SPHERICAL FOLDING TESSELLATIONS
BY KITES AND ISOSCELES TRIANGLES II

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Abstract: The classification of the dihedral folding tessellations of the sphere and the plane whose prototiles are a kite and an equilateral triangle were obtained in a recent paper, [1]. Concerning to isosceles triangles the classification is much harder, which is not surprising since more angles are involved. In this paper we extend this classification presenting all the dihedral folding tessellations of the sphere by kites and isosceles triangles in a particular case of adjacency. A list containing these tilings including its combinatorial structure is presente in Table 1.

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1. Introduction

By a folding tessellation or folding tiling of the sphere $S^2$ we mean an edge-to-edge pattern of spherical geodesic polygons that fills the whole sphere with no gaps and no overlaps, and such that the “underlying graph” has even valency at any vertex and the sums of alternate angles around each vertex are $\pi$. 

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Folding tilings are strongly related to the theory of isometric foldings on Riemannian manifolds. In fact, the set of singularities of any isometric folding on a surface \( S \) corresponds to a folding tiling in \( S \), see [11] for the foundations of this subject.

The study of these special class of tessellations was initiated in [3] with a complete classification of all spherical monohedral folding tilings (by congruent triangles, necessarily). Ten years latter, Ueno and Agaoka [12] have established the complete classification of all triangular spherical monohedral tilings (without any restriction on angles).

Dawson and Doyle have also been interested in special classes of spherical tilings, see [8], [9] and [10], for instance.

The complete classification of all spherical folding tilings by rhombi and triangles was obtained in 2005 [6]. A detailed study of the triangular spherical folding tilings by equilateral and isosceles triangles is presented in [7].

Spherical \( f \)-fillings by two non congruent classes of isosceles triangles have recently obtained, [4].

Here we discuss dihedral folding tessellations by spherical kites and isosceles spherical triangles.

A spherical kite \( K \) (Figure 1(a)) is a spherical quadrangle with two congruent pairs of adjacent sides, but distinct from each other. Let us denote by \((\alpha_1, \alpha_2, \alpha_1, \alpha_3)\), \( \alpha_2 > \alpha_3 \), the internal angles of \( K \) in cyclic order. The length sides are denoted by \( a \) and \( b \), with \( a < b \). From now on \( T \) denotes a spherical isosceles triangle with internal angles \( \beta \) and \( \gamma \) (\( \gamma \neq \beta \)), and length sides \( c \) and \( d \), see Figure 1(b).

We shall denote by \( \Omega(K, T) \) the set, up to isomorphism, of all dihedral folding tilings of \( S^2 \) whose prototiles are \( K \) and \( T \).

![Diagram of a spherical kite and a spherical isosceles triangle](image)

Figure 1: A spherical kite and a spherical isosceles triangle
Taking into account the area of the prototiles $K$ and $T$ we have

$$2\alpha_1 + \alpha_2 + \alpha_3 > 2\pi \quad \text{and} \quad \beta + 2\gamma > \pi.$$ 

As $\alpha_2 > \alpha_3$ we also have

$$\alpha_1 + \alpha_2 > \pi.$$ 

We begin by pointing out that any element of $\Omega (K, T)$ has at least two cells congruent to $K$ and $T$, respectively, such that they are in adjacent positions and in one and only one of the situations illustrated in Figure 2.

After certain initial assumptions are made, it is usually possible to deduce sequentially the nature and orientation of most of the other tiles. Eventually, either a complete tiling or an impossible configuration proving that the hypothetical tiling fails to exist is reached. In the diagrams that follow, the order in which these deductions can be made is indicated by the numbering of the tiles. For $j \geq 2$, the location of tiling $j$ can be deduced directly from the configurations of tiles $(1, 2, \ldots, j - 1)$ and from the hypothesis that the configuration is part of a complete tiling, except where otherwise indicated.

The first case has already been analyzed, [2]. In this paper we will consider the second case of adjacency. Our aim is to obtain all the dihedral $f$-tilings by kites and isosceles triangles. After the analysis of the cases of adjacency I and II, we believe that the study of the remaining cases will be handled more
briefly, since there is no more need to consider two adjacent tiles accordingly to the first two cases of adjacency.

2. Case of Adjacency II

Suppose that any element of $\Omega(K, T)$ has at least two cells congruent, respectively, to $K$ and $T$, such that they are in adjacent positions as illustrated in Figure 2–II. As $a = d$, using trigonometric formulas, we obtain

$$\frac{\cos \gamma (1 + \cos \beta)}{\sin \gamma \sin \beta} = \frac{\cos \frac{\alpha_2}{2} + \cos \alpha_1 \cos \frac{\alpha_3}{2}}{\sin \alpha_1 \sin \frac{\alpha_2}{2}}. \quad (1)$$

Concerning the angles of the triangle $T$ we have necessarily one of the following situations:

$$\gamma > \beta \quad \text{or} \quad \gamma < \beta.$$ 

In the following subsections we consider separately each one of these cases.

2.1. $\gamma > \beta$

As $\gamma > \beta$, we have $\gamma > \frac{\pi}{3}$ and $c < a = d < b$, and so $c \neq b$, which determines tile 3 (Figure 3). In the following proposition we prove that there are no f-tilings verifying such a condition.

**Figure 3: Local configuration**

**Proposition 1.** If $\gamma > \beta$, then $\Omega(K, T)$ is the empty set.

**Proof.** Suppose that any element of $\Omega(K, T)$ has at least two cells congruent, respectively, to $K$ and $T$, such that they are in adjacent positions as illustrated in Figure 2–II (tiles 1 and 2 in Figure 3).

We begin distinguishing the cases $\alpha_1 \geq \alpha_2 > \alpha_3$ and $\alpha_2 > \alpha_1, \alpha_3$ (includes the cases $\alpha_2 > \alpha_1 \geq \alpha_3$ and $\alpha_2 > \alpha_3 > \alpha_1$).
1. Suppose firstly that $\alpha_1 \geq \alpha_2 > \alpha_3$ ($\alpha_1 > \frac{\pi}{2}$). As $\alpha_1 + \gamma \leq \pi$, it follows that $\gamma < \frac{\pi}{2} < \alpha_1$ and $\beta < \gamma < \alpha_2$. At vertex $v_1$ (Figure 3) we have necessarily $\alpha_1 + \gamma + k\alpha_3 = \pi$, with $k \geq 0$. Taking into account the edge lengths, the last configuration extends to the one that follows (Figure 4(a)).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4}
\caption{Local configurations}
\end{figure}

We have $\theta_1 \in \{\gamma, \alpha_2, \beta\}$.

If $\theta_1 \in \{\gamma, \alpha_2\}$ (Figure 4(b)) and

(i) $\alpha_2 + \theta_1 = \pi$, then $\beta + \rho = \pi$ has no solution;

(ii) $\alpha_2 + \theta_1 < \pi$, we have $\alpha_2 + \theta_1 + t\alpha_3 = \pi$, $t \geq 1$. Now, if $\theta_1 = \gamma$, at vertex $v_2$ arises an incompatibility on the sides; on the other hand, if $\theta_1 = \alpha_2$, the sum of alternate angles containing $\beta$ must be of the form $\beta + \alpha_1 + (t - 1)\alpha_3 + \alpha_1 > 2\alpha_1 > \pi$.

If $\theta_1 = \beta$ (Figure 5(a)), at vertex $v_3$ we get $\gamma + \gamma + \gamma > \pi$, which is a contradiction.

2. Suppose now that $\alpha_2 > \alpha_1, \alpha_3$ ($\alpha_2 > \frac{\pi}{2}$) (Figure 3).

If $\alpha_1 + \gamma = \pi$ (Figure 5(b)), we must have $\theta_1 = \beta$. Consequently, taking into account the edge lengths, at vertex $v$ we obtain $\gamma + \gamma + \gamma > \pi$, which is not possible.

Therefore, $\alpha_1 + \gamma < \pi$. If $\alpha_1 \geq \gamma$ (see Figure 4(a)), we have $\theta_1 \in \{\gamma, \beta\}$. The analysis of both cases is analogous to the one made for 1.

Consequently, $\gamma > \alpha_1$ and, as $\alpha_1 + \gamma < \pi$, we also have $\alpha_1 < \frac{\pi}{2}$. Given the area of $K$, it follows that $\alpha_2 + \alpha_3 > \pi$, and so in a sum of alternate angles containing $\alpha_2$ can exist only angles $\beta$ (see Figure 6(a)). At vertex $v$ it follows that $\gamma + \gamma \leq \pi$. Taking into account that $\beta$ is the smallest angle and also
2γ + β > π, we conclude that γ + γ = π and so a = γ = \frac{π}{2}. Considering the kite K divided as shown in Figure 6(b), we obtain θ = \frac{π}{2}. But then α_1 > \frac{π}{2}, which is not possible.

2.2. γ < β

As γ < β, we have β > \frac{π}{3} and a < b, c. In the following propositions we consider separately each one of the following situations:

\[α_1 ≥ α_2 > α_3, \quad α_2 > α_1 ≥ α_3 \quad \text{and} \quad α_2 > α_3 > α_1.\]

**Proposition 2.** If α_1 ≥ α_2 > α_3, then Ω(K, T) ≠ ∅ iff

\[(i) \quad α_1 + γ = \pi, \quad α_2 = β = \frac{π}{2} \quad \text{and} \quad α_3 = \frac{π}{k}, \quad \text{with} \ k ≥ 3 \quad \text{and} \quad γ = γ_k = \arccos \left(\frac{\sqrt{2}}{2} \cos \frac{π}{2k}\right), \ \text{or}\]

Figure 5: Local configurations

Figure 6: Local configurations
(ii) $\alpha_1 + \gamma = \pi$, $\alpha_2 + \beta = \pi$ and $\gamma = \alpha_3 = \frac{\pi}{k}$, with $k \geq 3$ and $\beta = \beta_k = 2 \arcsin \left(2 \cos \frac{\pi}{2k} - \sec \frac{\pi}{2k}\right)$.

In the first case, for each $k \geq 3$, there is a unique f-tiling denoted by $E^k$. A planar representation of $E^k$ is given in Figure 15(b). 3D representations for $k = 3$ and $k = 4$ are given in Figure 16.

In the last situation there is a single f-tiling, $H^k$, for each $k \geq 3$. The corresponding planar and 3D representations, for $k = 3$ and $k = 4$, are given in Figure 20 and Figure 21, respectively.

Proof. Suppose that any element of $\Omega (K, T)$ has at least two cells congruent, respectively, to $K$ and $T$, such that they are in adjacent positions as illustrated in Figure 2-II and $\alpha_1 \geq \alpha_2 > \alpha_3 \ (\alpha_1 > \frac{\pi}{2})$. With the labeling of Figure 7(a), at vertex $v_1$ we have

$$\theta_1 = \alpha_1, \quad \theta_1 = \beta \quad \text{or} \quad \theta_1 = \alpha_2.$$  

Note that $\theta_1 \neq \gamma$, as the case of adjacency I analyzed in [2] (Proposition 2.3),

give not rise to any f-tiling.

1. Suppose firstly that $\theta_1 = \alpha_1$. We have $\alpha_1 + \beta \leq \pi \ (\beta < \frac{\pi}{2} < \alpha_1)$, and so at vertex $v_1$ we must have $\alpha_1 + \beta + k\alpha_3 = \pi$, with $k \geq 0$. It follows that the other sum of alternate angles containing $\alpha_2$ must be of the form $\alpha_2 + k\alpha_3 + \rho = \pi$, with $\rho \in \{\alpha_1, \gamma\}$. Nevertheless, each choice for $\rho$ results in a contradiction.

2. Suppose now that $\theta_1 = \beta$. We consider separately the cases $\beta + \beta < \pi$ and $\beta + \beta = \pi$.  

![Figure 7: Local configurations](image)
2.1 If \( \beta + \beta < \pi \), i.e. \( \beta < \frac{\pi}{2} \), then \( \beta + k_1 \alpha_2 + k_2 \alpha_3 = \pi \), with \( k_1, k_2 \geq 0 \) and \( k_1 + k_2 > 0 \).

If \( k_1 \neq 0 \), then \( \alpha_1 > \beta > \gamma > \alpha_2 > \alpha_3 \). As the set of vertices of valency four must be nonempty, \( \alpha_1 + \alpha_2 > \pi \) and \( 2\beta + \gamma > \pi \), we have necessarily \( \alpha_1 + \alpha_3 = \pi \).

Taking into account the edge lengths, we get \( \alpha_1 + \alpha_3 = \pi = \gamma + \alpha_3 \), and so \( \alpha_1 = \gamma \), which is not possible.

Thus, \( k_1 = 0 \), i.e., \( \beta + k_2 \alpha_3 = \pi \), \( k_2 \geq 1 \). The last configuration is extended to the one illustrated in Figure 7(b). The other sum of alternate angles at vertex \( v_1 \) must be \( \alpha_2 + \gamma + (k-1)\alpha_3 + \gamma = \pi \), and so \( \beta > \alpha_2 \) (implying tile 7 and \( \alpha_2 > \gamma \)). But an impossibility arises at vertex \( v_2 \) as \( \alpha_1 + \alpha_2 > \pi \).

2.2 If \( \beta + \beta = \pi \), then \( \alpha_2 + \rho = \pi \), with \( \rho \in \{ \alpha_2, \beta \} \) (Figure 8(a)). In any case, we have \( \alpha_2 = \beta = \frac{\pi}{2} \) and, observing Figure 8(b), we conclude that \( K = T \cup T' \). Therefore, \( \alpha_1 > \beta = \alpha_2 = \frac{\pi}{2} > \gamma, \alpha_3 \), and \( \gamma > \frac{\pi}{4} \). At vertex \( v_2 \) we have \( \alpha_1 + k\alpha_3 = \pi \), \( k \geq 1 \), or \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t \geq 0 \). The first case leads
to a contradiction, as the other sum of alternate angles at vertex \( v_2 \) must be \( \gamma + k\alpha_3 = \pi \), \( k \geq 1 \), implying \( \alpha_1 = \gamma \).

Thus, \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t \geq 0 \), and \( \theta_2 = \alpha_3 \) or \( \theta_2 = \gamma \).

If \( \theta_2 = \alpha_3 \) (Figure 9(a)), and so \( t > 0 \), then, using similar arguments, at vertex \( v_3 \) we must have \( \alpha_1 + \gamma + t\alpha_3 = \pi \). As we can observe in Figure 9(b), we reach a contradiction at vertex \( v_4 \).

Therefore we have \( \theta_2 = \gamma \). Now, we distinguish the cases \( \alpha_1 + \gamma < \pi \) and \( \alpha_1 + \gamma = \pi \), i.e., \( t > 0 \) and \( t = 0 \).

The first case leads to \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t > 0 \) (see Figure 10(a)). Note that tile 5 follows, by symmetry, the same analysis applied to tile 4. At vertex \( v_3 \) it follows that \( \alpha_1 + \alpha_1 > \pi \), which is a contradiction.

Then, \( t = 0 \) (see Figure 10(b)), and \( \theta_3 \in \{ \gamma, \beta, \alpha_2 \} \). We now analyze each possible choice for \( \theta_3 \).
2.2.1 If \( \theta_3 = \gamma \), the last configuration extends to the one illustrated in Figure 11(a). At vertex \( v_4 \) we obtain \( \alpha_1 + \gamma = \pi = \beta + \alpha_3 \), i.e., \( \alpha_3 = \frac{\pi}{2} \), which is an absurd.

2.2.2 Suppose now that \( \theta_3 = \beta \) (Figure 11(b)). If \( \theta_4 = \alpha_3 \), we get the configuration illustrated in Figure 12(a). But an incompatibility can not be avoided at vertex \( v_6 \) (note that the choice for tile 12 has analogous arguments to the ones used for tile 4).

Therefore, \( \theta_4 = \gamma \), and so \( \frac{\pi}{4} < \gamma \leq \frac{\pi}{3} \). If \( \gamma = \frac{\pi}{4} \), we have \( \alpha_1 = \frac{2\pi}{3} = \gamma + \gamma \) and the decomposition illustrated in Figure 12(b), \( K = T \cup T' \). It follows that \( \alpha_3 > \frac{\pi}{3} \) (area of \( T' = K \setminus T \)), and so \( \alpha_3 + \alpha_3 + \rho = \pi \) is a condition that becomes impossible to be satisfied. Thus, \( \gamma < \frac{\pi}{3} \) and \( \gamma + \gamma + \gamma + k\alpha_3 = \pi \),...
Figure 11: Local configurations

Figure 12: Local configurations

$k \geq 1$. Nevertheless, in this case we also obtain a contradiction, see vertex $v_6$ in Figure 13(a).

2.2.3 Considering $\theta_3 = \alpha_2$, we obtain the configuration of Figure 13(b).

Now, $\theta_4 = \alpha_2$ or $\theta_4 = \beta$.

2.2.3.1 If $\theta_4 = \alpha_2$, and

(i) $\theta_5 = \alpha_3$ (Figure 14(a)), at vertex $v_7$ we have $\alpha_1 + \gamma = \pi = \alpha_3 + \rho$, with $\rho \in \{\beta, \alpha_2\}$. But this implies $\alpha_3 = \frac{\pi}{2}$, which is not possible.

(ii) $\theta_5 = \gamma$ (Figure 14(b)), according to the edge lengths, it follows that $b = c$. 
As $v_N$ and $v_S$ are in antipodal positions, we conclude that $b = \frac{\pi}{3}$. Then, as $\cos b = \frac{\cos \beta + \cos^2 \gamma}{\sin^2 \gamma}$, we obtain $\gamma = \arctan \sqrt{2}$. Since $T'$ is an equilateral triangle (Figure 15(a)), we have $\alpha_1 - \gamma = \alpha_3$. Solving the system

$$
\begin{align*}
\alpha_1 + \gamma &= \pi \\
\alpha_1 - \gamma &= \alpha_3
\end{align*}
$$

we get $\alpha_3 = \pi - 2 \arctan \sqrt{2}$. At vertices $v_N$ and $v_S$ results that $\alpha_3 + \alpha_3 +$
\( \rho > \pi \), for all \( \rho \in \{ \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \} \).

2.2.3.2 In the case \( \theta_4 = \beta \) the last configuration extends to the one illustrated in Figure 15(b). For each \( k \geq 3 \), there is a unique f-tiling denoted by \( \mathcal{E}^k \).

3D representations of \( \mathcal{E}^3 \) and \( \mathcal{E}^4 \) are given in Figure 16.

From equation (1) one has \( \gamma = \arccos \left( \frac{\pi}{2} \cos \frac{\pi}{2k} \right) \).

3. Suppose finally that \( \theta_1 = \alpha_2 \) (Figure 7(a)).

3.1 If \( \alpha_2 + \beta < \pi \) and

(i) \( \alpha_2 \geq \gamma \), then \( \alpha_2 + \beta + k\alpha_3 = \pi \), \( k \geq 1 \) (Figure 17(a)). As tiles 3 and 5 form the subcase analyzed in [2] (Proposition 2.3) within the case of the adjacency \( I \), this local configuration will not give rise to any f-tiling.

(ii) \( \gamma > \alpha_2 \) (Figure 17(b)), it follows that \( \alpha_1 + k\alpha_3 = \pi \), \( k \geq 1 \). The other sum of alternate angles must be of the form \( \gamma + k\alpha_3 = \pi \), i.e., \( \gamma = \alpha_1 \), and so \( \pi > \beta + \alpha_2 > \gamma + \alpha_2 = \alpha_1 + \alpha_2 > \pi \), which is a contradiction.

3.2 Suppose now that \( \alpha_2 + \beta = \pi \) (Figure 18(a)). At vertex \( v_2 \) we have \( \alpha_1 + k\alpha_3 = \pi \), \( k \geq 1 \), or \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t \geq 0 \). The first case leads to a contradiction, as we get \( \alpha_1 = \gamma \).

Thus, \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t \geq 0 \). If \( \theta_2 = \alpha_3 \), and so \( t > 0 \) (Figure 18(b)), we reach a contradiction at vertex \( v_4 \).

Therefore, \( \theta_2 = \gamma \) and we distinguish the cases \( \alpha_1 + \gamma < \pi \) and \( \alpha_1 + \gamma = \pi \), i.e., \( t > 0 \) and \( t = 0 \).
Figure 16: $f$-tilings $\mathcal{E}^k$, cases $k = 3$ and $k = 4$

Figure 17: Local configurations

Figure 18: Local configurations

If $\alpha_1 + \gamma < \pi$, i.e., $\alpha_1 + \gamma + t\alpha_3 = \pi$, $t \geq 1$, then $\alpha_1 \geq \alpha_2 > \gamma > \alpha_3$ and $\alpha_1 > \beta > \gamma > \alpha_3$. We then obtain the configuration illustrated in
Figure 19(a). At vertex $v_5$ we have $\beta + \beta \leq \pi$. If $\beta + \beta < \pi$, we would have $\beta + k\alpha_3 = \pi$, $k \geq 1$. But then the other sum of alternate angles is greater than $\pi$. Therefore, $\beta = \frac{\pi}{2} = \alpha_2$ and it follows that $\alpha_1 = \gamma + \alpha_3$ (see Figure 19(b)). Then, $\alpha_1 + \gamma + t\alpha_3 \geq \alpha_1 + \gamma + \alpha_3 = \alpha_1 + \alpha_1 > \pi$, which is not possible.

On the other hand, if $\alpha_1 + \gamma = \pi$ the last configuration extends to the one in Figure 20. Note that if $\alpha_1 + \alpha_3 < \pi$ at vertex $v_1$, taking into account the edge lengths, we get $\alpha_1 + \alpha_3 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2, \beta, \gamma\}$. Also, if $\theta_2 = \gamma$ at vertex $v_2$, we would have $\alpha_1 + \gamma = \pi = \rho + \alpha_3$, with $\rho \in \{\alpha_2, \beta\}$, which is an absurd.

For each $k \geq 3$, we get a unique f-tiling denoted by $\mathcal{H}^k$. 3D representations for $k = 3$ and $k = 4$ are given in Figure 21.

Solving equation (1) with $\alpha_1 = \pi - \gamma$, $\alpha_2 = \pi - \beta$ and $\gamma = \alpha_3 = \frac{\pi}{k}$, we get

$$
\frac{\cos \frac{\pi}{k}(1 + \cos \beta)}{\sin \frac{\pi}{k} \sin \beta} = \frac{\cos \frac{\pi}{2k} - \cos \frac{\pi}{k} \sin \frac{\beta}{2}}{\sin \frac{\pi}{k} \cos \frac{\beta}{2}},
$$

and so one has $\beta = 2 \arcsin \left(2 \cos \frac{\pi}{2k} - \sec \frac{\pi}{2k}\right)$. \hfill \Box

**Proposition 3.** If $\alpha_2 > \alpha_1 \geq \alpha_3$, then $\Omega(K,T)$ is the empty set.

**Proof.** Suppose that any element of $\Omega(K,T)$ has at least two cells congruent, respectively, to $K$ and $T$, such that they are in adjacent positions as
Moreover, taking into account the kite’s area, \( \alpha \) and so \( \in \{ \rho \} \) which is also an absurd.

Figure 22(a), we have

\[
\alpha_1 = \beta \quad \text{or} \quad \alpha_1 = \gamma.
\]

1. Suppose firstly that \( \alpha_1 = \beta \). We must have \( \alpha_2 + \beta = \pi \), otherwise we get \( \alpha_2 + \beta + k\alpha_3 = \pi = \beta + \rho + (k - 1)\alpha_3 + \rho > \beta + 2\gamma > \pi \), with \( k \geq 1 \) and \( \rho \in \{ \alpha_1, \gamma \} \) (in this case we have \( \alpha_2 > \frac{\pi}{2} > \beta \) and \( \alpha_1 > \beta > \gamma \)). Now we have \( \theta_2 \in \{ \alpha_1, \alpha_3, \gamma \} \) (Figure 22(b)).

1.1 If \( \theta_2 = \alpha_1 \), it follows that \( \alpha_1 \leq \frac{\pi}{2} \). Taking into account the kite’s area, we have \( \alpha_2 + \alpha_3 > \pi \), and so \( \alpha_3 > \beta \) (\( \alpha_2 > \alpha_1 \geq \alpha_3 > \beta > \gamma \)). If \( \alpha_1 + \alpha_1 < \pi \), we get \( \alpha_1 + \alpha_1 + \rho > \beta + 2\gamma > \pi \), for all \( \rho \in \{ \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \} \). On the other hand, if \( \alpha_1 + \alpha_1 = \pi \), we obtain \( \gamma + \rho = \pi \), with \( \rho \in \{ \alpha_1, \gamma \} \), implying \( \alpha_1 = \gamma \), which is also an absurd.

1.2 If \( \theta_2 = \alpha_3 \), then \( \alpha_1 + \alpha_3 \leq \pi \). If \( \alpha_3 \geq \beta \), then \( \alpha_2 > \alpha_1 \geq \alpha_3 \geq \beta > \gamma \), and so \( \alpha_1 + \alpha_3 < \alpha_2 + \beta = \pi \). But \( \alpha_1 + \alpha_3 + \rho > \beta + 2\gamma > \pi \), for all \( \rho \in \{ \alpha_1, \alpha_2, \alpha_3, \beta, \gamma \} \). On the other hand, if \( \beta > \alpha_3 \), then \( \alpha_2 + \alpha_3 < \alpha_2 + \beta = \pi \). Moreover, taking into account the kite’s area, \( \alpha_1 > \frac{\pi}{2} > \beta \) (\( \alpha_2 > \alpha_1 > \frac{\pi}{2} > \beta > \gamma, \alpha_3 \)). Analyzing the configuration of Figure 23(a), at vertex \( v_2 \) we must have \( \alpha_1 + \gamma + t\alpha_3 = \pi \), \( t \geq 1 \), leading to a contradiction at vertex \( v_3 \) \( \alpha_1 + \alpha_1 > \pi \).

1.3 We finally consider that \( \theta_2 = \gamma \). We have \( \alpha_1 + \gamma < \alpha_2 + \beta = \pi \). If \( \alpha_3 \geq \beta \), then \( \alpha_2 > \alpha_1 \geq \alpha_3 \geq \beta > \gamma \), and so \( \alpha_1 + \gamma + \rho > \beta + 2\gamma > \pi \), for all...
Figure 21: f-tilings $\mathcal{H}^k$, cases $k = 3$ and $k = 4$

Figure 22: Local configurations

$\rho \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$. On the other hand, if $\beta > \alpha_3$, then $\alpha_2 + \alpha_3 < \alpha_2 + \beta = \pi$. Therefore, $\alpha_1 > \frac{\alpha_2}{2} > \beta$ ($\alpha_2 > \alpha_1 > \frac{\alpha_2}{2} > \beta > \gamma, \alpha_3$). Analyzing the configuration of Figure 23(b), at vertices $v_1$ and $v_1$ we must have $\alpha_1 + \gamma + t\alpha_3 = \pi$, $t \geq 1$, and so $\alpha_2 > \alpha_1 > \frac{\alpha_2}{2} > \beta > \gamma > \alpha_3$, $\gamma > \frac{\pi}{4}$. Nevertheless, we reach an incompatibility on the sides at vertex $v_2$.

2. Suppose now that $\theta_1 = \gamma$ (see Figure 22(a)). We distinguish the cases $\alpha_2 + \gamma < \pi$ and $\alpha_2 + \gamma = \pi$.

2.1 If $\alpha_2 + \gamma < \pi$ ($\gamma < \frac{\pi}{2} < \alpha_2$) and

(i) $\alpha_3 \geq \gamma$, then $\gamma$ is the smallest angle and, as $\beta + 2\gamma > \pi$, $v_1$ must have valency four, which is an absurd.

(ii) $\gamma > \alpha_3$, we have $\alpha_2 > \alpha_1 > \gamma > \alpha_3$ and $\beta > \gamma > \alpha_3$. Moreover, $\alpha_1 > \frac{\pi}{2}$ since $\alpha_2 + \alpha_3 < \pi$. 
(a) If $\beta \geq \alpha_2$, then $\beta \geq \alpha_2 > \alpha_1 > \frac{\pi}{2} > \gamma > \alpha_3$. As we always have vertices of valency four, we have $\beta + \gamma = \pi$ or $\beta + \alpha_3 = \pi$. But, taking into account the angles and edge lengths at vertex $v_1$, we have necessarily $\beta + \gamma + k\alpha_3 = \pi$, with $k \geq 1$, which is a contradiction.

(b) If $\alpha_2 > \beta$, at vertex $v_1$ we must have $\alpha_2 + \gamma + k\alpha_3 = \pi = \beta + \alpha_1 + k\alpha_3$, $k \geq 1$ (Figure 24(a)). Note that if $\theta_2 = \gamma$, we would have $\beta + \gamma + k\alpha_3 = \pi$, $k \geq 1$, implying $\beta = \alpha_2$, which is not possible. It follows that $\alpha_2 > \alpha_1 > \frac{\pi}{2} > \beta > \gamma > \frac{\pi}{4} > \alpha_3$. As the condition

Figure 23: Local configurations

Figure 24: Local configurations
\(\theta_3 = \gamma\) originates two adjacent tiles from the case of the adjacency \(I\) ([2], Proposition 2.4), that in the case \(\alpha_2 + \gamma < \pi\) give not rise to \(f\)-tilings, we conclude that \(\theta_3 = \beta\) and the configuration extends to the one illustrated in Figure 24(b). Note that if \(\theta_4 = \gamma\), at vertex \(v_2\) we would have \(\alpha_1 + \gamma + t\alpha_3 = \pi\), \(t > k\), implying at vertex \(v_3\) the condition \(\alpha_1 + \alpha_1 > \pi\). The other possible choice for \(\theta_4\) is \(\alpha_3\), but in this case we also obtain a similar contradiction at vertex \(v_3\). Note that after tile 12, we have \(\gamma + \gamma + \gamma + (k - 1)\alpha_3 = \pi\), \(k \geq 1\), and the case \(\theta_5 = \beta\) was already studied in 1, and no \(f\)-tilings were achieved.

2.2 If \(\alpha_2 + \gamma = \pi\) (Figure 22(a)), we have \(\beta + \rho = \pi\), with \(\rho \in \{\gamma, \alpha_1\}\). The condition \(\rho = \gamma\) gives rise to adjacent tiles of the adjacency case \(I\), that, under these conditions, do not originate \(f\)-tilings ([2], Proposition 2.4). Therefore, \(\rho = \alpha_1\) (Figure 25(a)). It follows that \(\alpha_2 > \alpha_1, \beta\) and \(\alpha_1, \beta > \gamma\). Now, \(\theta_2 = \alpha_1\), \(\theta_2 = \gamma\) or \(\theta_2 = \beta\).

2.2.1 If \(\theta_2 = \alpha_1\), then \(\alpha_1 \leq \frac{\pi}{2} \leq \beta\). Taking into account the kite’s area, we have \(\alpha_2 + \alpha_3 > \pi\), i.e., \(\alpha_3 > \gamma\), implying that \(v_2\) must have valency four \((\alpha_1, \alpha_2, \alpha_1, \gamma)\). But then an incompatibility on the sides arises at this vertex.

2.2.2 If \(\theta_2 = \gamma\), we obtain adjacent tiles of the adjacency case \(I\), that once again and under the current conditions, give not rise to \(f\)-tilings ([2], Proposition 2.4).

2.2.3 If \(\theta_2 = \beta\), the last configuration extends to the one illustrated in Figure 25(b), and \(\theta_3 \in \{\gamma, \alpha_1, \alpha_3\}\).

2.2.3.1 If \(\theta_3 = \gamma\) and

(i) \(\beta \geq \alpha_1\) (\(\alpha_1 \leq \frac{\pi}{2}\)), then \(\alpha_2 + \alpha_3 > \pi\), and so \(\alpha_3 > \gamma\). At vertex \(v_4\) we get \(\beta + \gamma < \alpha_2 + \gamma = \pi\). But \(\beta + \rho_1 + \rho_2 > \pi\), for all \(\rho \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}\).
(ii) $\alpha_1 > \beta$, then $\alpha_2 > \alpha_1 > \frac{\pi}{2} > \beta > \gamma > \frac{\pi}{4}$, and we obtain the configuration illustrated in Figure 26(a). At vertex $v_5$ we obtain a contradiction, as $\alpha_2 + \beta > \pi$.

![Diagram](image_url)

Figure 26: Local configurations

2.2.3.2 If $\theta_3 = \alpha_1$ (Figure 25(b)), it follows that $\alpha_1 \leq \frac{\pi}{2}$. By the kite’s area, we obtain $\alpha_3 > \gamma$ and $\alpha_2 > \beta \geq \frac{\pi}{2} \geq \alpha_1 \geq \alpha_3 > \gamma$. Consequently, at vertex $v_4$ we have $\beta + \alpha_3 = \pi$, as $\beta + \alpha_3 + \rho > \pi$, for all $\rho \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$. Nevertheless, there is no way to satisfy the angle-folding relation around this vertex.

2.2.3.3 If $\theta_3 = \alpha_3$, we obtain the configuration illustrated in Figure 26(b). Extending this configuration, the polygonal lines A, B and C close on themselves. Below the polygonal line B exists a great circle, and so $\alpha_1 + \gamma + \alpha_3 > \pi$. In fact, let $R$ be the region below line B, $n$ the number of sides of length $b$ and $\alpha = \alpha_1 + \gamma + \alpha_3$. Thus, $n\alpha - (n - 2)\pi > 2\pi$, implying $\alpha > \pi$. 
Now, we have $\alpha_1 + \alpha_3 < \pi$ (see edge lengths), and so $\alpha_1 + t\alpha_3 = \pi$, with $t \geq 2$. Given the edge lengths, it also follows that $\gamma + t\alpha_3 = \pi$. But then $\gamma = \alpha_1$, which is an absurd. \hfill \Box

**Proposition 4.** If $\alpha_2 > \alpha_3 > \alpha_1$, then $\Omega(K, T)$ is the empty set.

**Proof.** Suppose that any element of $\Omega(K, T)$ has at least two cells congruent, respectively, to $K$ and $T$, such that they are in adjacent positions as illustrated in Figure 2−II and $\alpha_2 > \alpha_3 > \alpha_1$. With the labeling of Figure 27(a), we have

$$\theta_1 = \beta \quad \text{or} \quad \theta_1 = \gamma.$$ 

![Local configurations](image)

Figure 27: Local configurations

1. Suppose firstly that $\theta_1 = \beta$. In this case $\alpha_2 + \beta \leq \pi$ and $\alpha_2 > \alpha_3 > \alpha_1 > \beta > \gamma$. As $\beta + 2\gamma > \pi$, it follows that $\alpha_2 + \beta = \pi$ (Figure 27(b)). $\theta_2$ can be $\gamma$, $\alpha_3$ or $\alpha_1$. All these cases lead to a contradiction as

(i) if $\theta_2 = \gamma$, then $\alpha_1 + \gamma < \alpha_2 + \beta = \pi$; nevertheless, $\alpha_1 + \gamma + \rho > \beta + 2\gamma > \pi$, for all $\rho \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$.

(ii) if $\theta_2 = \alpha_3$, it follows that $\alpha_1 + \alpha_3 = \pi$; taking into account the angles and edge lengths, we reach a contradiction.

(iii) $\theta_2 = \alpha_1$ leads to $\alpha_1 + \alpha_1 = \pi = \gamma + \rho < \pi$, with $\rho \in \{\gamma, \alpha_1\}$.

2. Suppose finally that $\theta_1 = \gamma$. As $\alpha_2 + \gamma \leq \pi$, we conclude that $\alpha_2 > \alpha_3 > \alpha_1 > \gamma$.

2.1 If $\alpha_2 + \gamma < \pi$, then $\alpha_2 + k\gamma = \pi$, with $k \geq 2$. The other sum of alternate angles containing $\beta$ is $\beta + \rho_1 + \rho_2 + \ldots \geq \beta + \gamma + \gamma > \pi$, for all $\rho_1, \rho_2 \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\}$, which is an absurd.
2.2 Suppose now that \( \alpha_2 + \gamma = \pi = \beta + \theta_2 \), with \( \theta_2 \in \{\gamma, \alpha_1\} \). If \( \theta_2 = \gamma \) (Figure 28(a)), then \( \beta = \alpha_2 > \alpha_3 > \alpha_1 > \gamma \). But there is no way to satisfy the angle-folding relation around vertex \( v_2 \). On the other hand, if \( \theta_2 = \alpha_1 \)

\[
\alpha_2 > \beta > \gamma
\]

(Figure 28(b)). As \( \beta + \rho_1 + \rho_2 > \beta + 2\gamma > \pi \), for all \( \rho_1, \rho_2 \in \{\alpha_1, \alpha_2, \alpha_3, \beta, \gamma\} \), we get \( \beta + \alpha_1 = \pi \) (tile 7). Consequently, \( \alpha_1 + \alpha_3 \leq \pi \) and \( \alpha_2 > \beta \geq \alpha_3 > \alpha_1 > \gamma \). Therefore, this case results analogous to the one studied in the previous proposition in 2.2.3.3.

Concerning to the combinatorial structure of each tiling obtained, we have that

(i) any symmetry of \( \mathcal{E}^k, k \geq 3 \), fixes \( N = (0, 0, 1) \) (and consequently \( S = -N \)) or maps \( N \) into \( S \) (and consequently \( S \) into \( N \)). The symmetries that fix \( N \) are generated, for instance, by the rotation \( R^z_{\pi} \) (around the \( z \) axis) and the reflection \( \rho^{yz} \) (on the coordinate plane \( yoz \)), giving rise to a subgroup of \( G(\mathcal{E}^k) \) isomorphic to \( D_{2k} \) (the dihedral group of order \( 4k \)).

Now, the map \( \phi = \rho^{xy} \) is a symmetry of \( \mathcal{E}^k \) that permutes \( N \) and \( S \) allowing us to get all the symmetries that map \( N \) into \( S \). Now, since \( \phi \) commutes with \( R^z_{\pi} \) and \( \rho^{yz} \), then it follows that \( G(\mathcal{E}^k) \) is isomorphic to \( C_2 \times D_{2k} \).

(ii) any symmetry of \( \mathcal{H}^k, k \geq 3 \), fixes \( N \) or maps \( N \) into \( S \). The symmetries that fix \( N \) are generated, for instance, by the rotation \( R^z_{\pi} \) and the reflection \( \rho^{yz} \), giving rise to a subgroup of \( G(\mathcal{H}^k) \) isomorphic to \( D_k \).

The map \( \phi = R^z_{\pi} \circ \rho^{xy} = \rho^{xy} \circ R^z_{\pi} \) is a symmetry of \( \mathcal{H}^k \) that changes \( N \) and \( S \). One has \( \phi^{2k-1} \circ \rho^{yz} = \rho^{yz} \circ \phi \) and \( \phi \) has order \( 2k \). It follows that \( \phi \) and \( \rho^{yz} \) generate \( G(\mathcal{H}^k) \). And so it is isomorphic to \( D_{2k} \).
3. Summary

In Table 1 is shown a list of the spherical dihedral f-tilings whose prototiles are a spherical kite and an isosceles spherical triangle, $K$ and $T$, of internal angles $(\alpha_1, \alpha_2, \alpha_1, \alpha_3)$, and $(\beta, \gamma, \gamma)$, respectively, in case of adjacency $II$. Our notation is as follows:

- $\gamma^k_0$ is the solution of equation (1), with $\alpha_2 = \beta = \frac{\pi}{2}$, $\alpha_1 = \pi - \gamma^k_0$ and $\alpha_3 = \frac{\pi}{k}$, with $k \geq 3$; $\beta^k_1$ is the solution of equation (1), with $\alpha_2 = \pi - \beta^k_1$, $\alpha_1 = \pi - \gamma$ and $\gamma = \alpha_3 = \frac{\pi}{k}$, with $k \geq 3$.

- $|V|$ is the number of distinct classes of congruent vertices;

- $N_1$ and $N_2$ are, respectively, the number of triangles congruent to $K$ and $T$, respectively, used in the dihedral f-tilings.

- $G(\tau)$ is the symmetry group of each tiling $\tau \in \Omega (K, T)$.

| $f$-tiling | $\alpha_1$ | $\alpha_2$ | $\alpha_3$ | $\beta$ | $\gamma$ | $|V|$ | $N_1$ | $N_2$ | $G(\tau)$ |
|-----------|-----------|-----------|-----------|---------|---------|------|------|------|---------|
| $E^k$, $k \geq 3$ | $\pi - \gamma^k_0$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | $\gamma^k_0$ | 3 \(4k\) \(4k\) | $C_2 \times D_{2k}$ |
| $H^k$, $k \geq 3$ | $\pi - \gamma$ | $\pi - \beta^k_1$ | $\frac{\pi}{2}$ | $\beta^k_1$ | $\gamma$ | 4 \(4k\) \(4k\) | | $D_{4k}$ |

Table 1: Combinatorial structure of dihedral $f$–tilings of $S^2$ by spherical kites and isosceles triangles in case of adjacency $II$

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