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FACIAL R-ACYCLIC EDGE-COLORINGS OF PLANE GRAPHS

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Abstract: An edge-coloring of a 2-connected plane graph G is a *facial r-acyclic edge-coloring* if every facial cycle C in G is colored with at least $\min\{|C|, r\}$ colors, in addition, no two face-adjacent edges (consecutive edges of a facial trail of some face) receive the same color. The minimum number of colors used in such a coloring of G is denoted by $a'_{fr}(G)$.

In this paper, we determine tight upper bounds for $a'_{fr}(G)$.

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1. Introduction

Our terminology and notation will be standard. The reader is referred to [2] for undefined terms. All graphs considered in this paper are simple, unless otherwise stated.

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We will adapt the convention that a graph is *planar* if it can be embedded in the plane without edges crossing, and *plane* if it is already embedded in the plane. Let the set of vertices and edges of a graph G be denoted by $V(G)$ and $E(G)$, respectively, or by V and E if G is known from the context. Let $\Delta(G)$ denote the maximum degree of G.

A k-edge-coloring of a graph $G = (V, E)$ is a mapping $c : E \to \{1, \dots, k\}$, in other words, an assignment of k colors to the edges of G. An edge-coloring is proper if adjacent edges receive distinct colors. The minimum number of colors used in a proper edge-coloring of G is denoted by $\chi'(G)$.

A proper edge-coloring of a graph G is *r-acyclic* if every cycle C contained in G is colored with at least $\min\{|C|, r\}$ colors. The *r-acyclic chromatic index* of a graph G, denoted by $a'_r(G)$, is the minimum number of colors used in an r-acyclic edge-coloring. Clearly, $a'_r(G) \geq \Delta(G)$.

If $r = 2$, then $\Delta(G) \leq a'_2(G) \leq \Delta(G)+1$ for every graph G, since a 2-acyclic edge-coloring coincides with a proper edge-coloring. A 3-acyclic edge-coloring is also known as an acyclic edge-coloring. The best already known upper bound for $a'_3(G)$ is due to Ndreca et al. [4], they proved that $a'_3(G) \leq 9.62\Delta(G)$ for any graph G. In view of the results mentioned above, we can see that $a'_r(G) = O(\Delta(G))$ for $r \leq 3$.

The r-acyclic chromatic index of G may not be linear in $\Delta(G)$ for $r \geq 4$. Greenhill and Pikhurko [3] proved that for $r \geq 4$ there are Δ-regular graphs G with $a'_r(G) \geq c\Delta^{\lfloor r/2 \rfloor}$, where c depends on r but is constant with respect to Δ.

For planar graphs G, Basavaraju et al. [1] proved that $a'_3(G) \leq \Delta(G) + 12$, Zhang et al. [7] showed that $a'_4(G) \leq 37\Delta(G)$. It is still open for which $r \geq 5$ is $a'_r(G)$ linear in $\Delta(G)$ for planar graphs.

The *facial r-acyclic edge-coloring* can be considered as a relaxation of the r-acyclic edge-coloring of plane graphs. We focus on facial cycles of plane graphs. This coloring has to satisfy the following two conditions:

1. every facial cycle C is colored with at least $\min\{|C|, r\}$ colors,

2. no two face-adjacent edges (consecutive edges of a facial trail of some face) receive the same color.

The *facial r-acyclic chromatic index* of a plane graph G, denoted by $a'_{fr}(G)$, is the minimum number of colors used in a facial r-acyclic edge-coloring.

In this paper, we determine tight upper bounds for $a'_{fr}(G)$.

2. Results

In this paper, we consider only 2-connected plane graphs since any non-2-connected plane graph contains a face whose boundary is not a cycle.

Clearly, for any 2-connected plane graph facial 1-acyclic edge-coloring coincides with facial 2-acyclic edge-coloring (since face-adjacent edges have different colors).

Lemma 1. *Let G be a 2-connected plane graph. Then $2 \leq a'_{f2}(G) \leq 4$. Moreover, these bounds are tight.*

Proof. The medial graph $M(G)$ of a plane graph G is obtained as follows. For each edge e of G insert a vertex $m(e)$ in $M(G)$. Join two vertices of $M(G)$ if the corresponding edges are face-adjacent. Clearly, $M(G)$ is also plane graph. Observe that every proper vertex-coloring of $M(G)$ corresponds to a facial 2-acyclic edge-coloring of G. By the Four Color Theorem, $M(G)$ has a proper vertex-coloring with at most 4 colors. Hence, $a'_{f2}(G) \leq 4$ for any 2-connected plane graph G.

Let C_n denote a cycle on n vertices. Clearly, $a'_{f2}(C_n) = 2$ for n even and $a'_{f2}(C_n) = 3$ for n odd. Plane graphs whose medial graphs have proper vertex-colorings with only 4 colors have facial 2-acyclic chromatic index 4; for example the wheel on six vertices (see Figure 1). □

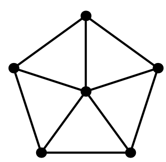

Figure 1: An example of a graph with no facial 2-acyclic edge-coloring using less than 4 colors.

In the following we will assume that $r \geq 3$.

The (geometric) *dual* $G^* = (V^*, E^*, F^*)$ of the plane graph $G = (V, E, F)$ can be defined as follows (see [2], pp. 252): Corresponding to each face f of G there is a vertex f^* of G^*, and corresponding to each edge e of G there is an edge e^* of G^*; two vertices f^* and g^* are joined by the edge e^* in G^* if and only if their corresponding faces f and g are separated by the edge e in G (an edge separates the faces incident with it).

Lemma 2. *Let G be a 2-connected plane graph and let G^* be its dual. Let $r \geq 3$ be an integer. Then $a'_{fr}(G) \leq \chi'(G^*)$.*

Proof. Every proper edge-coloring of the dual graph with $\chi'(G^*)$ colors induces a coloring of G with $\chi'(G^*)$ colors. In any such coloring of G, the edges bounding every face of G are colored distinctly, i.e. it is a facial $\chi'(G^*)$-acyclic edge-coloring. Therefore $a'_{fr}(G) \leq \chi'(G^*)$. □

Lemma 3. *Let G be a 2-connected plane graph and let G^* be its dual. Let $r \geq 3$ be an integer. Then $a'_{fr}(G) \leq \frac{3\Delta(G^*)}{2}$.*

Proof. The 2-connectedness of G implies that G^* contains no loop. Shannon [5] proved that every multigraph H has a proper edge-coloring with at most $\frac{3\Delta(H)}{2}$ colors. This result with Lemma 2 implies that $a'_{fr}(G) \leq \chi'(G^*) \leq \frac{3\Delta(G^*)}{2}$. □

Lemma 4. *Let G be a 3-connected plane graph and let G^* be its dual. Let $r \geq 3$ be an integer. Then $a'_{fr}(G) \leq \Delta(G^*) + 1$.*

Proof. Since G is 3-connected its dual is simple. By Vizing's theorem [6] $\chi'(G^*) \leq \Delta(G^*) + 1$. Using Lemma 2 we have $a'_{fr}(G) \leq \Delta(G^*) + 1$. □

2.1. Every Facial Cycle has Length at Most r

If there is no facial cycle in G of length at least $r+1$, then in any facial r-acyclic edge-coloring of G the edges bounding every face of G are colored distinctly. Consequently, any facial r-acyclic edge-coloring of G induces a proper edge-coloring of its dual G^*. Hence, $a'_{fr}(G) \geq \chi'(G^*)$. From this fact and Lemma 2 it follows that $a'_{fr}(G) = \chi'(G^*)$.

Theorem 5. *Let G be a 2-connected plane graph. Let $r \geq 3$ be an integer such that $r \geq |C|$ for any facial cycle C of G. Then $a'_{fr}(G) \leq \frac{3r}{2}$. Moreover, this bound is sharp.*

Proof. Observe that the maximum degree of G^* equals to the length of a longest facial cycle in G. Therefore, $\Delta(G^*) \leq r$. Consequently, $a'_{fr}(G) = \chi'(G^*) \leq \frac{3\Delta(G^*)}{2} \leq \frac{3r}{2}$.

To see that the bound is tight, it suffices to paste together two r-cycles on $\frac{r}{2} + 1$ vertices (if r is even). In this way we obtain a plane graph G which has three faces of size r. Thus, r different colors must appear on every face. Moreover, any two edges of G are incident with a common face, therefore no two edges are colored with the same color. The graph G has $\frac{3}{2}r$ edges, hence $a'_{fr}(G) = \frac{3}{2}r$.

If r is odd, then it is sufficient to paste together an r-cycle and an $(r-1)$-cycle on $\frac{r+1}{2}$ vertices. □

2.2. At Least one Cycle has Length Greater than r

For a face f of a plane graph G let $E(f)$ denote the set of edges incident with f.

Theorem 6. *Let G be a 2-connected plane graph. Let $r \geq 3$ be an integer. Then $a'_{fr}(G) \leq 2r - 1$.*

Proof. Every edge of G is incident with two faces, since G is 2-connected. Color the edges of G with colors $1, \dots, 2r-1$ as follows:

- Set $G_1 := G$ and $i := 1$.
- While $G_i \neq \emptyset$ choose an edge $e_i \in E(G_i)$, set $G_{i+1} := G_i \setminus \{e_i\}$ and increment i.
- Assume that e_i is incident with faces $f_{i,1}$ and $f_{i,2}$.
  - (i) If both $E(f_{i,1})$ and $E(f_{i,2})$ are colored with at least r colors, then we color e_i with the smallest color which does not occur on the edges face-adjacent to e_i.
  - (ii) If both $E(f_{i,1})$ and $E(f_{i,2})$ are colored with fewer than r colors, then we color the edge e_i with the smallest color which does not occur on $E(f_{i,1}) \cup E(f_{i,2})$.
  - (iii) If w.l.o.g. $E(f_{i,1})$ is colored with at least r colors and on $E(f_{i,2})$ appear fewer than r colors, then we color e_i with the smallest color which does not occur neither on $E(f_{i,2})$ nor on the edges face-adjacent to e_i.

Now we show that the above defined coloring is a facial r-acyclic edge-coloring which uses at most $2r - 1$ colors.

Let C be a facial cycle in G. If the length of C is at most r, then (ii) and (iii) ensure that all edges of C are colored distinctly. If the length of C is greater than r, then on the edges of C at least r colors occur. This is guaranteed by (ii) and (iii) as well.

Finally, we show that this coloring uses at most $2r - 1$ colors. When we apply the above defined coloring, new colors are used only in (ii) and (iii). In the case (ii), on the edges of $f_{i,1}$ and $f_{i,2}$ at most $2(r-1) = (2r-1)-1$ different colors occur. So there is a feasible color for e_i. In the case (iii), on the edges of $f_{i,2}$ and on the edges face-adjacent to e_i at most $(r-1)+2$ different colors occur. Since $r+1 < 2r-1$ for $r \geq 3$, there is a feasible color for e_i. □

2.3. Graphs with Few Big Faces

In this section we deal with plane graphs having property that the faces of a certain size are in a sense far from each other.

Two distinct faces f and g *touch* each other, if they share an edge (they are adjacent). Two distinct faces f and g *influence* each other, if they touch or there is a face h such that h touches both f and g.

Theorem 7. *Let G be a 3-connected plane graph in which no two faces of size at least k influence each other. Let $r \geq 3$ be an integer. Then $a'_{fr}(G) \leq r+k$.*

Proof. We say that a face is big if its size (the length of its facial cycle) is at least k, otherwise it is small.

Let $v_1v_2 \dots v_mv_1$, $m \geq k$, be a facial cycle of a big face f of G. We insert the diagonals v_1v_i, $i \in \{3, \dots, m-1\}$, to the face f. If we perform this extension on all big faces of G we obtain a graph H. Clearly, H has only small faces. Hence, the maximum degree of its dual H^* is not greater than $k-1$. The graph H^* is simple because H is 3-connected. By Vizing's theorem, H^* admits a proper edge-coloring with at most k colors. This coloring of H^* induces a coloring of H and also of G. On the small faces of G all edges have different colors, since each such face correspond to a small faces of H.

Now we recolor the edges incident with big faces with r colors (these r colors are different than the previous k ones) in such a way that face-adjacent edges receive different colors. Every small face of G is adjacent to at most one big face, otherwise G contains two big faces which influence each other. Consequently, we recolor at most one edge on every small face of G. In this way we obtain a facial r-acyclic edge-coloring of G which uses at most $r+k$ colors. □

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