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SUMS OF THE RATIONAL POWERS OF ROOTS OF THE POLYNOMIALS

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Abstract: In this paper the completely elementary methods of generating the trigonometric identities are presented. These methods are based on the discussion of sums of the rational powers of roots of the polynomials. Among investigated identities there are proven the identities connected with roots of Kepler's polynomial and generalizations of the known Ramanujan's equalities.

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1. Introduction

The current paper is a supplement of the authors' previous work [13].

Similarly as in paper [13] the direct reason of preparing the present work was a trial of the pure algebraic look on the following excellent Ramanujan formulae

$$\left(\frac{1}{9}\right)^{1/3} - \left(\frac{2}{9}\right)^{1/3} + \left(\frac{4}{9}\right)^{1/3} = \left(\sqrt[3]{2} - 1\right)^{1/3}, \tag{1}$$

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$$\Big(\cos\frac{2\,\pi}{7}\Big)^{1/3}+\Big(\cos\frac{4\,\pi}{7}\Big)^{1/3}+\Big(\cos\frac{8\,\pi}{7}\Big)^{1/3}=\Big(\frac{5-3\sqrt[3]{7}}{2}\Big)^{1/3}, \tag{2}$$

$$\Big(\cos\frac{2\,\pi}{9}\Big)^{1/3}+\Big(\cos\frac{4\,\pi}{9}\Big)^{1/3}+\Big(\cos\frac{8\,\pi}{9}\Big)^{1/3}=\Big(\frac{3\sqrt[3]{9}-6}{2}\Big)^{1/3}, \tag{3}$$

together with the identities inspired by formulae (1)–(2) and derived by the authors in recent papers [9, 10, 11], like for example:

$$\begin{aligned}
&\sqrt[3]{\frac{\cos(\beta)}{\cos(4\beta)}}\,\big(2\,\cos(\beta)\big)^{n}+\sqrt[3]{\frac{\cos(2\beta)}{\cos(\beta)}}\,\big(2\,\cos(2\beta)\big)^{n}+\\
&+\sqrt[3]{\frac{\cos(4\beta)}{\cos(2\beta)}}\,\big(2\,\cos(4\beta)\big)^{n}=-\Bigg(\sqrt[3]{\frac{\cos(2\beta)}{\cos(\beta)}}\,\big(2\,\cos(\beta)\big)^{n+1}+\\
&+\sqrt[3]{\frac{\cos(4\beta)}{\cos(2\beta)}}\,\big(2\,\cos(2\beta)\big)^{n+1}+\sqrt[3]{\frac{\cos(\beta)}{\cos(4\beta)}}\,\big(2\,\cos(4\beta)\big)^{n+1}\Bigg)= \qquad (4)\\
&=-\Bigg(\sqrt[3]{2\,\cos(2\beta)\,\big(2\,\cos(\beta)\big)^{3n+2}}+\sqrt[3]{2\,\cos(4\,\beta)\,\big(2\,\cos(2\,\beta)\big)^{3n+2}}+\\
&+\sqrt[3]{2\,\cos(\beta)\,\big(2\,\cos(4\beta)\big)^{3n+2}}\Bigg)=\sqrt[3]{9}\,\Psi_n,
\end{aligned}$$

where $\beta=\frac{2\pi}{9}$, $\Psi_0=-1$, $\Psi_1=1$, $\Psi_2=-4$ and

$$\Psi_{n+3}-3\,\Psi_{n+1}+\Psi_n=0, \qquad n\in\mathbb{Z}. \tag{5}$$

It is obvious that each of relations (4) is the Binet formula for the respective recurrence relation (5). Problem of deriving formulae of type (4) reduces in practice to decomposing the respective characteristic polynomials and finding the initial values. *Especially the latter is a fundamental problem.* In the current paper we present some technical tools enabling to generate the formulae of type (4) in specifically elementary way. Results of using these tools will be presented on the particular examples and seem to be equally attractive.

2. Basic Theorems

We present now our crucial technical results. Proofs of all theorems are given in paper [13].

Let $f\in\mathbb{C}[\mathbb{X}]$ be such that

$$f(\mathbb{X})=\mathbb{X}^3+p\,\mathbb{X}^2+q\,\mathbb{X}+r=(\mathbb{X}-\xi_1)(\mathbb{X}-\xi_2)(\mathbb{X}-\xi_3).$$

Theorem 1 (simple version). *If $p \in 3\sqrt[3]{r}$, then the values α, β, γ of complex roots $\sqrt[3]{\xi_1}$, $\sqrt[3]{\xi_2}$, $\sqrt[3]{\xi_3}$, respectively, can be chosen in such a way that*

$$\alpha + \beta + \gamma = 0.$$

In the sequel, if ξ_1, ξ_2, ξ_3 are the real numbers, then we can assume that also α, β, γ are the real numbers.

Let us present now the following version of Theorem 1 which holds for more general family of cubic polynomials.

Theorem 2 (general version). *If $3q \neq p^2$ or if $3q = p^2$ and $27r = p^3$, then there exists $c \in \mathbb{C}$ such that, for the respective $\alpha \in \sqrt[3]{\xi_1 + c}$, $\beta \in \sqrt[3]{\xi_2 + c}$ and $\gamma \in \sqrt[3]{\xi_3 + c}$, we have*

$$\alpha + \beta + \gamma = 0.$$

If ξ_1, ξ_2, $\xi_3 \in \mathbb{R}$ and $3\,q \neq p^2$, then the above equality holds for

$$c = \frac{p^3 - 27\,r}{9(p^2 - 3q)} \tag{6}$$

(all roots should be real). Whereas, if $3q = p^2$ and $27r \neq p^3$, then such constant $c \in \mathbb{C}$ does not exist.

Let $d \in \mathbb{R}$. If ξ_1, ξ_2, $\xi_3 \in \mathbb{R}$, then the following formula holds true

$$\sqrt[3]{\xi_1 + d} + \sqrt[3]{\xi_2 + d} + \sqrt[3]{\xi_3 + d} = \sqrt[3]{-P - 6\sqrt[3]{R} - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S} + \sqrt{\mathcal{T}}} + \sqrt[3]{\mathcal{S} - \sqrt{\mathcal{T}}}\right)},$$

where

$$\mathcal{S} := P\,Q + 6\,Q\,\sqrt[3]{R} + 6\,P\,\sqrt[3]{R^2} + 9\,R,$$
$$\mathcal{T} := P^2\,Q^2 - 4\,Q^3 - 4\,P^3\,R + 18\,P\,Q\,R - 27\,R^2,$$

$$\prod_{k=1}^{3}(\mathbb{X} - \xi_k - d) = \mathbb{X}^3 + P\,\mathbb{X}^2 + Q\,\mathbb{X} + R,$$

and

$$P = p - 3\,d,$$
$$Q = q + 3\,d^2 - 2\,d\,p,$$
$$R = r - d^3 + p\,d^2 - q\,d.$$

Additionally, we have

$$\sqrt[3]{\xi_1^2} + \sqrt[3]{\xi_2^2} + \sqrt[3]{\xi_3^2} = \sqrt[3]{-p_2 - 6\sqrt[3]{r_2} - \frac{3}{\sqrt[3]{2}}\left(\sqrt[3]{\mathcal{S}_2 + \sqrt{\mathcal{T}_2}} + \sqrt[3]{\mathcal{S}_2 - \sqrt{\mathcal{T}_2}}\right)}, \quad (7)$$

where

$$\begin{aligned} \mathcal{S}_2 &:= p_2\, q_2 + 6\, q_2 \sqrt[3]{r_2} + 6\, p_2 \sqrt[3]{r_2^2} + 9\, r_2, \\ \mathcal{T}_2 &:= p_2^2\, q_2^2 - 4\, q_2^3 - 4\, p_2^3\, r_2 + 18\, p_2\, q_2\, r_2 - 27\, r_2^2, \\ p_2 &= 2\, q - p^2, \\ q_2 &= q^2 - 2\, p\, r, \\ r_2 &= -r^2. \end{aligned}$$

In the next theorem we present the recurrence sequence which is important for the further discussion.

Theorem 3 (special case of the Newton-Girard formula)**.** *If for the certain values* $\alpha \in \xi_1^{1/3}$, $\beta \in \xi_2^{1/3}$, $\gamma \in \xi_3^{1/3}$ *we have*

$$\alpha + \beta + \gamma = 0$$

and we set

$$S_n := \alpha^n + \beta^n + \gamma^n, \qquad n = 0, 1, 2, ...,$$

then from the Newton-Girard formula (see Remark 1 in [10]) we get

$$S_{n+3} = \alpha\, \beta\, \gamma\, S_n + \frac{1}{2}\, S_2\, S_{n+1}, \qquad n = 0, 1, 2, \ldots$$

Here we have $\alpha\, \beta\, \gamma \in -\sqrt[3]{r}$, *whereas* S_2 *belongs to the right side of (7) for the respective values of all five complex roots appearing in this root of the third order (both* $\sqrt{\mathcal{T}}$ *are chosen with the same value).*

In the sequel, if ξ_1, ξ_2, $\xi_3 \in \mathbb{R}$ *and* $\mathcal{T}_2 \geq 0$, *then we can assume that all the above roots are real.*

Remark 4. More information about generalizations of sequence S_n, its importance and its properties can be found in paper [3].

3. Application of Fundamental Theorems

Similarly as in paper [13] we want to give many specific examples of applications of the indicated theorems. In other words, we present several examples of polynomials with the known prime factors decompositions to which our fundamental theorems will be applied. It will lead us to generate new original equalities and identities, of trigonometric nature especially. Collection of these examples completes the appropriate collection given in paper [13]. It will be also used in our activity within the framework of OEIS–Wiki. Henceforward we will denote

$$\boldsymbol{\alpha} = \frac{2\pi}{7} \qquad \text{and} \qquad \boldsymbol{\beta} = \frac{2\pi}{9}.$$

1^0 We have the following decomposition

$$\mathbb{X}^3 + \mathbb{X}^2 - 2\,\mathbb{X} - 1 = \prod_{k=0}^{2} \big(\mathbb{X} - 2\cos(2^k\boldsymbol{\alpha})\big). \tag{8}$$

Hence we also receive

$$\mathbb{X}^3 - 3\,\mathbb{X}^2 - 46\,\mathbb{X} - 1 = \prod_{k=0}^{2} \Big(\mathbb{X} - \frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+1}\boldsymbol{\alpha})}\big(2\,\cos(2^k\boldsymbol{\alpha})\big)^3\Big) \tag{9}$$

which, by Theorem 1, implies

$$\sum_{k=0}^{2} \sqrt[3]{\frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+1}\boldsymbol{\alpha})}}\,\cos(2^k\boldsymbol{\alpha}) = 0. \tag{10}$$

Furthermore, from Theorem 3 it can be generated the formula

$$S_{n+3} = \sqrt[3]{49}\,S_{n+1} + S_n, \tag{11}$$

where for $n = 0, 1, 2, ...$:

$$S_n := \sum_{k=0}^{2}\left(\sqrt[3]{\frac{\cos(2^k\boldsymbol{\alpha})}{\cos(2^{k+1}\boldsymbol{\alpha})}}\,2\cos(2^k\boldsymbol{\alpha})\right)^n = \left(\sqrt[3]{\frac{\cos(\frac{\pi}{7})}{\cos(\frac{2\pi}{7})}}\,2\cos\left(\frac{\pi}{7}\right)\right)^n$$
$$+\left(-\sqrt[3]{\frac{\cos(\frac{2\pi}{7})}{\cos(\frac{3\pi}{7})}}\,2\cos\left(\frac{2\pi}{7}\right)\right)^n + \left(-\sqrt[3]{\frac{\cos(\frac{3\pi}{7})}{\cos(\frac{\pi}{7})}}\,2\cos\left(\frac{3\pi}{7}\right)\right)^n.$$

One can also obtain the following decomposition (which can be proven by induction):

$$S_n = \widetilde{a}_n + \widetilde{b}_n\sqrt[3]{7} + \widetilde{c}_n\sqrt[3]{49}, \qquad n = 0, 1, 2, ...,$$

where, by (11), we have

$$\widetilde{a}_{n+3} = \widetilde{a}_n + 7\,\widetilde{b}_{n+1}, \quad \widetilde{b}_{n+3} = 7\,\widetilde{c}_{n+1} + \widetilde{b}_n, \quad \widetilde{c}_{n+3} = \widetilde{a}_{n+1} + \widetilde{c}_n,$$

$$\begin{array}{llllll} \widetilde{a}_0 = 3, & \widetilde{a}_1 = 0, & \widetilde{a}_2 = 0, & \widetilde{a}_3 = 3, & \widetilde{a}_4 = 0, & \widetilde{a}_5 = 0, \\ \widetilde{b}_0 = 0, & \widetilde{b}_1 = 0, & \widetilde{b}_2 = 0, & \widetilde{b}_3 = 0, & \widetilde{b}_4 = 14, & \widetilde{b}_5 = 0, \\ \widetilde{c}_0 = 0, & \widetilde{c}_1 = 0, & \widetilde{c}_2 = 2, & \widetilde{c}_3 = 0, & \widetilde{c}_4 = 0, & \widetilde{c}_5 = 5, \end{array}$$

which implies, after simple calculation, the following recurrence relation

$$A_{n+9} - 3A_{n+6} - 46A_{n+3} - A_n = 0, \tag{12}$$

for every $A \in \{a, b, c\}$ and $n \in \mathbb{N}_0$. Hence and from the respective initial conditions we deduce that

$$\begin{array}{lll} a_n \neq 0 & \Longleftrightarrow & 3|n, \\ b_n \neq 0 & \Longleftrightarrow & n \geq 4 \textit{ and } 3|(n+2), \\ c_n \neq 0 & \Longleftrightarrow & 3|(n+1), \end{array}$$

and

$$S_{3n} = a_{3n}, \qquad S_{3n+1} = b_{3n+1}\sqrt[3]{7}, \qquad S_{3n+2} = c_{3n+2}\sqrt[3]{49}.$$

Let us notice that relation (12) holds also for $A_n := S_n$.

2^0 We have decomposition of the form (see [7]):

$$\mathbb{X}^3 - \sqrt{7}\,\mathbb{X}^2 + \sqrt{7} = \prod_{k=0}^{2} \big(\mathbb{X} - 2\sin(2^k \boldsymbol{\alpha})\big). \tag{13}$$

Thus, the equality $\mathbb{X}^3 - \sqrt{7}\,\mathbb{X}^2 + \sqrt{7} = 0$ implies that

$$\mathbb{X}^3 = \sqrt{7}\big(\mathbb{X}^2 - 1\big) \tag{14}$$

which concludes

$$\sqrt[3]{7} = \sum_{k=0}^{2} \sqrt[3]{4\sin^2(2^k\boldsymbol{\alpha}) - 1} = \sum_{k=0}^{2} \sqrt[3]{1 - 2\cos(2^k\boldsymbol{\alpha})}. \tag{15}$$

On the other hand, from (14) we obtain

$$8\sin^3(2^k\boldsymbol{\alpha}) = \sqrt{7}(3\sin^2(2^k\boldsymbol{\alpha}) - \cos^2(2^k\boldsymbol{\alpha})),$$
$$8\sin(2^k\boldsymbol{\alpha}) = \sqrt{7}(3 - \cot^2(2^k\boldsymbol{\alpha})),$$

for every $k = 0, 1, 2$, which implies

$$4\sum_{k=0}^{2} 2\sin(2^k\boldsymbol{\alpha}) = 4\sqrt{7} = \sqrt{7}(9 - \sum_{k=0}^{2}\cot^2(2^k\boldsymbol{\alpha})),$$

i.e.

$$5 = \sum_{k=0}^{2}\cot^2(2^k\boldsymbol{\alpha}). \tag{16}$$

Moreover, from (13) and (6) we get

$$\sum_{k=0}^{2}\sqrt[3]{9\sqrt{7}\sin(2^k\boldsymbol{\alpha}) - 17} = 0. \tag{17}$$

3^0 From (13) we receive

$$\mathbb{X}^3 - 7\,\mathbb{X} - 7 = \prod_{k=0}^{2}\Big(\mathbb{X} + \frac{\sqrt{7}}{2}\csc(2^k\boldsymbol{\alpha})\Big) \tag{18}$$

which implies by (6):

$$\sum_{k=0}^{2}\sqrt[3]{1 - \frac{\sqrt{7}}{2}\csc(2^k\boldsymbol{\alpha})} = 0.$$

4^0 Additionally, from (13) we can obtain the Johannes Kepler polynomial of the form

$$\mathbb{X}^3 - 7\,\mathbb{X}^2 + 14\,\mathbb{X} - 7 = \prod_{k=0}^{2}\big(\mathbb{X} - 4\sin^2(2^k\boldsymbol{\alpha})\big) \tag{19}$$

(numbers $4\sin^2(2^k\boldsymbol{\alpha})$, $k = 0, 1, 2$, are equal to the squares of chords A_1A_2, A_1A_3 and A_1A_4 of the regular heptagon $A_1A_2...A_7$ inscribed into the unit circle - see for example [1, 2]).

Next, from the equality $\mathbb{X}^3 - 7\,\mathbb{X}^2 + 14\,\mathbb{X} - 7 = 0$ we receive

$$\mathbb{X}^3 = 7\,(\mathbb{X} - 1)^2,$$

which leads to (compare with equality (15)):

$$\sqrt[3]{49} = \sum_{k=0}^{2}\sqrt[3]{\big(4\sin^2(2^k\boldsymbol{\alpha}) - 1\big)^2} = \sum_{k=0}^{2}\sqrt[3]{\big(1 - 2\cos(2^k\boldsymbol{\alpha})\big)^2}. \tag{20}$$

Remark 5. We note that (compare with (19)):

$$\mathbb{X}^3 - 7\,\mathbb{X}^2 + 14\,\mathbb{X} - \mathbf{8} = (\mathbb{X}-1)(\mathbb{X}-2)(\mathbb{X}-4),$$

which implies by (6):

$$\sqrt[3]{1-\frac{127}{63}} + \sqrt[3]{2-\frac{127}{63}} + \sqrt[3]{4-\frac{127}{63}} = 0.$$

But the terms are indeed equal to $-\frac{4}{\sqrt[3]{63}}$, $-\frac{1}{\sqrt[3]{63}}$ and $\frac{5}{\sqrt[3]{63}}$, respectively. So, it is not the interesting case.

Remark 6. From (15) and (20) we get

$$\prod_{k=0}^{2}\Big(\mathbb{X} - \sqrt[3]{1-2\cos(2^k\boldsymbol{\alpha})}\Big) = \mathbb{X}^3 - \sqrt[3]{7}\mathbb{X}^2 + 1,$$

since $(a+b+c)^2 = a^2+b^2+c^2+2(ab+ac+bc)$ and

$$(a+b+c)^2 = a^2+b^2+c^2, \qquad abc \overset{(8)}{=} -1,$$

for $a = \sqrt[3]{1-2\cos(\boldsymbol{\alpha})}$, $b = \sqrt[3]{1-2\cos(2\boldsymbol{\alpha})}$, $c = \sqrt[3]{1-2\cos(4\boldsymbol{\alpha})}$. Moreover we have

$$4 = \sum_{k=0}^{2}\Big(1-2\cos(2^k\boldsymbol{\alpha})\Big).$$

The following formula can be easily derived

$$\sum_{k=0}^{2}\Big(1-2\cos(2^k\boldsymbol{\alpha})\Big)^{n/3} = a_n + b_n\sqrt[3]{7} + c_n\sqrt[3]{49},$$

for $n = 1,\ldots,100$, where $a_n, b_n, c_n \in \mathbb{Z}$ and

$$\begin{array}{lll} a_n \neq 0 & \text{only for} & 3|n, \\ b_n \neq 0 & \text{only when} & 3|(n+2), \\ c_n \neq 0 & \text{only when} & 3|(n+1), \end{array}$$

(numerically tested).

5^0 We have decomposition of the form (see [7]):

$$\mathbb{X}^3 - 3\,\mathbb{X} + \sqrt{3} = \big(\mathbb{X} - 2\sin(\boldsymbol{\beta})\big)\big(\mathbb{X} + 2\sin(2\boldsymbol{\beta})\big)\big(\mathbb{X} - 2\sin(4\boldsymbol{\beta})\big). \tag{21}$$

Hence, we obtain by (6):

$$\sqrt[3]{6\sin(\boldsymbol{\beta})-\sqrt{3}}+\sqrt[3]{6\sin(4\boldsymbol{\beta})-\sqrt{3}}=\sqrt[3]{6\sin(2\boldsymbol{\beta})+\sqrt{3}}. \tag{22}$$

Thus, by Theorems 2 and 3, the formula can be genareted

$$S_{n+3}=\sqrt[3]{3}\,S_{n+1}-\frac{\sqrt{3}}{3}\,S_n,\quad n=0,1,2,...,$$

where

$$S_n:=\left(2\,\sin(\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3}+\left(-2\,\sin(2\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3} +\left(2\,\sin(4\boldsymbol{\beta})-\frac{\sqrt{3}}{3}\right)^{n/3}.$$

In the current paper the further discussion of this sequence will be omitted.

Furthermore, from the equality $\mathbb{X}^3-3\,\mathbb{X}+\sqrt{3}=0$ we get

$$\left(\mathbb{X}^2-\frac{3}{2}\right)^2=\frac{9}{4}-\sqrt{3}\,\mathbb{X},$$

which implies

$$\sqrt{1-\frac{8}{9}\sqrt{3}\sin(\boldsymbol{\beta})}\;+\;\sqrt{1+\frac{8}{9}\sqrt{3}\sin(2\boldsymbol{\beta})}\;=\;1\;+\;\sqrt{1-\frac{8}{9}\sqrt{3}\sin(4\boldsymbol{\beta})} \tag{23}$$

(compare with the equality

$$\sqrt{1-\frac{8}{9}\cos(\boldsymbol{\beta})}+\sqrt{1-\frac{8}{9}\cos(4\boldsymbol{\beta})}=1+\sqrt{1-\frac{8}{9}\cos(2\boldsymbol{\beta})}, \tag{24}$$

given in [13]).

4. Generalizations

All the previously discussed problems can be easily generalized to the polynomials of degree higher than 3. For example, the following result holds true.

Theorem 7. *Let p, q, $r \in \mathbb{C}$ and suppose that $\left(\frac{p}{4}\right)^4 = r$.*
If we set

$$f(\mathbb{X}) = \mathbb{X}^4 + p\,\mathbb{X}^3 + \frac{3}{8}p^2\mathbb{X}^2 + q\,\mathbb{X} + r = \prod_{k=1}^{4}(\mathbb{X} - \xi_k),$$

then there exist $\alpha_k \in \sqrt[4]{\xi_k}$, $k = 1, ..., 4$, such that

$$\sum_{k=1}^{4} \alpha_k = 0.$$

Proof. We propose the following sketch of proof. We note that

$$f(\mathbb{X}) = 0 \qquad \Longleftrightarrow \qquad \left(\mathbb{X} + \frac{p}{4}\right)^4 + \left(q - \frac{p^3}{16}\right)\mathbb{X} = 0,$$

which implies

$$\xi_k + \frac{p}{4} \in \sqrt[4]{\left(\frac{p^3}{16} - q\right)\xi_k}, \qquad k = 1, \ldots, 4,$$

i.e.

$$0 = p - p = p + \sum_{k=1}^{4} \xi_k \in \sum_{k=1}^{4} \sqrt[4]{\left(\frac{p^3}{16} - q\right)\xi_k},$$

and

$$0 \in \sum_{k=1}^{4} \sqrt[4]{\xi_k}.$$

□

Example 8. Let us set

$$\begin{aligned} f_3(\mathbb{X}) &= \mathbb{X}^4 + \left(\frac{5}{3}\mathbb{X} - \frac{8}{3}\right)^3 \\ &= (\mathbb{X} - 1)(\mathbb{X} + 8)\left(\mathbb{X} - \frac{8}{27}\big(4 - i\sqrt{11}\big)\right)\left(\mathbb{X} - \frac{8}{27}\big(4 + i\sqrt{11}\big)\right). \end{aligned}$$

Hence we receive

$$f_3(\mathbb{X}) = 0 \qquad \Longleftrightarrow \qquad \mathbb{X}^4 = \left(\frac{8}{3} - \frac{5}{3}\mathbb{X}\right)^3,$$

which implies the relations

$$\Big(\frac{1}{2}(7+i5\sqrt{11})\Big)^3+\Big(\frac{1}{2}(7-i5\sqrt{11})\Big)^3=\big(4-i\sqrt{11}\big)^4+\big(4+i\sqrt{11}\big)^4$$

and

$$-\Big(\frac{5}{3}\Big)^3=\alpha+\beta+\gamma+\delta,$$

for certain complex values $\alpha\in\sqrt[4]{1}$, $\beta\in 8\sqrt[4]{1}$, $\gamma\in\frac{4}{27}\sqrt[4]{2(7+i5\sqrt{11})^3}$ and $\delta\in\frac{4}{27}\sqrt[4]{2\,(7-i\,5\sqrt{11})^3}$.

Example 9. We have (see [7]):

$$\begin{aligned}
f_4(\mathbb{X})&=\mathbb{X}^4-\mathbb{X}^3-4\,\mathbb{X}^2-4\,\mathbb{X}+1\\
&=\Big[\Big(\mathbb{X}-2\cos\frac{2\pi}{15}\Big)\Big(\mathbb{X}-2\cos\frac{8\pi}{15}\Big)\Big]\Big[\Big(\mathbb{X}-2\cos\frac{4\pi}{15}\Big)\Big(\mathbb{X}-2\cos\frac{14\pi}{15}\Big)\Big]\\
&=\Big(\mathbb{X}^2-2\big(\cos\frac{\pi}{5}\big)\mathbb{X}+2\big(\cos\frac{\pi}{5}-1\big)\Big)\Big(\mathbb{X}^2+\big(2\cos\frac{\pi}{5}-1\big)\mathbb{X}-1-2\cos\frac{\pi}{5}\Big)\\
&=(\mathbb{X}^2-\varphi\,\mathbb{X}+\varphi-2)(\mathbb{X}^2+(\varphi-1)\,\mathbb{X}-\varphi-1),
\end{aligned}$$

where $\varphi:=2\cos\frac{\pi}{5}=\frac{1+\sqrt{5}}{2}$ is the golden ratio and

$$f_4(\mathbb{X})=0\quad\Leftrightarrow\quad\Big(\mathbb{X}^2-\frac{1}{2}\mathbb{X}-\frac{17}{8}\Big)^2=\Big(\frac{15}{8}\Big)^2\Big(1-\frac{8}{15}\mathbb{X}\Big).$$

Thus, we get

$$\begin{aligned}
&\sqrt{1-\frac{16}{15}\cos\frac{2\pi}{15}}+\sqrt{1-\frac{16}{15}\cos\frac{14\pi}{15}}\\
&\qquad=\sqrt{1-\frac{16}{15}\cos\frac{4\pi}{15}}+\sqrt{1-\frac{16}{15}\cos\frac{8\pi}{15}}\\
&\qquad=\frac{4}{15}\Big(\cos\frac{2\pi}{15}-\cos\frac{8\pi}{15}+3\cos\frac{4\pi}{15}-3\cos\frac{14\pi}{15}\Big)\\
&\qquad=\frac{2}{15}\left(\sqrt{\frac{15-3\sqrt{5}}{2}}+3\sqrt{\frac{15+3\sqrt{5}}{2}}\right).
\end{aligned}\tag{25}$$

We note that

$$\cos\frac{2\pi}{15}=\frac{1}{4}\left(\frac{1+\sqrt{5}}{2}+\sqrt{\frac{15-3\sqrt{5}}{2}}\right),$$

$$\cos\frac{4\pi}{15} = \frac{1}{4}\left(\frac{1-\sqrt{5}}{2} + \sqrt{\frac{15+3\sqrt{5}}{2}}\right),$$

$$\cos\frac{8\pi}{15} = \frac{1}{4}\left(\frac{1+\sqrt{5}}{2} - \sqrt{\frac{15-3\sqrt{5}}{2}}\right),$$

$$\cos\frac{14\pi}{15} = \frac{1}{4}\left(\frac{1-\sqrt{5}}{2} - \sqrt{\frac{15+3\sqrt{5}}{2}}\right).$$

Final Remark

Our colleague from Russia Sergey Markelov informed us in private correspondence about deriving several new formulae of the type discussed in the current paper. Among others, he found the following ones

$$\sqrt[3]{\sin(\boldsymbol{\alpha})} + \sqrt[3]{\sin(2\boldsymbol{\alpha})} + \sqrt[3]{\sin(4\boldsymbol{\alpha})}$$
$$= \sqrt[3]{\frac{\sqrt[3]{7}}{3} - 2 + \sqrt[3]{3\sqrt[3]{7} - 4} + \sqrt[3]{3\sqrt[3]{7} - 5}}\ \sqrt[3]{\frac{3}{2}}\sqrt[6]{7}.$$

More information can be found in http://ru-math.livejournal.com/79774.html – the Russian internet forum where sums of this type are considered and where some efforts are made to present them in the general context.

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