International Journal of Pure and Applied Mathematics
Volume 85 No. 1 2013, 23-31
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i1.3
ijpam.eu

THREE STEP ITERATION SCHEME FOR M-CONTRACTIVE CONDITION IN BANACH SPACE

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Abstract: Our aim in this paper is to prove a strong convergence theorem by three-step iteration scheme with respect to contractive condition in a Banach space. Our results obtained in this paper is new extension as well as refinement of previous known results.

AMS Subject Classification: 47H05, 47H10, 49M05
Key Words: three-step iteration scheme, contractive condition, Banach space, strong convergence, common fixed point

1. Introduction

Let X be a Banach space, K be a nonempty closed convex subset of X and $T : K \to K$ be a self mapping of K. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers and $F(T) \neq \phi$ i.e., $F(T) = \{x \in K : Tx = x\}$. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in real number $[0, 1]$.

Received: August 27, 2012

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The Picard and Mann [7] iteration schemes for a mapping $T : K \to K$ are defined by

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = Tx_n \end{cases} \tag{1}$$

and

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \in \mathbb{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is in $(0,1)$.

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N} \end{cases} \tag{3}$$

is known as the Ishikawa iteration process [5].

In this paper, we define a three-step iteration scheme as follows:

Let K be a nonempty, closed and convex subset of a Banach space X. Suppose $T : K \to K$ be two nonlinear operators and $\{x_n\}_{n=0}^{\infty}$ be the sequence in $[0,1]$.

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n Tz_n \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \\ z_n = (1-\gamma_n)y_n + \gamma_n Ty_n, n \in \mathbb{N}, \end{cases} \tag{4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$.

We note that, when $\beta_n = 0$, then the iteration scheme (4) reduce to the iteration given by Thianwan [11],

$$\begin{cases} x_1 = x_0 \in K, \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n Ty_n \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, n \in \mathbb{N}, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

When $\beta_n = 0$ the iteration (5) reduces to the Mann iteration (2).

The following theorem was obtained by the Zamfirescu [12].

Theorem 1. *Let* (X, d) *be complete metric space and* $T : X \to X$ *a map for which there exist the real number a, b, and c satisfying* $0 < a < 1$, $b, c \in (0, 1/2)$ *such that for each pair x,y in X, at least one the following is true:*
(z_1) $d(Tx, Ty) \le ad(x, y)$
(z_2) $d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)]$,
(z_3) $d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)]$.
Then T *is a Picard operator* $x_{n+1} = Tx_n$.

An operator T satisfying the contractive condition (z_1),(z_2) and (z_3) in Theorem 1, above, is called a Zamfirescu operator.

2. Preliminaries

In recent years, Mann and Ishikawa iteration schemes have been studied extensively by many authors to solve one parameter nonlinear operator equations as well as variational inequalities in Hilbert space and Banach spaces. Many authors study Rhoades ([9], [10]) emplyed the Zamfirescu condition to prove several convergence results for Mann and Ishikawa iteration process in a uniformly convex Banach space. The results of Rhoades ([9], [10]) were also extended by Berinde [3] to an arbitrary Banach space for the same fixed point iteration processes. Recently, many research papers have been published on the iterative approximation of fixed points for contractive type, quasi-contractive and Zamfirescu operators using several iteration schemes, for example see ([3], [4], [6], [8].

Suppose K be a nonempty subset of a metric space (X, d). A mapping $T : K \to K$ is said to be contractive if there exists $0 \le \alpha < 1$ such that for all $x, y \in X$,

$$d(Tx, Ty) \le \alpha d(x, y). \tag{6}$$

The constant α is called the contractivity coefficient. It is well known that Banach's fixed point theorem asserts that if $K = X$, T is contractive and (X, d) is complete, then T has a unique fixed point p in X, and for any $x_0 \in X$ the sequence $\{T^n(x_0)\}$ converges to p. This result has been extended by several authors to some classes of mappings by changing the contractive condition (6). It is well known that the Banach contraction principle [1] is a very useful and classical tool in nonlinear analysis.

Our purpose in this paper to found common fixed point theorem by using a more general contractive condition and three-step iteration scheme (4) and employ the following cotractive condition:

Let K be a nonempty closed convex subset of a Banach space X and $T : K \to K$ be a self-mapping of K such that for each $x, y \in X$ and $k_n \in [0, 1)$, we define the following contractive condition (say M-contraction):

$$\|Tx - Ty\| \leq e^{M\|x-Tx\|}(k_n\|x - y\|), \tag{7}$$

where $M \geq 0$ and e^x denotes the exponential function of $x \in X$.

Example. Let X be the real line with the usual norm $\|.\|$ and suppose $K = [0, 1]$. Define $T : K \to K$ by

$$Tx = \frac{x+1}{2} \text{ and } Ty = \frac{3y-1}{2}$$

for all $x, y \in K$. Obviously T is self-mapping with the common fixed point 1 for all $x, y \in K$. Now we check that our condition (7) is true. If $x, y \in [0, 1]$. Moreover, $\|Tx - Ty\| \leq e^{M\|x-Tx\|}(k_n\|x - y\|)$ for all $x, y \in [0, 1]$. In fact

$$\begin{aligned}\|Tx - Ty\| &= \|\frac{(x+1)}{2} - \frac{(3y-1)}{2}\| = \|\frac{(x+1)-(3y-1)}{2} \\ &= \|\frac{x-3y+2}{2}\|,\end{aligned}$$

and

$$\begin{aligned}e^{M\|x-Tx\|}(k_n\|x - y\|) &= e^{M\|x-\frac{x+1}{2}\|}(k_n\|x - y\|) \\ &= e^{M\|\frac{x-1}{2}\|}(k_n\|x - y\|)\end{aligned}$$

Clearly, if we choose $x = 0, y = 1$, then contractive condition (7) is satisfied because

$$\|Tx - Ty\| = \|\frac{x-3y+2}{2}\| = \|\frac{-1}{2}\| = \frac{1}{2},$$

and for $L \geq 0$ and $k_n \in [0, 1)$, we chose $M = 0$ and $k_n = 0.5$, then

$$\begin{aligned}e^{M\|x-Tx\|}(k_n\|x - y\|) &= e^{M\|x-\frac{x+1}{2}\|}(k_n\|x - y\|) = e^{M\|\frac{x-1}{2}\|}(k_n\|x - y\|) \\ &= e^{M(1/2)}(k_n\|0 - 1\|) = e^{0(1/2)}(\frac{1}{2}(1)) = \frac{1}{2}.\end{aligned}$$

Therefore $\|Tx - Ty\| \leq e^{M\|x-Tx\|}(k_n\|x - y\|)$. Hence T is a self mapping with common fixed and satisfying contractive condition (7).

In order to prove our result, we need the following lemma [2]:

Lemma 1. *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{t_n\}$ *be the sequences of nonnegative numbers satisfying:*

$$a_{n+1} \leq (1-\omega_n)a_n + b_n + t_n \text{ for all } n \geq 0 \text{ where } \{\omega_n\}_{n=0}^{\infty} \subset [0,1].$$

If $\sum_{n=0}^{\infty} \omega_n = \infty, b_n = O(\omega_n)$ *and* $\sum_{n=0}^{\infty} t_n < \infty$,*then* $\lim_{n\to\infty} a_n = 0$.

3. Main Results

In this section, we have proved strong convergence theorem and find approximate common fixed points of self-mapping T. In the consequence, F denotes the set of common fixed point of the mapping T.

Theorem 2. *Let* K *be a nonempty closed convex subset of Banach space* X *and* $T : K \to K$ *be a self-mapping of* K *satisfying contractive condition (7) with* $F(T) \neq \phi$, *where* $F(T)$ *is the set of fixed points of* $T : K \to K$ *be continuous. Suppose* $\{x_n\}$ *be the sequence defined by iteration (4) for arbitrary* $x_0 \in K$, *where* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *with* $\alpha_n+\beta_n+\gamma_n = 1$ *such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$. *Then* $\{x_n\}$ *converges strongly to a common fixed point of* T.

Proof. Suppose $F(T) \neq \phi$ and $p \in F(T)$. Since T is satisfying the contractive condition (7). For any $x, y \in K$. Suppose p is a fixed point of T now, put $x = p$ and $y = z_n$ in (7), we obtain

$$\|Tz_n - p\| \leq e^{M\|p-Tp\|}(k_n\|z_n - p\|) \tag{8}$$

Suppose $\{x_n\}_{n=0}^{\infty}$ be the sequences defined by (4). Then we have

$$\begin{aligned}\|x_{n+1} - p\| &= \|(1-\alpha_n)z_n + \alpha_n Tz_n - p\| \\ &= \|(1-\alpha_n)(z_n - p) + \alpha_n(Tz_n - p)\| \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n\|Tz_n - p\|.\end{aligned} \tag{9}$$

From (8) and (9), we get

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n e^{M\|p-Tp\|}(k_n\|z_n - p\|) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n e^{M\|p-p\|}(k_n\|z_n - p\|) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n e^{M(0)}(k_n\|z_n - p\|) \\ &\leq (1-\alpha_n)\|z_n - p\| + \alpha_n k_n\|z_n - p\| \\ &\leq (1-\alpha_n + \alpha_n k_n)\|z_n - p\|.\end{aligned} \tag{10}$$

Taking $x = p$ and $y = y_n$ in (7), we obtain

$$\|Ty_n - p\| \leq e^{M\|p-Tp\|}(k_n\|y_n - p\|) \tag{11}$$

Now,

$$\begin{aligned}\|z_n - p\| &= \|(1-\gamma_n)y_n + \gamma_n Ty_n - p\| \\ &= \|(1-\gamma_n)(y_n - p) + \gamma_n(Ty_n - p)\| \\ &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n\|Ty_n - p\|. \end{aligned} \tag{12}$$

Now From (11) and (12), we obtain

$$\begin{aligned}\|z_n - p\| &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n e^{M\|p-Tp\|}(k_n\|y_n - p\|) \\ &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n e^{M\|p-p\|}(k_n\|y_n - p\|) \\ &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n e^{M(0)}(k_n\|y_n - p\|) \\ &\leq (1-\gamma_n)\|y_n - p\| + \gamma_n k_n\|y_n - p\| \\ &\leq (1-\gamma_n + \gamma_n k_n)\|y_n - p\|. \end{aligned} \tag{13}$$

Substitute (13) into (10) implies that

$$\|x_{n+1} - p\| \leq (1-\alpha_n + \alpha_n k_n)(1-\gamma_n + \gamma_n k_n)\|y_n - p\|. \tag{14}$$

Now, put $x = p$ and $y = x_n$ in (7), we obtain

$$\|Tx_n - p\| \leq e^{M\|p-Tp\|}(k_n\|x_n - p\|) \tag{15}$$

Now,

$$\begin{aligned}\|y_n - p\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\| \\ &= \|(1-\beta_n)(x_n - p) + \beta_n(Tx_n - p)\| \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\|. \end{aligned} \tag{16}$$

Now From (15) and (16), we get

$$\begin{aligned}\|y_n - p\| &\leq (1-\beta_n)\|x_n - p\| + \beta_n e^{M\|p-Tp\|}(k_n\|x_n - p\|) \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n e^{M\|p-p\|}(k_n\|x_n - p\|) \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n e^{M(0)}(k_n\|x_n - p\|) \\ &\leq (1-\beta_n)\|x_n - p\| + \beta_n k_n\|x_n - p\| \\ &\leq (1-\beta_n + \beta_n k_n)\|x_n - p\|. \end{aligned} \tag{17}$$

Substitute (17) into (14), we get

$$\begin{aligned}\|x_{n+1}-p\| &\le (1-\alpha_n+\alpha_n k_n)(1-\beta_n+\beta_n k_n)(1-\gamma_n+\gamma_n k_n)\|x_n-p\| \\ &\le [1-\alpha_n(1-k_n)][1-\beta_n(1-k_n)][1-\gamma_n(1-k_n)]\|x_n-p\| \quad (18)\end{aligned}$$

i.e.,

$$\|x_{n+1}-p\| \le [1-\alpha_n(1-k_n)]\|x_n-p\|.$$

Since since $0 \le k_n < 1$, $\alpha_n \in [0,1]$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$ and setting $a_n = \|x_n-p\|, \omega_n = \alpha_n(1-k_n)$ by lemma 1, we obtain

$$\lim_{n\to\infty} \|x_{n+1}-p\| = 0.$$

So that $\{x_n\}_{n=0}^{\infty}$ converges strongly to the common fixed point of T.

Proof of the Uniqueness. suppose $p_1, p_2 \in F(T)$, where $F(T)$ is the set of fixed points of T in X such that $p_1 = Tp_1$ and $p_2 = Tp_2$.
Let on the contrary that $p_1 \ne p_2$. Then, using contractive condition (7), we get

$$\begin{aligned}\|p_1-p_2\| = \|Tp_1-Tp_2\| &= e^{M\|\mathbf{p_1}-T\mathbf{p_1}\|}(k_n\|p_1-p_2\|) \\ &= e^{M\|\mathbf{p_1}-\mathbf{p_1}\|}(k_n\|p_1-p_2\|) \\ &= e^{M(0)}(k_n\|p_1-p_2\|) \\ &= k_n\|p_1-p_2\| < \|p_1-p_2\|,\end{aligned}$$

which is a contradiction. Therefore, $p_1 = p_2$.

This complete the proof. □

The following results are immediate sequel of our strong convergence theorem.

Corollary 3. *Let K be a nonempty closed convex subset of Banach space X and $T : K \to K$ be a self-mapping of K satisfying contractive condition (7) with $F(T) \ne \phi$, where $F(T)$ is the set of fixed points of $T : K \to K$ be continuous. Suppose $\{x_n\}$ be the sequence defined by iteration (2) for arbitrary $x_0 \in K$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T.*

Corollary 4. *Let K be a nonempty closed convex subset of Banach space X and $T : K \to K$ be a self-mapping of K satisfying contractive condition (7) with $F(T) \ne \phi$, where $F(T)$ is the set of fixed points of $T : K \to K$ be continuous. Suppose $\{x_n\}$ be the sequence defined by iteration (3) for arbitrary*

$x_0 \in K$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T.

When $\beta_n = 0$ then given by Thianwan [11] iteration scheme is corollary of our result.

Corollary 5. *Let K be a nonempty closed convex subset of Banach space X and $T : K \to K$ be a self-mapping of K satisfying contractive condition (7) with $F(T) \neq \phi$, where $F(T)$ is the set of fixed points of $T : K \to K$ be continuous. Suppose $\{x_n\}$ be the sequence defined by iteration (5) for arbitrary $x_0 \in K$, where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n+\beta_n+\gamma_n = 1$ such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to a common fixed point of T.*

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