CHROMATIC EQUIVALENCE OF A FAMILY OF 
\( K_4 \)-HOMEOMORPHS WITH GIRTH 9

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Abstract: For a graph \( G \), let \( P(G, \lambda) \) denote the chromatic polynomial of \( G \). Two graphs \( G \) and \( H \) are chromatically equivalent (or simply \( \chi \)-equivalent), denoted by \( G \sim H \), if \( P(G, \lambda) = P(H, \lambda) \). A graph \( G \) is chromatically unique (or simply \( \chi \)-unique) if for any graph \( H \) such as \( H \sim G \), we have \( H \cong G \), i.e, \( H \) is isomorphic to \( G \). A \( K_4 \)-homeomorph is a subdivision of the complete graph \( K_4 \). In this paper, we determine when two \( K_4 \)-homeomorphs of the form \( K_4(2, 3, 4, d, e, f) \) and \( K_4(1, 2, 6, d', e', f') \) are chromatically equivalent. The result obtained can be extended in the study of chromatic equivalence classes of \( K_4(2, 3, 4, d, e, f) \) and chromatic uniqueness of \( K_4 \)-homeomorphs with girth 9.

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1. Introduction

All graphs considered here are simple graphs. For such a graph \( G \), let \( P(G, \lambda) \) denote the chromatic polynomial of \( G \). Two graphs \( G \) and \( H \) are chromatically equivalent (or simply \( \chi \)-equivalent), denoted by \( G \sim H \), if \( P(G, \lambda) = P(H, \lambda) \).
A graph $G$ is chromatically unique (or simply $\chi-$unique) if for any graph $H$ such as $H \sim G$, we have $H \cong G$, i.e, $H$ is isomorphic to $G$.

A $K_4$-homeomorph is a subdivision of the complete graph $K_4$. Such a homeomorph is denoted by $K_4(a, b, c, d, e, f)$ if the six edges of $K_4$ are replaced by the six paths of length $a, b, c, d, e, f$, respectively, as shown in Figure 1. So far, the chromaticity of $K_4$-homeomorphs with girth $g$, where $3 \leq g \leq 7$ has been studied by many authors (see [2,6,7,8]). Recently, Zhao et al. [9] studied the chromaticity of one family of $K_4$-homeomorphs with girth 8, i.e., $K_4(2, 3, 3, d, e, f)$. In [10], Shi has solved completely the chromaticity of $K_4$-homeomorphs with girth 8. When referring to the chromaticity of $K_4$-homeomorphs with girth 9, we know that ten types of $K_4$-homeomorphs need to be solved, i.e.,

$K_4(1, 2, 6, d, e, f), \ K_4(1, 3, 5, d, e, f), \ K_4(1, 4, 4, d, e, f), \ K_4(2, 2, 5, d, e, f),$

$K_4(2, 3, 4, d, e, f), \ K_4(1, 2, c, 2, e, 4), \ K_4(1, 2, c, 4, e, 2), \ K_4(1, 2, c, 3, e, 3),$

$K_4(1, 3, c, 2, e, 3) \text{ and } K_4(2, 2, c, 2, e, 3).$

In this paper, we consider one family of $K_4$-homeomorphs with girth 9, i.e., $K_4(2, 3, 4, d, e, f)$.

Hasni et al. [3,4] characterized chromatically equivalence pairs of $K_4$-homeomorphs, $K_4(1, 3, 5, d, e, f)$ with $K_4(1, 3, 5, d', e', f')$; and $K_4(1, 4, 4, d, e, f)$.
with \(K_4(1, 4, 4, d', e', f')\). In this paper, we shall discuss a chromatically equivalence pair of \(K_4\)-homeomorphs, \(K_4(2, 3, 4, d, e, f)\) (as shown in Figure 2) and \(K_4(1, 2, 6, d', e', f')\). Our main aim is to provide a result which can be extended to the study of the chromatic equivalence classes of \(K_4(2, 3, 4, d, e, f)\). Such results are an indispensable tool in the study of the chromatic uniqueness of \(K_4\)-homeomorphs with girth 9.

2. Preliminary Results

In this section, we give some known results used in the sequel.

**Lemma 2.1.** Assume that \(G\) and \(H\) are \(\chi\)-equivalent. Then

1. \(|V(G)| = |V(H)|, |E(G)| = |E(H)|\) (see [5]);

2. \(G\) and \(H\) has the same girth and same number of cycles with length equal to their girth (see [12]);

3. If \(G\) is a \(K_4\)-homeomorph, then \(H\) must itself be a \(K_4\)-homeomorph (see [1]);

4. Let \(G = K_4(a, b, c, d, e, f)\) and \(H = K_4(a', b', c', d', e', f')\), then
In this section, we present our main result.

\textbf{Theorem 3.1.} (Hasni et al. [3]) Let $K_4(1, 3, 5, d, e, f)$ and $K_4(1, 3, 5, d', e', f')$ be chromatically equivalent, then

\begin{align*}
K_4(1, 3, 5, i, i + 6, i + 1) &\sim K_4(1, 3, 5, i + 2, i, i + 5), \\
K_4(1, 3, 5, i, i + 1, i + 4) &\sim K_4(1, 3, 5, i + 2, i + 3, i).
\end{align*}

\textbf{Theorem 2.2.} (Hasni et al. [4]) Let $K_4(1, 4, 4, d, e, f)$ and $K_4(1, 4, 4, d', e', f')$ be chromatically equivalent, then

\begin{align*}
K_4(1, 4, 4, i, i + 1, i + 5) &\sim K_4(1, 4, 4, i + 2, i, i + 4).
\end{align*}

\textbf{Theorem 2.3.} (Zhao et al. [9]) $K_4$-homeomorph $K_4(2, 3, 3, d, e, f)$ with girth 8 is not $\chi$-unique if and only if it is isomorphic to $K_4(2, 3, 3, 1, 6, \delta)$ ($\delta \geq 6$), $K_4(2, 3, 3, 1, \beta, \beta + 2)$ ($\beta \geq 4$), or $K_4(2, 3, 3, 1, 5, 6)$.

\section{3. Main Results}

In this section, we present our main result.

\textbf{Theorem 3.1.} If $G$ is in the type of $K_4(2, 3, 4, d, e, f)$, and $H$ is in the type of $K_4(1, 2, 6, d', e', f')$, then $G \sim H$ if $G$ is isomorphic to $K_4(2, 3, 4, 1, 7, f)$, $K_4(2, 3, 4, 5, 1, 6)$, $K_4(2, 3, 4, 7, 1, 5)$ or $K_4(2, 3, 4, 4, 1, 6)$, or $K_4(2, 3, 4, 1, 7, 6)$, or $K_4(2, 3, 4, 1, 5, 8)$, where $f \geq 7$.

\textbf{Proof.} Let $G$ and $H$ be two graphs such that $G \cong K_4(2, 3, 4, d, e, f)$ and $H \cong K_4(1, 2, 6, d', e', f')$. Since the girth of $G$ is 9, there is at most 1 among $d, e$ and $f$.

Let

\begin{align*}
Q(K_4(a, b, c, d, e, f)) &= -(s + 1)(s^a + s^b + s^c + s^d + s^e + s^f) + s^{a+d} + s^{b+f} +
\end{align*}
Let $s = 1 - \lambda$ and $x$ is the number of edges in $G$. From [11], we have the chromatic polynomial of $K_4$-homeomorphs $K_4(a, b, c, d, e, f)$ is as follows:

$$P(K_4(a, b, c, d, e, f)) = (-1)^{x-1} \frac{s}{(s-1)^2} \left[(s^2 + 3s + 2) + Q(K_4(a, b, c, d, e, f)) - s^{x-1}\right].$$

Hence $P(G) = P(H)$ if and only if $Q(G) = Q(H)$. We solve the equation $Q(G) = Q(H)$ to get all solutions. Let the lowest remaining power and the highest remaining power be denoted by l.r.p. and h.r.p., respectively.

$$Q(G) = -(s + 1)(s^2 + s^3 + s^4 + s^d + s^e + s^f) + s^{2+d} + s^{3+f} + s^{4+e} + s^{5+e} + s^{7+d} + s^{6+f} + s^{d+e+f}.$$  

$$Q(H) = -(s + 1)(s + s^2 + s^6 + s^{d'} + s^{e'} + s^{f'}) + s^{1+d'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'} + s^{d'+e'+f'}.$$  

We can assume $e \leq f$. Since $K_4(2, 3, 4, d, e, f)$ has exactly 1 path of length 1 and $e \leq f$, we have $\min \{d, e, f\} = \{d, e\} = 1$. From Lemma 2.1 (1),

$$d + e + f = d' + e' + f'.$$

There are two cases to be considered.

**Case 1.** $\min \{d, e\} = d = 1$. We obtain the following after simplification.

$$Q_1(G) = -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8 + s^{6+f},$$  

$$Q_1(H) = -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{c'} - s^f + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'}.$$  

After comparing the h.r.p in $Q_1(G)$ and the h.r.p in $Q_1(H)$, we have the h.r.p in $Q_1(G)$ is $6 + f$. Considering the h.r.p in $Q_1(G)$ and the h.r.p in $Q_1(H)$, we know that there are three cases to be considered.

**Case 1.1** $\max \{6 + e', 8 + d', 7 + f'\} = 6 + e' = 6 + f$. So $e' = f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

$$Q_2(G) = -2s^4 - s^5 - s^{e+1} - s^e + s^{4+e} + s^{5+e} + s^8,$$  

$$Q_2(H) = -s^7 - s^{f'+1} - s^6 - s^{d'} - s^f + s^{2+f'} + s^{8+d'} + s^{7+f'}.$$
Consider the $-2s^4$ in $Q_2(G)$. Since $-2s^4$ cannot be cancelled by the terms in $Q_2(G)$, there are two terms in $Q_2(H)$ which are equal to $-s^4$. So $f' + 1 = d = 4$ or $f' = d' = 4$.

If $f' + 1 = d' = 4$, from Equa (1), we get $e = 6$. So $Q_2(G) \neq Q_2(H)$, a contradiction.

If $f' = d' = 4$, from Equa (1), we get $e = 7$. In this case, we obtain a solution where $G$ is isomorphic to $K_4(2, 3, 4, 1, 7, f)$ and $H$ isomorphic to $K_4(1, 2, 6, 4, f, 4)$, i.e., $K_4(2, 3, 4, 1, 7, f) \sim K_4(1, 2, 6, 4, f, 4)$.

**Case 1.2** max $\{6 + e', 8 + d', 7 + f'\} = 7 + f' = 6 + f$. So $1 + f' = f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

\[
\begin{aligned}
Q_3(G) &= -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e + s^{4+e} + s^{5+e} + s^8, \\
Q_3(H) &= -s^7 - s^{e'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{3+e} + s^{8+d'}. \\
\end{aligned}
\]

Assume $6 + e' < 6 + f = 7 + f'$ since $6 + e' = 6 + f$ has been discussed in Case 1.1. As the term $8 + d' \leq 7 + f'$, the term $s^{2+f'}$ cannot be cancelled by any negative terms in $Q_3(H)$, then none of the terms in $Q_3(H)$ are equal to the term $-s^{f+1}$ in $Q_3(G)$ by noting $f+1 = f'+2$. Therefore, $2s^{2+f'}(or -2s^{f+1}) \in Q_3(G)$. We also get $4 + e = 8 = 2 + f'$. Thus, $e = 4$, $f' = 6$ and $f = 7$. Then $-3s^4 \in Q_3(G)$, but $-3s^4 \notin Q_3(G)$, a contradiction.

**Case 1.3** max $\{6 + e', 8 + d', 7 + f'\} = 8 + d' = 6 + f$. After discussing the case $6 + e' = 6 + f$ in Case 1.1, then we suppose that $6 + e' < 6 + f$. From $Q_1(G)$ and $Q_1(H)$, we obtain the following after simplification.

\[
\begin{aligned}
Q_4(G) &= -2s^4 - s^5 - s^{e+1} - s^{f+1} - s^e - s^f + s^{4+e} + s^{5+e} + s^8, \\
Q_4(H) &= -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{7+f'}. \\
\end{aligned}
\]

Comparing the l.r.p in $Q_4(G)$ and the l.r.p in $Q_4(H)$, we have $d' = e' = 4$ or $d' = f' = 4$ or $e' = f' = 4$.

**Case 1.3.1** If $d' = e' = 4$, we obtain the following after simplification.

\[
\begin{aligned}
Q_5(G) &= -s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8, \\
Q_5(H) &= -s^7 - s^{f'+1} - s^6 - s^{f'} + s^{2+f'} + s^{10} + s^{7} + s^{7+f'}. \\
\end{aligned}
\]

Comparing the h.r.p in $Q_5(G)$ and the h.r.p in $Q_5(H)$, we have $7 + f' = 3 + f$ or $7 + f' = 5 + e$.

If $7 + f' = 3 + f$, from Equa (1), we get $e = 3$. Then $Q_5(G) \neq Q_5(H)$, a contradiction.

If $7 + f' = 5 + e$, from Equa (1), we get $f = 5$. Then $Q_5(G) \neq Q_5(H)$, a contradiction.
Case 1.3.2 If \( d' = f' = 4 \), we obtain the following after simplification.

\[
\begin{align*}
Q_6(G) &= -s^{e+1} - s^{f+1} - s^e - s^f + s^{3+f} + s^{4+e} + s^{5+e} + s^8, \\
Q_6(H) &= -s^7 - s^{e'+1} - s^{e'} + s^{6+e'} + s^{3+e'} + s^{11}.
\end{align*}
\]

Comparing the h.r.p in \( Q_6(G) \) and the h.r.p in \( Q_6(H) \), we have \( 6 + e' = 3 + f \) or \( 6 + e' = 5 + e \).

If \( 6 + e' = 3 + f \), from Equa (1), we get \( e = 4 \). Then \( Q_6(G) \neq Q_6(H) \), a contradiction.

If \( 6 + e' = 5 + e \), from Equa (1), we get \( f = 6 \). Then \( e' = 6 \) and \( e = 7 \). So we obtain a solution where \( G \cong K_4(2, 3, 4, 1, 7, 6) \) and \( H \cong K_4(1, 2, 6, 4, 4, 5) \).

Case 1.3.3 If \( e' = f' = 4 \), from Equa (1), we get \( e = 5 \). We obtain the following after simplification.

\[
Q_7(G) = -s^{f+1} - s^f + s^{3+f} + s^9 + s^8, \quad Q_7(H) = -s^{d'} + s^6 + s^{11}.
\]

It is clear that \( d' = 6 \) and \( f = 8 \). So we obtain a solution where \( G \cong K_4(2, 3, 4, 1, 5, 8) \) and \( H \cong K_4(1, 2, 6, 4, 4) \).

Case 2. \min \{d, e\} = e = 1. Since \( d + e \geq 6 \), \( e + f \geq 7 \), we have \( d \geq 5 \) and \( f \geq 6 \). We obtain the following after simplification.

\[
\begin{align*}
Q_8(G) &= -s^3 - 2s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}, \\
Q_8(H) &= -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{d'} - s^{e'} - s^{f'} + s^{2+f'} + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{7+f'}.
\end{align*}
\]

Consider the l.r.p in \( Q_8(G) \) and the l.r.p in \( Q_8(H) \), we have \( \min \{d', e', f'\} = 3 \).

Case 2.1 \( d' = 3 \). From \( Q_5(G) \) and \( Q_5(H) \), we obtain the following after simplification.

\[
\begin{align*}
Q_9(G) &= -2s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f}, \\
Q_9(H) &= -s^7 - s^{e'+1} - s^{f'+1} - s^6 - s^{e'} - s^{f'} + s^{2+f'} + s^6 + s^{3+e'} + s^{11} + s^{7+f'}.
\end{align*}
\]

Consider the \(-2s^4\) in \( Q_9(G) \). Since \( Q_9(G) = Q_9(H) \), there are two terms in \( Q_9(H) \) which are equal to \(-s^4\). So we have \( e' = f' = 4 \) or \( e' = f' + 1 = 4 \) or \( f' = e' + 1 = 4 \).

If \( e' = f' = 4 \), then we obtain the following after simplification.

\[
\begin{align*}
Q_{10}(G) &= -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^{7+d} + s^{6+f}, \\
Q_{10}(H) &= -s^7 - s^5 - s^6 + s^{10} + s^7 + s^{11} + s^{11}.
\end{align*}
\]
Consider the h.r.p in \( Q_{10}(G) \) and the h.r.p in \( Q_{10}(H) \). If \( 7 + d = 11 \), then \( d = 4 \) which contradicts \( d \geq 5 \). If \( 6 + f = 11 \), then \( f = 5 \) which contradicts \( f \geq 6 \).

If \( e' = f' + 1 = 4 \), then we obtain the following after simplification.
\[
Q_{11}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f},
Q_{11}(H) = -s^7 - s^5 - s^6 - s^3 + s^5 + s^{10} + s^7 + s^{11} + s^{10}.
\]

Note that \( d \geq 5 \). Then \(-s^3 \in Q_{11}(H)\) but not in \( Q_{11}(G)\), a contradiction.

If \( f' = e' + 1 = 4 \), then we obtain the following after simplification.
\[
Q_{12}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f},
Q_{12}(H) = -s^7 - s^5 - s^3 - s^3 + s^9 + s^6 + s^{11} + s^{11}.
\]

Then \(-2s^3 \in Q_{12}(H)\) but not in \( Q_{12}(G)\), a contradiction.

**Case 2.2** \( e' = 3 \). From \( Q_8(G) \) and \( Q_8(H) \), we obtain the following after simplification.
\[
Q_{13}(G) = -s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^{7+d} + s^{6+f},
Q_{13}(H) = -s^7 - s^{f+1} - s^6 - s^d - s^f + s^{2+f'} + s^9 + s^{8+d'} + s^{7+f'}.
\]

Consider the l.r.p in \( Q_{13}(G) \) and the l.r.p in \( Q_{13}(H) \). So we have \( d' = 4 \) or \( f' = 4 \).

**Case 2.2.1** \( d' = 4 \). We obtain the following after simplification.
\[
Q_{14}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^{7+d} + s^{6+f},
Q_{14}(H) = -s^7 - s^{f+1} - s^6 - s^d - s^f + s^{2+f'} + s^9 + s^{12} + s^{7+f'}.
\]

Consider the h.r.p in \( Q_{14}(G) \) and the h.r.p in \( Q_{14}(H) \). So we have \( 7 + f' = 7 + d \) or \( 7 + f' = 6 + f \).

If \( 7 + f' = 7 + d \), from Eq. (1), we get \( f = 6 \). We have \( Q_{14}(G) = Q_{14}(H) \).

Thus, \( G \cong H \).

If \( 7 + f' = 6 + f \), from Eq. (1), we get \( d = 5 \). We obtain the following after simplification.
\[
Q_{15}(G) = -s^5 + s^7 + s^{3+f}, \quad Q_{15}(H) = -s^7 - s^{f'} + s^{2+f'} + s^9.
\]

It is easy to see that \( f' = 5 \) and \( f = 6 \). Thus we obtain a solution where \( G \) is isomorphic to \( K_4(2, 3, 4, 5, 1, 6) \) and \( H \) is isomorphic to \( K_4(1, 2, 6, 4, 3, 5) \).

**Case 2.2.2** \( f' = 4 \). We obtain the following after simplification.
\[
Q_{16}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^{7+d} + s^{6+f},
Q_{16}(H) = -s^7 - s^5 - s^{d'} + s^9 + s^{8+d'} + s^{11}.
\]

Consider the h.r.p in \( Q_{16}(G) \) and the h.r.p in \( Q_{16}(H) \). So we have \( 8 + d' = 7 + d \) or \( 8 + d' = 6 + f \).
If $8 + d' = 7 + d$, from Equa (1), we get $f = 5$. We obtain the following after simplification.

$$Q_{17}(G) = -s^{d+1} - s^6 - s^d + s^{2+d} + s^8, \quad Q_{17}(H) = -s^7 - s^{d'} + s^9.$$ 

It is easy to see that $d' = 6$ and $d = 7$. Thus we obtain a solution where $G$ is isomorphic to $K_4(2, 3, 4, 7, 1, 5)$ and $H$ is isomorphic to $K_4(1, 2, 6, 6, 3, 4)$.

If $8 + d' = 6 + f$, from Equa (1), we get $d = 4$. We obtain the following after simplification.

$$Q_{18}(G) = -s^{f+1} - s^4 - s^f + s^6 + s^{3+f} + s^{6+f}, \quad Q_{18}(H) = -s^7 - s^{d'} + s^9 + s^{12}.$$ 

It is easy to see that $d' = 4$ and $f = 6$. Thus we obtain $G \cong H$.

**Case 2.3** $f' = 3$. From $Q_8(G)$ and $Q_8(H)$, we obtain the following after simplification.

$$Q_{19}(G) = -s^4 - s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f},$$
$$Q_{19}(H) = -s^7 - s^{e'+1} - s^6 - s^{d'} - s^{e'} + s^5 + s^{6+e'} + s^{3+e'} + s^{8+d'} + s^{11}.$$ 

Consider the l.r.p in $Q_{19}(G)$ and the l.r.p in $Q_{19}(H)$. So we have $d' = 4$ or $e' = 4$.

**Case 2.3.1** $d' = 4$. We obtain the following after simplification.

$$Q_{20}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f},$$
$$Q_{20}(H) = -s^7 - s^{e'+1} - s^6 - s^{e'} + s^5 + s^{6+e'} + s^{3+e'} + s^{12} + s^{10}.$$ 

Consider the h.r.p in $Q_{20}(G)$ and the h.r.p in $Q_{20}(H)$. We obtain $7 + d = 6 + e'$ or $6 + f = 6 + e'$.

If $6 + f = 6 + e'$, from Equa (1), we get $d = 6$. Then $Q_{20}(G) \neq Q_{20}(H)$, a contradiction.

If $7 + d = 6 + e'$, from Equa (1), we get $7 = 6$. Then $Q_{20}(G) \neq Q_{20}(H)$, a contradiction.

**Case 2.3.2** $e' = 4$. We obtain the following after simplification.

$$Q_{21}(G) = -s^{d+1} - s^{f+1} - s^d - s^f + s^{2+d} + s^{3+f} + s^6 + s^{7+d} + s^{6+f},$$
$$Q_{21}(H) = -s^6 - s^{d'} + s^{10} + s^{8+d'} + s^{10}.$$ 

Consider the h.r.p in $Q_{21}(G)$ and the h.r.p in $Q_{21}(H)$. We obtain $7 + d = 8 + d'$ or $6 + f = 8 + d'$.

If $7 + d = 8 + d'$, from Equa (1), we get $f = 5$. Then $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.
If $6 + f = 8 + d'$, from Equa (1), we get $d = 4$. Then $Q_{21}(G) \neq Q_{21}(H)$, a contradiction.

So far, we have solved the equation $Q(G) = Q(H)$ and obtained the solutions as follows:

$K_4(2, 3, 4, 1, 7, a) \sim K_4(1, 2, 6, 4, a, 4),$

$K_4(2, 3, 4, 5, 1, 6) \sim K_4(1, 2, 6, 4, 3, 5),$

$K_4(2, 3, 4, 7, 1, 5) \sim K_4(1, 2, 6, 6, 3, 4),$

$K_4(2, 3, 4, 1, 7, 6) \sim K_4(1, 2, 6, 4, 6, 4),$

$K_4(2, 3, 4, 1, 5, 8) \sim K_4(1, 2, 6, 6, 4, 4),$

where $a \geq 4$.

The proof is completed.

We close the paper with the following open problems.

**Problem 1.** Study the chromatic equivalence of the graph $G$ is in the type of $K_4(2, 3, 4, d, e, f)$ and $H$ is in the type of the following graphs:

$K_4(1, 3, 5, d, e, f), \ K_4(1, 4, 4, d, e, f), \ K_4(2, 2, 5, d, e, f), \ K_4(2, 3, 4, d, e, f),$

$K_4(1, 2, c, 2, e, 4), \ K_4(1, 2, c, 4, e, 2), \ K_4(1, 2, c, 3, e, 3), \ K_4(1, 3, c, 2, e, 3)$

and $K_4(2, 2, c, 2, e, 3)$.

**Problem 2.** Study the chromatic uniqueness of the graph $K_4(2, 3, 4, d, e, f)$, where $d + e \geq 6, \ d + f \geq 5$ and $e + f \geq 7$.

References


