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CONVOLUTION IDENTITIES FOR CENTRAL BINOMIAL NUMBERS

Roman Wituła[1 §], Edyta Hetmaniok[2],
Damian Słota[3], Natalia Gawrońska[4]
[1,2,3,4]Institute of Mathematics
Silesian University of Technology
Kaszubska 23, 44-100 Gliwice, POLAND

Abstract: Scope of this paper is the generalization of certain convolution-type identities for the central binomial numbers given by Lehmer. Some classical Lehmer's and Riordan-Stadler's formulae for these numbers in the new original expression and their application are also presented.

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1. Introduction

This paper is a follow-up of the research initiated by the authors in [8], inspired, first and foremost, by Lehmer's publication [4] where from many attractive results the new questions arose.

The paper is divided into four sections. In the second one, the basic known and some new convolution identities for central binomial numbers are presented. In the third section, an overview of certain Lehmer's formulae from [4] is given. And finally, in the fourth section, the new polynomial identities are proven in view of the discussion of certain Lehmer's identities.

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§Correspondence author

2. Basic Identities

In the derivation of convolution identities for central binomial coefficients $a_n := \binom{2n}{n}$, $n = 0, 1, 2, \ldots$ a basic role is played by the generating function of a_n (see [2, 5]; moreover, for interesting context see [1]):

$$f(x) := \frac{1}{\sqrt{1-4\,x}} = \sum_{n=0}^{\infty} a_n\, x^n, \qquad x \in \left[-\tfrac{1}{4}, \tfrac{1}{4}\right). \tag{1}$$

Let us present now a set of some known and probably new convolution-type identities for a_n.

Proposition 1. *We have*

$$\sum_{k=0}^{n} a_k\, a_{n-k} = 4^n, \tag{2}$$

$$\sum_{k=0}^{n} (-1)^k\, a_k\, a_{n-k} = \begin{cases} 0, & \text{when } n \text{ is odd;} \\ 2^n\, a_{n/2}, & \text{when } n \text{ is even,} \end{cases} \tag{3}$$

$$4^n \sum_{k=0}^{n} \left(-\tfrac{1}{4}\right)^k a_k = \sum_{k=0}^{\lfloor n/2 \rfloor} 4^k\, a_k\, a_{n-2k}, \tag{4}$$

and

$$\sum_{k=0}^{n} a_k\, a_{n-k}\, x^k\, y^{n-k} = \left(-\frac{4\,x\,y}{(x-y)^2}\right)^{1/2} \cdot \left(-\frac{8\,x\,y}{x+y}\right)^n \sum_{k=\lfloor (n+1)/2 \rfloor}^{\infty} a_k \binom{2\,k}{n} \left(\frac{x+y}{2\,(y-x)}\right)^{2k}, \tag{5}$$

whenever $|x+y| < |x-y|$.

Proof. (4). We have

$$\begin{aligned} f(x)\, f(4\,x^2) &= \big(f(x)\big)^2\, f(-x) \\ &= \Big(\sum_{n=0}^{\infty} (4\,x)^n\Big) \Big(\sum_{n=0}^{\infty} a_n\, (-x)^n\Big) = \sum_{n=0}^{\infty} 4^n \Big(\sum_{k=0}^{n} \left(-\tfrac{1}{4}\right)^k a_k\Big)\, x^n. \end{aligned}$$

On the other hand, we obtain

$$f(x)\,f(4\,x^2) \;=\; \Big(\sum_{n=0}^{\infty} a_n\,x^n\Big)\Big(\sum_{n=0}^{\infty} 4^n\,a_n\,x^{2n}\Big) \;=\; \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{\lfloor n/2\rfloor} 4^k\,a_k\,a_{n-2k}\Big)\,x^n.$$

For proving (5), first we obtain

$$f(\alpha\,x)\,f(\beta\,x) = \Big(\sum_{n=0}^{\infty} a_n\,(\alpha\,x)^n\Big)\Big(\sum_{n=0}^{\infty} a_n\,(\beta\,x)^n\Big)$$
$$= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} a_k\,a_{n-k}\alpha^k\,\beta^{n-k}\Big)\,x^n. \quad (6)$$

Next, we generate the formula

$$f(\alpha\,x)\,f(\beta\,x) = \Big(\frac{-4\,\alpha\,\beta}{(\alpha-\beta)^2}\Big)^{1/2} f\Big[\Big(\frac{4\,\alpha\,\beta}{\alpha-\beta}\Big(x-\frac{\alpha+\beta}{8\,\alpha\,\beta}\Big)\Big)^2\Big]$$
$$= \Big(\frac{-4\,\alpha\,\beta}{(\alpha-\beta)^2}\Big)^{1/2}\sum_{n=0}^{\infty} a_n\Big[\frac{4\,\alpha\,\beta}{\alpha-\beta}\Big(x-\frac{\alpha+\beta}{8\,\alpha\,\beta}\Big)\Big]^{2n} = \Big(\frac{-4\,\alpha\,\beta}{(\alpha-\beta)^2}\Big)^{1/2}$$
$$\cdot\sum_{n=0}^{\infty}\Big[\sum_{k=\lfloor (n+1)/2\rfloor}^{\infty} a_k\Big(\frac{4\,\alpha\,\beta}{\alpha-\beta}\Big)^{2k}\binom{2\,k}{n}\Big(-\frac{\alpha+\beta}{8\,\alpha\,\beta}\Big)^{2k-n}\Big]\,x^n,$$

which, by (6), implies formula (5). □

In this place we note that the Stirling formula leads to the following asymptotic expansion (see [3]):

$$\binom{2\,n}{n} = \frac{4^n}{\sqrt{\pi\,n}}\Big(1-\frac{1}{8\,n}+\frac{1}{128\,n^2}+O\big(\tfrac{1}{n^3}\big)\Big), \quad (7)$$

which implies that the series

$$\sum_{k=\lfloor (n+1)/2\rfloor}^{\infty} a_k\binom{2\,k}{n}\Big(-\frac{\alpha+\beta}{2\,(\alpha-\beta)}\Big)^{2k}$$

is convergent if and only if the condition $|\alpha-\beta| > |\alpha+\beta|$ is satisfied.

3. Lehmer's Convolution-Type Formula

We have also

$$\int f\left(\tfrac{x^2}{4}\right) dx = \arcsin x + C,$$

which, in view of (1), implies the Maclaurin expansion of $\arcsin x$:

$$\arcsin x = \sum_{n=0}^{\infty} \frac{1}{4^n} \binom{2\,n}{n} \frac{x^{2n+1}}{2\,n+1}, \qquad x \in [-1, 1]. \tag{8}$$

Hence, we derive the identity given below

$$\begin{aligned} x\, f\left(\tfrac{x^2}{4}\right) \arcsin x &= \Big(\sum_{n=0}^{\infty} 4^{-n}\, a_n\, x^{2n+1}\Big) \Big(\sum_{n=0}^{\infty} 4^{-n}\, a_n \frac{x^{2n+1}}{2\,n+1}\Big) \\ &= \sum_{n=0}^{\infty} \Big(4^{-n} \sum_{k=0}^{n} \frac{a_k\, a_{n-k}}{2\,k+1}\Big)\, x^{2(n+1)}. \end{aligned} \tag{9}$$

Moreover, we have the following Lehmer's formula (see [4, formula (15)]):

$$\sum_{n=1}^{\infty} \frac{(2\,x)^{2n}}{a_n} = \frac{x^2}{1-x^2} + \frac{x \arcsin x}{(1-x^2)^{3/2}}, \tag{10}$$

which implies

$$\begin{aligned} x\, f\left(\tfrac{x^2}{4}\right) \arcsin x &= (1-x^2) \sum_{n=1}^{\infty} \frac{(2\,x)^{2n}}{a_n} - x^2 \\ &= x^2 + \sum_{n=2}^{\infty} \Big(\frac{4}{a_n} - \frac{1}{a_{n-1}}\Big)\, 2^{2(n-1)}\, x^{2n}. \end{aligned}$$

Hence, by (9), the following identities can be obtained.

Proposition 2. *We get*

$$\sum_{k=0}^{n} \frac{a_k\, a_{n-k}}{2\,k+1} = 4^{2n} \Big(\frac{4}{a_{n+1}} - \frac{1}{a_n}\Big) = \frac{2^{4n+1}}{(n+1)\, a_{n+1}} = \frac{16^n}{(2\,n+1)\, a_n}, \tag{11}$$

$n = 1, 2, \ldots,$ *since we have the recurrence identity*

$$(n+1)\, a_{n+1} = (4\,n+2)\, a_n,$$

and

$$2\,x\,f\big(\tfrac{x^2}{4}\big)\ \arcsin x = \sum_{n=0}^{\infty} \frac{2^{2n+2}}{(n+1)\,a_{n+1}} = \sum_{n=1}^{\infty} \frac{(2\,x)^{2n}}{n\,a_n}, \tag{12}$$

which is formula (9) from [4].

Remark 3. Originally, in paper [4] the proof of formula (12) is given first and after that formula (10) is deduced.

In the next section we prove (11) in the other way. In fact we prove the functional identity (17) which is more general than identity (11).

4. Some New Identities

First we introduce some auxiliary functions

$$A_n(t) := \sum_{k=0}^{n} a_k\, a_{n-k}\, t^k, \qquad n = 0, 1, \ldots$$

It is easy to verify that

$$f(x)\,f(t\,x) = \sum_{n=0}^{\infty} A_n(t)\,x^n. \tag{13}$$

Since

$$\frac{d}{dt}\big(f(t\,x)\big) = \frac{2\,x}{(1-4\,t\,x)^{3/2}},$$

thus, from (13), we get

$$\frac{d}{dt}\big(f(x)\,f(t\,x)\big) = \sum_{n=0}^{\infty} \frac{d}{dt} A_n(t)\,x^n.$$

Yet, conversely, we have

$$\begin{aligned}\frac{d}{dt}\big(f(x)\,f(t\,x)\big) &= f(x)\,f(t\,x)\,\frac{2\,x}{1-4\,t\,x}\\ &= x\Big(\sum_{n=0}^{\infty} A_n(t)x^n\Big)\Big(\sum_{n=0}^{\infty} 2(4t)^n x^n\Big) = \sum_{n=0}^{\infty}\Big(2\sum_{k=0}^{n}(4t)^{n-k}A_k(t)\Big)x^{n+1}.\end{aligned}$$

Finally, we derive the following identity

$$\frac{d}{dt} A_{n+1}(t) = 2\sum_{k=0}^{n} (4\,t)^{n-k}\,A_k(t). \tag{14}$$

In [5], as well as in [6], the identity given below was proved

$$4^{-n} A_n\big((2x-1)^2\big) = \sum_{k=0}^{n} \binom{n}{k}^2 x^{2k} (1-x)^{2(n-k)}, \tag{15}$$

which is equivalent to the following one

$$A_n(t^2) = \sum_{k=0}^{n} \binom{n}{k}^2 (1+t)^{2k} (1-t)^{2(n-k)}. \tag{16}$$

Hence, for example, we obtain for $t=0$:

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2,$$

and next for $t = 2, 3$, respectively, we get the formulae

$$A_n(4) = \sum_{k=0}^{n} \binom{n}{k}^2 9^k \quad \text{and} \quad A_n(9) = 4^n \sum_{k=0}^{n} \binom{n}{k}^2 4^k.$$

Remark 4. The following identities come from [5]:

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k}^2 x^k &= \sum_{k=0}^{n} \binom{n+k}{2k} a_k\, x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} a_k\, x^k (1+x)^{n-2k} \\ &= 2^{-n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} a_{n-k} (1-x)^{2k} (1+x)^{n-2k}.\end{aligned}$$

By integrating (16) over t (from 1 to t) the following identity can be obtained, which in fact is the main result of this paper.

Theorem 5. *We have*

$$\begin{aligned}\sum_{k=0}^{n} \frac{a_k\, a_{n-k}}{2k+1} \big(t^{2k+1} - 1\big) &= \sum_{k=0}^{n} \binom{n}{k}^2 \int_1^t (1+\tau)^{2k} (1-\tau)^{2(n-k)}\, d\tau \\ &= \sum_{k=0}^{n} \binom{n}{k}^2 \Omega_{2k,2n}(t), \end{aligned} \tag{17}$$

where

$$\Omega_{k,n}(t) := \int_1^t (1+\tau)^k\,(1-\tau)^{n-k}\,d\tau = \frac{(1+\tau)^{k+1}(1-\tau)^{n-k}}{k+1}\Big|_1^t + \frac{n-k}{k+1}\int_1^t (1+\tau)^{k+1}(1-\tau)^{n-k-1}\,d\tau,$$

which implies the recurrence formula ($k+1 \le n$):

$$(k+1)\,\Omega_{k,n}(t) = (1+t)^{k+1}\,(1-t)^{n-k} + (n-k)\,\Omega_{k+1,n}(t), \tag{18}$$

and, respectively, the following explicit formula

$$(n+1)\binom{n}{k}\,\Omega_{k,n}(t) = -\sum_{i=0}^{k}\binom{n+1}{i}\,(1+t)^i\,(1-t)^{n+1-i}, \tag{19}$$

for $k = 0, 1, \ldots, n$. We note that

$$\Omega_{0,n}(t) = -\frac{(1-t)^{n+1}}{n+1}, \tag{20}$$

$$\Omega_{n,n}(t) = \frac{1}{n+1}\big((1+t)^{n+1} - 2^{n+1}\big). \tag{21}$$

Proof. The proof of (19) follows by simple induction. □

Corollary 6. *From (17) and (19), we obtain*

$$-2\sum_{k=0}^{n}\frac{a_k\,a_{n-k}}{2\,k+1} = \sum_{k=0}^{n}\binom{n}{k}^2\Omega_{2k,2n}(-1) = -\frac{2^{2n+1}}{2n+1}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2\,n}{2\,k}^{-1} = -\frac{2^{2n+1}}{2n+1}\,\frac{1}{a_n}\sum_{k=0}^{n}a_k\,a_{n-k} \overset{(2)}{=} -\frac{2^{2n+1}}{2n+1}\,\frac{1}{a_n}\,4^n = -\frac{2^{4n+1}}{(2\,n+1)\,a_n},$$

which is compatible with formula (11). More precisely, in this manner we obtain a new proof of relation (11) and, in consequence, we get Lehmer's identities (12) and (10).

Corollary 7. *From (17) and (19) for $t = 0$ we get*

$$\sum_{k=0}^{n} \frac{a_n\, a_{n-k}}{2\,k+1} = \frac{1}{2n+1} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n}{2k}^{-1} \sum_{i=0}^{2k} \binom{2n+1}{i}$$

$$= \frac{1}{2n+1}\, \frac{1}{a_n} \sum_{k=0}^{n} a_k\, a_{n-k} \sum_{i=0}^{2k} \binom{2n+1}{i}.$$

Then we can generate the following formula

$$2^{2n}\left(2^{2n}-1\right) = \sum_{k=0}^{n} a_k\, a_{n-k} \sum_{i=1}^{2k} \binom{2n+1}{i}.$$

Final Remark. The authors are in the course of preparing the second part of this paper concerning the convolution identities for inverses of the central binomial coefficients. The new paper bases mostly on the authors' former paper [7].

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