International Journal of Pure and Applied Mathematics
Volume 84 No. 5 2013, 557-572
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v84i5.9

ijpam.eu

ON FUNCTIONAL EQUATION OF MIXED TYPE IN FUZZY NORMED SPACES

Ick-Soon Chang[1 §], Hark-Mahn Kim[2], Yang-Hi Lee[3]

[1]Department of Mathematics
Mokwon University
Daejeon, 302-729, REPUBLIC OF KOREA

[2]Department of Mathematics
Daejeon, 305-764, REPUBLIC OF KOREA

[3]Department of Mathematics Education
Gongju National University of Education
Gongju, 314-711, REPUBLIC OF KOREA

Abstract: In this paper, we investigate the stability of functional equation deriving from additive-quadratic mapping in fuzzy normed space. More precisely, we show under suitable conditions that a fuzzy q-almost additive-quadratic mapping can be approximated by additive-quadratic mapping.

AMS Subject Classification: 39B52, 46S40
Key Words: additive-quadratic mapping, fuzzy almost quadratic-additive mapping, fuzzy normed space

1. Introduction and Preliminaries

Katsaras [6] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Later, some mathematicians have introduced several types of fuzzy norm in different points of view. Bag and Samanta

Received: December 4, 2012

[§]Correspondence author

[3], following Cheng and Mordeson [4], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [7].

On the other hand, the stability problem of functional equations has originally been formulated by Ulam [12] : *given an approximately linear mapping f, when does a linear mapping L estimating f exist*? Hyers [5] gave an affirmative answer the problem of Ulam in the context of Banach spaces. The theorem of Hyers was extended by Aoki [1] for approximately additive mappings and by Rassias [11] for approximately linear mappings. During the last decades a number of papers have been published on the stability of functional equations and several stability problems have been investigated. Recently, the fuzzy stability problem for the additive functional equation and the quadratic functional equation is studied, see in [8, 9]. Quite recently, we considered the following functional equation of additive-quadratic type and proved a fuzzy stability in the reference [2] :

$$f(x+y) - f(-x-y) - f(x) + f(-x) - 4f(y) + f(2y) = 0. \tag{1}$$

We first take account of the functional equation of additive-quadratic type which generalize than the equation (1)

$$\sum_{1\leq i,j\leq n, i\neq j} [f(x_i + x_j) + f(x_i - x_j)] - (n-1)\sum_{j=1}^{n} f(2x_j) = 0. \tag{2}$$

In this case, a mapping f satisfying (2) is called an *additive-quadratic mapping*.

In this paper, we establish the stability of the functional equation (2) in a fuzzy sense (cf. [8]). More precisely, we show under suitable conditions that a fuzzy q-almost additive-quadratic mapping can be approximated by additive-quadratic mapping.

2. Main Results

We first demonstrate the following definition for a fuzzy normed space given in [3] and the fundamental concepts :

Definition 2.1. ([3]) Let X be a real linear space. A function $N : \mathcal{X} \times \mathbb{R} \to [0,1]$ (the so-called *fuzzy subset*) is said to be *a fuzzy norm* on $\mathcal{X}$ if for all $x, y \in \mathcal{X}$ and all $s, t \in \mathbb{R}$,

(1) $N(x,c) = 0$ for $c \leq 0$;

(2) $x = 0$ if and only if $N(x, c) = 1$ for all $c > 0$;

(3) $N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;

(4) $N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;

(5) $N(x, \cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.

In this case, the pair $(\mathcal{X}, N)$ is called a *fuzzy normed space*.

The examples of fuzzy norms and the properties of fuzzy normed linear spaces are given in [8, 9, 10].

Definition 2.2. Let $(\mathcal{X}, N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in $\mathcal{X}$. Then $\{x_n\}$ is said to be *convergent* if there exists $x \in \mathcal{X}$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the *limit* of the sequence $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

Definition 2.3. Let $(\mathcal{X}, N)$ be a fuzzy normed linear space. A sequence $\{x_n\}$ in $\mathcal{X}$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.

It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called *a fuzzy Banach space*.

Now let $(\mathcal{X}, N)$ and $(\mathcal{Y}, N')$ be a fuzzy normed space and a fuzzy Banach space. For a given mapping $f : \mathcal{X} \to \mathcal{Y}$, we write

$$Df(x_1, x_2, \cdots, x_n) := \sum_{1\leq i,j\leq n, i\neq j} [f(x_i + x_j) + f(x_i - x_j)] - (n - 1)\sum_{j=1}^{n} f(2x_j)$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, where $n > 2$ is a fixed integer. The mapping f is said to be a *fuzzy q-almost additive-quadratic mapping* if for given $q > 0$, the inequality

$$N'(Df(x_1, \cdots, x_n), t_1 + \cdots + t_n) \geq \min\{N(x_1, t_1^q), \cdots, N(x_n, t_n^q)\} \qquad (3)$$

holds for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and all $t_1, t_2, \cdots, t_n \in [0, \infty)$. Now we deal with the stability of functional equation (2) in fuzzy normed space as follows.

Theorem 2.4. *Let q be a positive real number with $q \notin \{\frac{1}{2}, 1\}$. Suppose that f is a fuzzy q-almost additive-quadratic mapping from a fuzzy normed space $(\mathcal{X}, N)$ into a fuzzy Banach space $(\mathcal{Y}, N')$. Then there exists a unique additive-quadratic mapping $F : \mathcal{X} \to \mathcal{Y}$ satisfying*

$$N'(F(x)-f(x),t) \geq \begin{cases} \sup_{s<t}\{N(x,(2-2^p)^q(n-1)^q s^q)\} & (q>1), \\ \sup_{s<t}\{N(x,2^{-q}(4-2^p)^q(2^p-2)^q(n-1)^q s^q)\} & (\frac{1}{2}<q<1), \\ \sup_{s<t}\{N(x,(2^p-4)^q(n-1)^q s^q)\} & (0<q<\frac{1}{2}) \end{cases} \tag{4}$$

for all $x \in \mathcal{X}$ *and* $t>0$, *where* $p=\frac{1}{q}$.

Proof. By (3), we see that

$$N'(f(0),t) = N'(Df(0,\cdots,0), n(n-1)t) \geq \min\{N(0,(n-1)^q t^q)\} = 1$$

for all $t>0$. So we find that $f(0)=0$.

Now, we are in the position to show our result. We will consider three different cases for $q>1$, $\frac{1}{2}<q<1$ and $0<q<\frac{1}{2}$:

Case 1. Let $q>1$. Suppose that $J_m f : \mathcal{X} \to \mathcal{Y}$ is a mapping defined by

$$J_m f(x) = \frac{f(2^m x) + f(-2^m x)}{2\cdot 4^m} + \frac{f(2^m x) - f(-2^m x)}{2^{m+1}}$$

for all $x \in \mathcal{X}$. Then $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \frac{(2^{j+1}+1)Df(2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)} - \frac{(2^{j+1}-1)Df(-2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)} \tag{5}$$

for all $x \in \mathcal{X}$ and $j \geq 0$. This equation and (3) imply that if $m'+m>m\geq 0$, then

$$\begin{aligned} &N'\Bigg(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j t\Bigg) \\ &\geq N'\Bigg(\sum_{j=m}^{m'+m-1}\Big(J_j f(x) - J_{j+1} f(x)\Big), \sum_{j=m}^{m'+m-1} \frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j t\Bigg) \\ &\geq \min \bigcup_{j=m}^{m'+m-1} \left\{N'\Bigg(J_j f(x) - J_{j+1} f(x), \frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j t\Bigg)\right\} \\ &\geq \min \bigcup_{j=m}^{m'+m-1} \left\{N'\Bigg(\frac{(2^{j+1}+1)Df(2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)}, \frac{(2^{j+1}+1)2^{jp} t}{2^{2j+3}(n-1)}\Bigg),\right. \end{aligned}$$

$$N'\Big(-\frac{(2^{j+1}-1)Df(-2^jx,0,\cdots,0)}{2^{2j+3}(n-1)},\frac{(2^{j+1}-1)2^{jp}t}{2^{2j+3}(n-1)}\Big)\Big\}$$

$$\geq \min\bigcup_{j=m}^{m'+m-1}\left\{N(2^jx,2^js^q),N\left(0,2^j(n-1)^{-q}(t-s)^q\right)\right\}$$

$$= N(x,s^q)$$

for all $x \in \mathcal{X}$ and $t > 0$, where $0 < s < t$. Hence we have the following inequality

$$N'\Big(J_mf(x)-J_{m'+m}f(x),\sum_{j=m}^{m'+m-1}\frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^jt\Big)\geq\sup_{0<s<t}N(x,s^q) \quad (6)$$

for all $x \in \mathcal{X}$ and $t > 0$. Let $\varepsilon > 0$ be given. Since $\lim_{t\to\infty}N(x,t^q)=1$, there is $t_0 > 0$ such that $N(x,t_0^q)\geq 1-\varepsilon$. We observe that for some $\tilde{t} > t_0$, the series $\sum_{j=0}^{\infty}\frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j\tilde{t}$ converges for $p=\frac{1}{q}<1$. It guarantees that, for an arbitrary given $c > 0$, there exists $m_0 \geq 0$ such that $\sum_{j=m}^{m'+m-1}\frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j\tilde{t}<c$ for each $m \geq m_0$ and $m' > 0$. By (6), we have

$$\begin{aligned}N'(J_mf(x)-J_{m'+m}f(x),c) &\geq N'\Big(J_mf(x)-J_{m'+m}f(x),\\ &\qquad\sum_{j=m}^{m'+m-1}\frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j\tilde{t}\Big)\\ &\geq \sup_{0<s<\tilde{t}}N(x,s^q)\\ &\geq N(x,t_0^q)\\ &\geq 1-\varepsilon\end{aligned}$$

for all $x \in \mathcal{X}$. Hence $\{J_mf(x)\}$ is a Cauchy sequence in the fuzzy Banach space $(\mathcal{Y},N')$ and so we can define a mapping $F:\mathcal{X}\to\mathcal{Y}$ by

$$F(x) := N'-\lim_{m\to\infty}J_mf(x).$$

Moreover, if we put $m'=0$ in (6), then we have

$$N'(f(x)-J_{m'}f(x),t)\geq\sup_{0<s<t}N\left(x,\frac{s^q}{\left(\sum_{j=0}^{m'-1}\frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j\right)^q}\right) \quad (7)$$

for all $x \in \mathcal{X}$.

Next, we shall assert that F is an additive-quadratic mapping. We feel that

$$\begin{aligned}
&N'(DF(x_1, \cdots, x_n), t) \qquad (8)\\
&\geq \min\Big\{ \min \bigcup_{1\leq i,j\leq n, i\neq j} \Big\{N'\Big((F - J_m f)(x_i + x_j),\\
&\quad \frac{t}{4n(n-1)}\Big), N'\Big((F - J_m f)(x_i - x_j),\\
&\quad \frac{t}{4n(n-1)}\Big)\Big\}, \min \bigcup_{j=1}^{n} \Big\{N'\Big((n-1)(J_m f - F)(2x_j), \frac{t}{4n}\Big)\Big\},\\
&\quad N'\Big(DJ_m f(x_1, \cdots, x_n), \frac{t}{4}\Big)\Big\}
\end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and $m \in \mathbb{N}$. The first three terms on the right hand side of (8) tend to 1 as $m \to \infty$ by the definition of F and the last term holds

$$\begin{aligned}
&N'\Big(DJ_m f(x_1, x_2, \cdots, x_n), \frac{t}{4}\Big)\\
&\geq \min\Big\{N'\Big(\frac{Df(2^m x_1, \cdots, 2^m x_n)}{2^{2m+1}}, \frac{t}{16}\Big), N'\Big(\frac{Df(-2^m x_1, \cdots, -2^m x_n)}{2^{2m+1}}, \frac{t}{16}\Big),\\
&\quad N'\Big(\frac{Df(2^m x_1, \cdots, 2^m x_n)}{2^{m+1}}, \frac{t}{16}\Big), N'\Big(\frac{Df(-2^m x_1, \cdots, -2^m x_n)}{2^{m+1}}, \frac{t}{16}\Big)\Big\}
\end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and $t > 0$. By virtue of (3), we obtain

$$\begin{aligned}
&N'\Big(\frac{Df(\pm 2^m x_1, \cdots, \pm 2^m x_n)}{2^{2m+1}}, \frac{t}{16}\Big)\\
&\geq \min\{N(x_1, 2^{(2q-1)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(2q-1)m-3q} n^{-q} t^q)\}
\end{aligned}$$

and

$$\begin{aligned}
&N'\Big(\frac{Df(\pm 2^m x_1, \cdots, \pm 2^m x_n)}{2^{m+1}}, \frac{t}{16}\Big)\\
&\geq \min\{N(x_1, 2^{(q-1)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(q-1)m-3q} n^{-q} t^q)\}
\end{aligned}$$

for all $x_1, \cdots, x_n \in \mathcal{X}$ and $n \in \mathbb{N}$. Since $q > 1$, we can deduce that the last term of (8) tends to 1 as $m \to \infty$. It follows from (8) that

$$N'(DF(x_1, x_2, \cdots, x_n), t) = 1,$$

which means that $DF(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in \mathcal{X}$ and $t > 0$.

We now approximate the difference between f and F in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since F is the limit of $\{J_m f(x)\}$, there is $m \in \mathbb{N}$ such that

$$N'\left(F(x) - J_m f(x), t - t'\right) \geq 1 - \varepsilon.$$

By (7), we have

$$\begin{aligned} N'(F(x) - f(x), t) &\geq \min\{N'(F(x) - J_m f(x), t - t'), N'(J_m f(x) - f(x), t')\} \\ &\geq \min\left\{1 - \varepsilon, \sup_{0<s<t'}\left\{N\left(x, \frac{s^q}{\left(\sum_{j=0}^{m-1} \frac{1}{2(n-1)}\left(\frac{2^p}{2}\right)^j\right)^q}\right)\right\}\right\} \\ &\geq \min\left\{1 - \varepsilon, \sup_{0<s<t'}\{N(x, (n-1)^q(2-2^p)^q s^q)\}\right\}. \end{aligned}$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (4).

To prove the uniqueness of F, let us assume that $F' : \mathcal{X} \to \mathcal{Y}$ be another additive-quadratic mapping satisfying (4). Then by (5), we get

$$\begin{cases} F(x) - J_m F(x) = \sum_{j=0}^{m-1}(J_j F(x) - J_{j+1}F(x)) = 0, \\ F'(x) - J_m F'(x) = \sum_{j=0}^{m-1}(J_j F'(x) - J_{j+1}F'(x)) = 0 \end{cases} \tag{9}$$

for all $x \in \mathcal{X}$ and $m \in \mathbb{N}$. It follows by (4) and (9) that

$$\begin{aligned} N'(F(x) - F'(x), t) &= N'(J_m F(x) - J_m F'(x), t) \\ &\geq \min\left\{N'\Big((J_m F - J_m f)(x), \frac{t}{2}\Big), N'\Big((J_m f - J_m F')(x), \frac{t}{2}\Big)\right\} \\ &\geq \min\left\{N'\left(\frac{(F - f)(2^m x)}{2^{2m+1}}, \frac{t}{8}\right), N'\Big(\frac{(f - F')(2^m x)}{2^{2m+1}}, \frac{t}{8}\Big),\right. \\ &\quad N'\Big(\frac{(F - f)(-2^m x)}{2^{2m+1}}, \frac{t}{8}\Big), N'\Big(\frac{(f - F')(-2^m x)}{2^{2m+1}}, \frac{t}{8}\Big), \\ &\quad N'\Big(\frac{(F - f)(2^m x)}{2^{m+1}}, \frac{t}{8}\Big), N'\Big(\frac{(f - F')(2^m x)}{2^{m+1}}, \frac{t}{8}\Big), \\ &\quad \left. N'\Big(\frac{(F - f)(-2^m x)}{2^{m+1}}, \frac{t}{8}\Big), N'\Big(\frac{(f - F')(-2^m x)}{2^{m+1}}, \frac{t}{8}\Big)\right\} \\ &\geq \sup_{s<t} N\{(x, 2^{(q-1)m-2q}(2-2^p)^q(n-1)^q s^q)\} \end{aligned}$$

for all $x \in \mathcal{X}$ and $m \in \mathbb{N}$. Observe that for $q = \frac{1}{p} > 1$, the last term of the previous inequality goes to 1 as $m \to \infty$. Thus we conclude that

$$N'(F(x) - F'(x), t) = 1$$

and so we get $F = F'$.

Case 2. Let $\frac{1}{2} < q < 1$. Suppose that $J_m f : \mathcal{X} \to \mathcal{Y}$ is a mapping defined by

$$J_m f(x) = \frac{f(2^m x) + f(-2^m x)}{2^{2m+1}} + \frac{2^m}{2}\Big(f\Big(\frac{x}{2^m}\Big) - f\Big(-\frac{x}{2^m}\Big)\Big)$$

for all $x \in \mathcal{X}$. Then we have $J_0 f(x) = f(x)$. Moreover, we see that

$$\begin{aligned} J_j f(x) - J_{j+1} f(x) &= \frac{Df(2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)} + \frac{Df(-2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)} \\ &- \frac{2^{j-1}}{n-1}\Big(Df\Big(\frac{x}{2^{j+1}}, 0, \cdots, 0\Big) - Df\Big(-\frac{x}{2^{j+1}}, 0, \cdots, 0\Big)\Big) \end{aligned}$$

for all $x \in \mathcal{X}$ and $j \geq 0$. If $m' + m > m \geq 0$, then we have

$$\begin{aligned} & N'\left(J_m f(x) - J_{m'+m} f(x), \sum_{j=m}^{m'+m-1} \left(\frac{1}{4}\left(\frac{2^p}{4}\right)^j + \frac{1}{2^p}\left(\frac{2}{2^p}\right)^j\right) \frac{t}{n-1}\right) \\ & \geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N'\left(\frac{Df(2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)}, \frac{2^{jp} t}{2^{2j+3}(n-1)}\right), \right. \\ & N'\left(\frac{Df(-2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)}, \frac{2^{jp} t}{2^{2j+3}(n-1)}\right), \\ & N'\left(-\frac{2^{j-1}}{n-1} Df\left(\frac{x}{2^{j+1}}, 0, \cdots, 0\right), \frac{2^{j-1} t}{2^{(j+1)p}(n-1)}\right), \\ & \left. N'\left(\frac{2^{j-1}}{n-1} Df\left(-\frac{x}{2^{j+1}}, 0, \cdots, 0\right), \frac{2^{j-1} t}{2^{(j+1)p}(n-1)}\right) \right\} \\ & \geq \min \bigcup_{j=m}^{m'+m-1} \left\{ N(2^j x, 2^j s^q), N(0, 2^j (n-1)^{-q} (t-s)^q), \right. \\ & \left. N\left(\frac{x}{2^{j+1}}, \frac{s^q}{2^{j+1}}\right), N\left(0, \frac{(t-s)^q}{2^{j+1}(n-1)^q}\right) \right\} \\ & = N(x, s^q) \end{aligned}$$

for all $x \in \mathcal{X}$ and $t > 0$, where $0 < s < t$. Employing the similar way following (6) of the previous case, we can define the limit

$$F(x) := N' - \lim_{m\to\infty} J_m f(x)$$

of the Cauchy sequence $\{J_m f(x)\}$ in the Banach fuzzy space $\mathcal{Y}$. Moreover, put $m = 0$ in the above inequality and then we have

$$N'(f(x) - J_{m'} f(x), t) \ge \sup_{s<t} N\left(x, \frac{(n-1)^q s^q}{\left(\sum_{j=0}^{m'-1}\left(\frac{1}{4}\left(\frac{2^p}{4}\right)^j + \frac{1}{2^p}\left(\frac{2}{2^p}\right)^j\right)\right)^q}\right) \tag{10}$$

for each $x \in \mathcal{X}$ and $t > 0$.

To prove that F is an additive-quadratic mapping, we have enough to show that the last term of (8) in Case 1 tends to 1 as $m \to \infty$. Due to (3), we get

$$\begin{aligned}
&N'\left(DJ_m f(x_1, x_2, \cdots, x_n), \frac{t}{4}\right)\\
&\ge \min\left\{N'\left(\frac{Df(2^m x_1, \cdots, 2^m x_n)}{2^{2m+1}}, \frac{t}{16}\right), N'\left(\frac{Df(-2^m x_1, \cdots, -2^m x_n)}{2^{2m+1}}, \frac{t}{16}\right),\right.\\
&\quad \left. N'\left(\frac{2^m}{2} Df\left(\frac{x_1}{2^m}, \cdots, \frac{x_n}{2^m}\right), \frac{t}{16}\right), N'\left(-\frac{2^m}{2} Df\left(\frac{-x_1}{2^m}, \cdots, \frac{-x_n}{2^m}\right), \frac{t}{16}\right)\right\}\\
&\ge \min\{N(x_1, 2^{(2q-1)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(2q-1)m-3q} n^{-q} t^q),\\
&\quad N(x_1, 2^{(1-q)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(1-q)m-3q} n^{-q} t^q)\}.
\end{aligned}$$

Observe that all the terms on the right hand side of this inequality tend to 1 as $m \to \infty$. Hence, together with the similar argument after (8), we can say that $DF(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in \mathcal{X}$ and $t > 0$. Recall, in Case 1, the inequality (4) follows from (7). By the same reasoning, we get (4) from (10).

Now, to prove the uniqueness of F, we assume that F' be an additive-quadratic mapping satisfying (4). In view of (4) and (9), we have

$$\begin{aligned}
&N'(F(x) - F'(x), t) = N'(J_m F(x) - J_m F'(x), t)\\
&\ge \min\left\{N'\Big((J_m F - J_m f)(x), \frac{t}{2}\Big), N'\Big((J_m f - J_m F')(x), \frac{t}{2}\Big)\right\}\\
&\ge \min\left\{N'\Big(\frac{(F-f)(2^m x)}{2^{2m+1}}, \frac{t}{8}\Big), N'\Big(\frac{(f-F')(2^m x)}{2^{2m+1}}, \frac{t}{8}\Big),\right.\\
&\quad N'\Big(\frac{(F-f)(-2^m x)}{2^{2m+1}}, \frac{t}{8}\Big), N'\Big(\frac{(f-F')(-2^m x)}{2^{2m+1}}, \frac{t}{8}\Big),
\end{aligned}$$

$$N'\Big(\frac{2^m}{2}(F-f)\Big(\frac{x}{2^m}\Big),\frac{t}{8}\Big),N'\Big(\frac{2^m}{2}(f-F')\Big(\frac{x}{2^m}\Big),\frac{t}{8}\Big),$$
$$N'\Big(\frac{2^m}{2}(F-f)\Big(\frac{-x}{2^m}\Big),\frac{t}{8}\Big),N'\Big(\frac{2^m}{2}(f-F')\Big(\frac{-x}{2^m}\Big),\frac{t}{8}\Big)\Big\}$$
$$\geq \min\Big\{\sup_{s<t}N(x,2^{(2q-1)m-3q}(n-1)^q(4-2^p)^q(2^p-2)^q s^q),$$
$$\sup_{s<t}N(x,2^{(1-q)m-3q}(n-1)^q(4-2^p)^q(2^p-2)^q s^q)\Big\}$$

for all $x \in \mathcal{X}$ and $m \in \mathbb{N}$. Since both terms on the right hand side of the above inequality tend to 1 as $m \to \infty$, we find that

$$N'(F(x)-F'(x),\ t)=1.$$

That is, $F(x)=F'(x)$ for all $x \in \mathcal{X}$.

Case 3. Take $0<q<\frac{1}{2}$. Suppose that $J_m f : \mathcal{X} \to \mathcal{Y}$ is a mapping defined by

$$J_m f(x)=\frac{4^m(f(2^{-m}x)+f(-2^{-m}x))+2^m(f(2^{-m}x)-f(-2^{-m}x))}{2}$$

for all $x \in \mathcal{X}$. Then we have $J_0 f(x)=f(x)$. In addition,

$$J_j f(x)-J_{j+1}f(x)=\frac{-2^{2j}-2^j}{2(n-1)}Df\Big(\frac{x}{2^{j+1}},0,\cdots,0\Big)$$
$$+\frac{-2^{2j}+2^j}{2(n-1)}Df\Big(-\frac{x}{2^{j+1}},0,\cdots,0\Big),$$

which implies that if $m'+m>m\geq 0$, then

$$N'\left(J_m f(x)-J_{m'+m}f(x),\sum_{j=m}^{m'+m-1}\left(\frac{4}{2^p}\right)^{j+1}\frac{t}{2(n-1)}\right)$$
$$\geq \min\bigcup_{j=m}^{m'+m-1}\left\{N'\left(\frac{-2^j(2^j+1)}{2(n-1)}Df\left(\frac{x}{2^{j+1}},0,\cdots,0\right),\frac{2^j(2^j+1)t}{2^{(j+1)p}(n-1)}\right),\right.$$
$$\left.N'\left(\frac{-2^j(2^j-1)}{2(n-1)}Df\left(-\frac{x}{2^{j+1}},0,\cdots,0\right),\frac{2^j(2^j-1)t}{2^{(j+1)p}(n-1)}\right)\right\}$$
$$\geq \min\bigcup_{j=m}^{m'+m-1}\left\{N\left(\frac{x}{2^{j+1}},\frac{s^q}{2^{j+1}}\right),N\left(0,\frac{(t-s)^q}{2^{j+1}(n-1)^q}\right)\right\}$$

$$= N(x, s^q)$$

for all $x \in \mathcal{X}$ and $t > 0$, where $0 < s < t$. Similar to the previous cases, it leads us to define the mapping $F : \mathcal{X} \to \mathcal{Y}$ by

$$F(x) := N' - \lim_{n\to\infty} J_m f(x).$$

Setting $m = 0$ in the above inequality, then we yield that

$$N'(f(x) - J_{m'} f(x), t) \geq \sup_{s<t} N\left(x, \frac{(n-1)^q s^q}{\left(\sum_{j=0}^{m'-1} \frac{1}{2^p}\left(\frac{4}{2^p}\right)^j\right)^q}\right) \tag{11}$$

for all $x \in \mathcal{X}$ and $t > 0$. Note that

$$\begin{aligned}
&N'\left(DJ_m f(x_1, x_2, \cdots, x_n), \frac{t}{4}\right)\\
&\geq \min\left\{N'\left(\frac{2^{2m}}{2}Df\left(\frac{x_1}{2^m}, \cdots, \frac{x_n}{2^m}\right), \frac{t}{16}\right), N'\left(\frac{2^{2m}}{2}Df\left(\frac{-x_1}{2^m}, \cdots, \frac{-x_n}{2^m}\right),\right.\right.\\
&\qquad \left.\frac{t}{16}\right),\\
&\quad \left.N'\left(\frac{2^m}{2}Df\left(\frac{x_1}{2^m}, \cdots, \frac{x_n}{2^m}\right), \frac{t}{16}\right), N'\left(-\frac{2^m}{2}Df\left(\frac{-x_1}{2^m}, \cdots, \frac{-x_n}{2^m}\right), \frac{t}{16}\right)\right\}\\
&\geq \min\{N(x_1, 2^{(1-2q)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(1-2q)m-q} n^{-3q} t^q),\\
&\quad N(x_1, 2^{(1-q)m-3q} n^{-q} t^q), \cdots, N(x_n, 2^{(1-q)m-3q} n^{-q} t^q)\}.
\end{aligned}$$

Since $0 < q < \frac{1}{2}$, terms on the right hand side tend to 1 as $m \to \infty$, which implies that the last term of (8) tends to 1 as $m \to \infty$. Therefore, one can conclude that $DF(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, \cdots, x_n \in \mathcal{X}$ and $t > 0$. As we did in the argument after (8) in Case 1, we get the inequality (4) from (11).

To prove the uniqueness of F, assume that $F' : \mathcal{X} \to \mathcal{Y}$ be an additive-quadratic mapping satisfying (4). Then we have by (9)

$$\begin{aligned}
N'(F(x) - F'(x), t) &\geq \min\left\{N'\Big((J_m F - J_m f)(x), \frac{t}{2}\Big), N'\Big((J_m f - J_m F')(x),\right.\\
&\qquad \left.\frac{t}{2}\Big)\right\}\\
&\geq \min\left\{N'\Big(\frac{2^{2m}}{2}(F - f)\Big(\frac{x}{2^m}\Big), \frac{t}{8}\Big), N'\Big(\frac{2^{2m}}{2}(f - F')\Big(\frac{x}{2^m}\Big),\right.\\
&\qquad \frac{t}{8}\Big),
\end{aligned}$$

$$N'\Big(\frac{2^{2m}}{2}(F-f)\Big(\frac{-x}{2^m}\Big),\frac{t}{8}\Big), N'\Big(\frac{2^{2m}}{2}(f-F')\Big(\frac{-x}{2^m}\Big),\frac{t}{8}\Big),$$
$$N'\Big(\frac{2^{m}}{2}(F-f)\Big(\frac{x}{2^m}\Big),\frac{t}{8}\Big), N'\Big(\frac{2^{m}}{2}(f-F')\Big(\frac{x}{2^m}\Big),\frac{t}{8}\Big),$$
$$N'\left(\frac{2^{m}}{2}(F-f)\left(\frac{-x}{2^m}\right),\frac{t}{8}\right), N'\left(\frac{2^{m}}{2}(f-F')\left(\frac{-x}{2^m}\right),\right.$$
$$\left.\left.\frac{t}{8}\right)\right\}$$
$$\geq \sup_{s<t} N(x, 2^{(1-2q)m-2q}(2^p-4)^q(n-1)^q s^q)$$

for all $x \in \mathcal{X}$ and $m \in \mathbb{N}$. Observe that for $0 < q < \frac{1}{2}$ the last term tends to 1 as $m \to \infty$, which implies that

$$N'(F(x) - F'(x), t) = 1.$$

Therefore we have $F(x) = F'(x)$ for all $x \in \mathcal{X}$. □

Corollary 2.5. *If an odd mapping f satisfies all of the conditions of Theorem 2.4, then there is a unique additive mapping $F : \mathcal{X} \to \mathcal{Y}$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{s<t} N\left(x, |2-2^p|^q(n-1)^q s^q\right) \tag{12}$$

for all $x \in \mathcal{X}$ and $t > 0$, where $q \neq 1$.

Proof. Let $J_m f$ be defined as in Theorem 2.4. Then, by the oddness of f, we get

$$J_m f(x) = \begin{cases} \frac{f(2^m x) - f(-2^m x)}{2^{m+1}} & (q > 1), \\ \frac{2^m\left(f(2^{-m}x) - f(-2^{-m}x)\right)}{2} & (0 < q < 1) \end{cases}$$

for all $x \in \mathcal{X}$. We observe that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x)$$
$$= \begin{cases} -\frac{Df(2^j x, 0, \cdots, 0)}{2^{j+2}(n-1)} + \frac{Df(-2^j x, 0, \cdots, 0)}{2^{j+2}(n-1)} & (q > 1), \\ -\frac{2^{j-1}}{n-1} Df\left(\frac{x}{2^{j+1}}, 0, \cdots, 0\right) + \frac{2^{j-1}}{n-1} Df\left(-\frac{x}{2^{j+1}}, 0, \cdots, 0\right) & (0 < q < 1) \end{cases}$$

for all $x \in \mathcal{X}$ and $j \in \mathbb{N} \cup \{0\}$. Using the similar fashion in Theorem 2.4, there exists a unique additive-quadratic mapping F satisfying (12). In this case, F is an odd mapping and $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ for all $x \in \mathcal{X}$. Furthermore, $DF(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Thus we have

$$F(x+y) - F(x) - F(y)$$
$$= \frac{1}{2}(DF(x,y,0,\cdots,0) - DF(x,0,\cdots,0) - DF(y,0,\cdots,0)) = 0$$

for all $x, y \in \mathcal{X}$, which yields that F is an additive mapping. □

Corollary 2.6. *If an even mapping f satisfies all of the conditions of Theorem 2.4, then there is a unique quadratic mapping $F : \mathcal{X} \to \mathcal{Y}$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{s<t} N\left(x, |2^p - 4|^q (n-1)^q s^q\right) \tag{13}$$

for all $x \in \mathcal{X}$ and $t > 0$, where $q \neq \frac{1}{2}$.

Proof. Let $J_m f$ be defined as in Theorem 2.4. Since f is an even mapping, we obtain

$$J_m f(x) = \begin{cases} \frac{f(2^m x) + f(-2^m x)}{2^{2m+1}} & (q > \frac{1}{2}), \\ \frac{2^{2m}\left(f(2^{-m}x) + f(-2^{-m}x)\right)}{2} & (0 < q < \frac{1}{2}) \end{cases}$$

for all $x \in \mathcal{X}$. Note that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x)$$
$$= \begin{cases} -\frac{Df(2^j x, 0, \cdots, 0)}{2^{2j+3}(n-1)} - \frac{Df(-2^j x, 0, \cdots, 0))}{2^{2j+3}(n-1)} & (q > \frac{1}{2}), \\ \frac{4^j}{2(n-1)} Df\left(\frac{x}{2^{j+1}}, 0, \cdots, 0\right) + \frac{4^j}{2(n-1)} Df\left(-\frac{x}{2^{j+1}}, 0, \cdots, 0\right) & (0 < q < \frac{1}{2}) \end{cases}$$

for all $x \in \mathcal{X}$ and $j \in \mathbb{N} \cup \{0\}$. Employing the similar method in Theorem 2.4, there exists a unique additive-quadratic mapping F satisfying (13). In particular, we remark that F is an even mapping and $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ for all $x \in \mathcal{X}$. Moreover, $DF(x_1, x_2, \cdots, x_n) = 0$ for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$. Hence we get

$$F(x+y) + F(x-y) - 2F(x) - 2F(y) = \frac{1}{2}(DF(x,y,0,\cdots,0)$$
$$- DF(x,0,\cdots,0) - DF(y,0,\cdots,0)) = 0$$

for all $x, y \in \mathcal{X}$. So F is a quadratic mapping. □

Remark. Let $(\mathcal{X}, \|\cdot\|)$ be a normed space. Then we can define a fuzzy norm $N_{\mathcal{X}}$ on $\mathcal{X}$ by

$$N_{\mathcal{X}}(x,t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in \mathcal{X}$ and $t \in \mathbb{R}$ (see [8]). If $f : \mathcal{X} \to \mathcal{Y}$ is a mapping into a Banach space $(\mathcal{Y}, |||\cdot|||)$ such that

$$|||Df(x_1, x_2, \cdots, x_n)||| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$

for all $x_1, \cdots, x_n \in \mathcal{X}$, where $p > 0$ and $p \neq 1, 2$. Let $N_{\mathcal{Y}}$ be a fuzzy norm on $\mathcal{Y}$. Then

$$N_{\mathcal{Y}}(Df(x_1, \cdots, x_n), t_1 + \cdots + t_n) = \begin{cases} 0, & t_1 + \cdots + t_n \leq |||Df(x_1, \cdots, x_n)||| \\ 1, & t_1 + \cdots + t_n > |||Df(x_1, \cdots, x_n)|||| \end{cases}$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and $t_1, \cdots, t_n \in \mathbb{R}$. Consider the case

$$N_{\mathcal{Y}}(Df(x_1, \cdots, x_n), t_1 + \cdots + t_n) = 0.$$

This implies that

$$\|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p \geq |||Df(x_1, x_2, \cdots, x_n)||| \geq t_1 + \cdots + t_n$$

and so one of i satisfies $\|x_i\|^p \geq t_i$ in this case. Hence, for $q = \frac{1}{p}$, we have

$$\min\{N_{\mathcal{X}}(x_1, t_1^q), \cdots, N_{\mathcal{X}}(x_n, t_n^q)\} = 0$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$ and $t_1, \cdots, t_n > 0$. Therefore, in every case, the inequality

$$N_{\mathcal{Y}}(Df(x_1, \cdots, x_n), t_1 + \cdots + t_n) \geq \min\{N_{\mathcal{X}}(x_1, t_1^q), \cdots, N_{\mathcal{X}}(x_n, t_n^q)\}$$

holds. It means that f is a fuzzy q-almost additive-quadratic mapping.

Based on Theorem 2.4 and Remark, we obtain the following for a classical result in the framework of normed spaces.

Corollary 2.7. *Let $(\mathcal{X}, \|\cdot\|)$ and $(\mathcal{Y}, |||\cdot|||)$ be a normed space and a Banach space, respectively. Suppose that $n > 2$ is an integer and $f : \mathcal{X} \to \mathcal{Y}$ satisfies the inequality*

$$|||Df(x_1, x_2, \cdots, x_n)||| \leq \|x_1\|^p + \|x_2\|^p + \cdots + \|x_n\|^p$$

for all $x_1, x_2, \cdots, x_n \in \mathcal{X}$, where $p > 0$ and $p \neq 1, 2$. Then there is a unique additive-quadratic mapping $F : \mathcal{X} \to \mathcal{Y}$ such that

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{\|x\|^p}{(2^p-4)(n-1)} & (p > 2), \\ \frac{\|x\|^p}{(2^p-2)(n-1)} + \frac{\|x\|^p}{(4-2^p)(n-1)} & (1 < p < 2), \\ \frac{\|x\|^p}{(2-2^p)(n-1)} & (p < 1) \end{cases}$$

for all $x \in \mathcal{X}$.

Acknowledgments

The authors would like to thank the referees for giving useful suggestions which led to the improvement of this manuscript. This work was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012-0002410 and No. 2012R1A1A4A01002971).

References

[1] T. Aoki, On the stability of the linear mapping in Banach spaces, *J. Math. Soc. Japan*, **2** (1950), 64-66.

[2] I.-S. Chang, Y.H. Lee, Additive and quadratic type of functional equation and its fuzzy stability, *Results Math.*, To Appear.

[3] T. Bag, S.K. Samanta, Finite dimensional fuzzy normed linear spaces, *J. Fuzzy Math.*, **3** (2003), 687-705.

[4] S.C. Cheng, J.N. Mordeson, Fuzzy linear operator and fuzzy normed linear spaces, *Bull. Calcutta Math. Soc.*, **86** (1994), 429-436.

[5] D.H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. USA*, **27** (1941), 222-224.

[6] A.K. Katsaras, Fuzzy topological vector spaces II, *Fuzzy Sets and Systems*, **12** (1984), 143-154.

[7] I. Kramosil, J. Michalek, Fuzzy metric and statistical metric spaces, *Kybernetica*, **11** (1975), 326-334.

[8] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy almost quadratic functions, *Results Math.*, **52** (2008), 161-177.

[9] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy versions of Hyers-Ulam-Rassias theorem, *Fuzzy Sets and Systems*, **159** (2008), 720-729.

[10] A.K. Mirmostafaee, M. Mirzavaziri, M.S. Moslehian, Fuzzy stability of the Jensen functional equation, *Fuzzy Sets and Systems*, **159** (2008), 730-738

[11] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[12] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York (1968).