that A is a $(0,2)$-bi-Γ-ideal of S. □

Theorem 11. *Let S be a Γ-semigroup. If A is a 0-minimal $(0,2)$-bi-Γ-ideal of S, then exactly one of the following cases occurs:*

(i) $A = \{0, a\}$, $a^2 = \{0\}$, $a\Gamma S\Gamma a = \{0\}$;

(ii) $A = \{0, a\}, a^2 = \{0\}, a\Gamma S\Gamma a = A$;

(iii) $\forall a \in A \setminus \{0\}, S\Gamma a^2 = A$.

Proof. Assume that A is a 0-minimal $(0,2)$-bi-Γ-ideal of S. Let $a \in A \setminus \{0\}$. Then $S\Gamma a^2 \subseteq A$. Since $S\Gamma a^2$ is a left Γ-ideal of S, we have $S\Gamma a^2$ is a $(0,2)$-bi-Γ-ideal of S, and so $S\Gamma a^2 = \{0\}$ or $S\Gamma a^2 = A$. Assume that $S\Gamma a^2 = \{0\}$. Since $a^2 \subseteq A$, so

$$a^2 = \{a\} \text{ or } a^2 = \{0\} \text{ or } (a^2 \neq \{0\} \text{ and } a^2 \neq \{a\}).$$

If $a^2 = \{a\}$, then $a^3 = \{a\}$, and so $a = 0$. This is a contradiction. Assume that $a^2 \neq \{0\}$ and $a^2 \neq \{a\}$. Then $\{0\} \cup a^2$ is a $(0,2)$-bi-Γ-ideal of S contained in A. Clearly, $\{0\} \cup a^2 \neq \{0\}$ and $\{0\} \cup a^2 \neq A$. This is also a contradiction to the minimality of A. Therefore, $a^2 = \{0\}$ and so $A = \{0, a\}$ by Lemma 10. Using $a^2 = \{0\}$, we obtain $a\Gamma S\Gamma a$ is a $(0,2)$-bi-Γ-ideal of S contained in A. By the maximality of A, $a\Gamma S\Gamma a = \{0\}$ or $a\Gamma S\Gamma a = A$. □

Corollary 12. *Let S be a Γ-semigroup with a zero element 0. If A is a 0-minimal $(0,2)$-bi-Γ-ideal of S such that $A^2 \neq \{0\}$, then $A = S\Gamma a^2$ for every $a \in A \setminus \{0\}$.*

Definition 13. A Γ-semigroup S with a zero element 0 is said to be 0-$(0,2)$-*bisimple* if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper $(0,2)$-bi-Γ-ideal of S.

Corollary 14. *Let S be a Γ-semigroup with a zero element 0. Then S is a 0-$(0,2)$-bisimple if and only if $S\Gamma a^2 = S$ for all $a \in S \setminus \{0\}$.*

Proof. If S is a 0-$(0,2)$-bisimple, then $S^2 \neq \{0\}$ and S is a 0-minimal $(0,2)$-bi-Γ-ideal. By Corollary 12, $S = S\Gamma a^2$ for every $a \in S \setminus \{0\}$.

Conversely, assume that $S = S\Gamma a^2$ for every $a \in S \setminus \{0\}$. Let A is $(0,2)$-bi-Γ-ideal of S such that $A \neq \{0\}$. Let $a \in A \setminus \{0\}$, then $S = S\Gamma a^2 \subseteq S\Gamma A^2 \subseteq A$, and so $S = A$. Since $S = S\Gamma a^2 \subseteq S^2$, we have $\{0\} \neq A = S = S^2$. Therefore S is 0-$(0,2)$-bisimple. □

A Γ-semigroup S with a zero element 0 is said to be *left* 0-*simple* if $\{0\}$ is the only proper left Γ-ideal of S.

Theorem 15. *Let S be a Γ-semigroup with a zero element 0. Then S is 0-$(0,2)$-bisimple if and only if S is left 0-simple.*

Proof. Assume that S is a 0-$(0,2)$-bisimple. Then $S^2 \neq \{0\}$ and $\{0\}$ is the proper $(0,2)$-bi-Γ-ideal of S. If A is a left Γ-ideal of S such that $A \neq \{0\}$, then A is a $(0,2)$-bi-Γ-ideal of S, and so $A = S$ since S is a 0-$(0,2)$-bisimple. Hence S is left 0-simple.

Conversely, if S is left 0-simple then $S\Gamma a^2 = S$ for every $a \in S \setminus \{0\}$, and hence, by Corollary 14, S is 0-$(0,2)$-bisimple. □

Theorem 16. *Let S be a Γ-semigroup with a zero element 0. If A is a 0-minimal $(0,2)$-bi-Γ-ideal of S, then either $A^2 = \{0\}$ or A is left 0-simple.*

Proof. Assume that A is a 0-minimal $(0,2)$-bi-Γ-ideal of S such that $A^2 \neq \{0\}$. By Corollary 12, $S\Gamma a^2 = A$ for every $a \in A \setminus \{0\}$. Let $a \in A \setminus \{0\}$. Since A is a bi-Γ-ideal of S, $A\Gamma a^2$ is a bi-Γ-ideal of S. Since A is a $(0,2)$-Γ-ideal of S, $A\Gamma a^2$ is a $(0,2)$-Γ-ideal of S. Hence $A\Gamma a^2$ is a $(0,2)$-bi-Γ-ideal of S contained in A and so $A\Gamma a^2 = \{0\}$ or $A\Gamma a^2 = A$. If $A\Gamma a^2 = \{0\}$, then $A^2 = \{0\}$. This is a contradiction. Therefore, $A\Gamma a^2 = A$. By Corollary 14 and Theorem 15, we have A is left 0-simple. □

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