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$t-$DERIVATIONS ON $TM-$ALGEBRAS

T. Ganeshkumar[1], M. Chandramouleeswaran[2 §]

[1]Department of Mathematics
M.S.S. Wakf Board College
Madurai, 625020, INDIA

[2]Department of Mathematics
SBK College
Arupukottai, 626101, INDIA

Abstract: Recently an algebra based on proportional calculi was introdued by Tamilarasi and Mekalai in the year 2010 known as $TM-$algebras [3]. Kandaraj and Chandramouleeswaran [5] introduced the notion of derivation on $d-$algebras. In [6], we introduced the notion of derivations on TM-algebras. In this paper, we introduce the notion of $t-$derivation on TM-algebras. We study the properties of regular $t-$derivations on a TM-algebra and prove that the set of all $t-$derivations on a TM-algebra forms a semigroup under a suitable binary composition.

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1. Introduction

It is well known that BCK and BCI-algebras are two classes of algebras of logic. They were introduced by Imai and Iseki [1] and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper sub class of the BCI-algebras. J.Neggers and H.S.Kim [2] introduced the notion of $d-$algebras which is another generalization of BCK-algebras.

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§Correspondence author

Recently another algebra based on proportional calculi was introdued by Tamilarasi and Mekalai in the year 2010 known as $TM-$algebras. In their paper [3] they claimed that $TM-$algebra was the generalization of BCK and BCI algebras. But this was proved wrong in [4], by giving counter examples.

Motivated by the notion of derivations on rings and near-rings Jun and Xin [7]studied the notion of derivation on BCI-algebras. In [5], the authors introduced the notion of derivation on $d-$algebras, another generalisation of BCK-algebras.

In our paper [6], we introduced the notion of derivation on $TM-$algebras. In this paper, we introduce the notion of $t-$derivation on TM-algebras. We study the properties of regular $t-$derivations on X and prove that the set of all $t-$derivations on a TM-algebra X forms a semigroup under a suitable binary composition.

2. Preliminaries

In this section, we recall some basic definitions and results that are needed for our work.

Definition 2.1. A $TM-$algebra $(X, *, 0)$ is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * 0 = x$
2. $(x * y) * (x * z) = z * y \ \forall \ x, y, z \in X$.

Lemma 2.2. The following properties hold in a $TM-$algebra X.

1. $x * x = 0$.
2. $(x * y) * x = 0 * y$.
3. $x * (x * y) = y$.
4. $(x * y) * z = (x * z) * y$.
5. $x * 0 = 0 \ \Rightarrow x = 0$. In other words $x \leq 0 \ \Rightarrow x = 0$.
6. $0 * (x * y) = y * x = (0 * x) * (0 * y)$.
7. $(x * z) * (y * z) = (x * y)$.

Remark 2.3. In a $TM-$algebra X, by definition, $x \wedge y = y * (y * x)$. However, by property (3) above, we have $x = y * (y * x)$. Hence, in a TM-algebra we have $x \wedge y = x \ \ \forall \ x, y \in X$.

Definition 2.4. In any $TM-$algebra X, we define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$.

Definition 2.5. A non-empty subset S of a $TM-$algebra $(X, *, 0)$ is said to be a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 2.6. Let $(X, *, 0)$ be a $TM-$algebra. A self map $d : X \to X$ is said to be a $(l, r)-$derivation on X, if $d(x * y) = (d(x) * y) \wedge (x * d(y))$. d is said to be a $(r, l)-$derivation on X, if $d(x * y) = (x * d(y)) \wedge (d(x) * y)$. It is said to be a derivation on X if d is both a $(l, r)-$derivation and a $(r, l)-$derivation on X.

3. $t-$Derivations on $TM-$Algebra

Definition 3.1. A $TM-$algebra X is said to be associative if $(x * y) * z = x * (y * z)$ for all $x, y, z \in X$.

Example 3.2. Let $(X, *, 0)$ be a $TM-$algebra with the Cayley table.

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then X is an associative $TM-$algebra.

Definition 3.3. Let X be a $TM-$algebra. Then for any $t \in X$, we define a self map $d_t : X \to X$ by $d_t(x) = x * t$ for all $x \in X$.

Definition 3.4. Let X be a $TM-$algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a $(l, r) - t-$derivation of X if it satisfies the condition $d_t(x * y) = (d_t(x) * y) \wedge (x * d_t(y))$ for all $x, y \in X$.

Example 3.5. Let $(X, *, 0)$ be a $TM-$algebra with the following Cayley table.

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Define, when $t = 0$, $d_t(x) = x \ \forall\ x \in X$.

when $t = 1$, $d_t(0) = 2$, $d_t(1) = 0$, $d_t(2) = 1$.

when $t = 2$, $d_t(0) = 1$, $d_t(1) = 2$, $d_t(2) = 0$.

For each $t \in X$, d_t is a $(l, r) - t-$derivation of X.

Remark 3.6. In a $TM-$algebra, $x \wedge y = y * (y * x) = x \ \forall\ x, y \in X$. We can observe that by using the above property we take d_t is a $(l, r) - t-$derivation of X then $d_t(x * y) = d_t(x) * y$.

Definition 3.7. Let X be a $TM-$algebra. Then for any $t \in X$ a self map $d_t : X \to X$ is called a $(r, l) - t-$derivation of X if it satisfies the condition $d_t(x * y) = (x * d_t(y)) \wedge (d_t(x) * y)$ for all $x, y \in X$.

Remark 3.8. We can observe that, if d_t is a $(r, l) - t-$derivation of X, then $d_t(x * y) = x * d_t(y)$ for all $x, y \in X$.

Definition 3.9. Let X be a $TM-$algebra. Then for any $t \in X$, a self map $d_t : X \to X$ is called a $t-$derivation on X if d_t is both a $(l, r) - t-$derivation and a $(r, l) - t-$derivation on X.

Example 3.10. Consider the TM-algebra $(X, *, 0)$ in 3.2. Define the mapping d_t as follows:

Whenr $t = 0, d_t(x) = x \ \forall\ x \in X$.

When $t = 1$, $d_t(0) = 1$, $d_t(1) = 0$, $d_t(2) = 3$, $d_t(3) = 2$.

When $t = 2$, $d_t(0) = 2$, $d_t(1) = 3$, $d_t(2) = 0$, $d_t(3) = 1$.

When $t = 3$, $d_t(0) = 3$, $d_t(1) = 2$, $d_t(2) = 1, d_t(3) = 0$.

For each $t \in X$, d_t is a $t-$derivation of X.

Remark 3.11. Any self map d_t of a $TM-$algebra X is a $(l, r) - t-$derivation on X.

Proposition 3.12. Let d_t be a self map of an associative $TM-$algebra X. Then d_t is a $(r, l) - t-$derivation of X.

Proof. Let X be an associative $TM-$algebra. Then we have

$$\begin{aligned}
d_t(x * y) &= (x * y) * t \\
&= (x * t) * y \qquad (\because\ (x * y) * z = (x * z) * y) \\
&= ((x * t) * y) * 0 \\
&= ((x * t) * y) * (((x * t) * y) * ((x * t) * y)) \\
&\quad (\because\ x * x = 0) \\
&= (((x * t) * y) * (((x * t) * y) * ((x * y) * t)) \\
&\quad (\because\ (x * y) * z = (x * z) * y)
\end{aligned}$$

$$
\begin{aligned}
&= ((x * t) * y) * (((x * t) * y) * (x * (y * t))) \\
&\quad (\because \ X \ is \ a \ associative \ TM - algebra \ ) \\
&= (x * (y * t)) \wedge ((x * t) * y) \\
&= (x * d_t(y)) \wedge (d_t(x) * y)
\end{aligned}
$$

$\therefore$ d_t is a $(r, l) - t-$derivation of X.

By combining the remark 3.11 and proposition 3.12, we get the following theorem.

Theorem 3.13. Let X be an associative $TM-$algebra. For ant $t \in X$, a self map d_t is a $t-$derivation on X.

Definition 3.14. A self map d_t of a $TM-$algebra X is said to be $t-$regular if $d_t(0) = 0$.

Example 3.15. In example 3.10 d_t is a regular $t-$derivation on X when $t = 0$. However, $t = 1$ or $t = 2$ or $t = 3$, d_t is not a regular $t-$derivation of X.

Proposition 3.16. For any self map d_t of a $TM-$algebra X, the following holds:

1. If d_t is a $(l, r) - t-$derivation of X, $d_t(x) = d_t(x) \wedge x \ \forall \ x \in X$.
2. If d_t is a $(r, l) - t-$derivation of X, $d_t(x) = x \wedge d_t(x)$ for all $x \in X$ if and only if d_t is $t-$regular.

Proof.

1. Let d_t be a $(l, r) - t-$derivation of X. Then we have

$$
\begin{aligned}
d_t(x) &= d_t(x * 0) \\
&= (d_t(x) * 0) \wedge (x * d_t(0)) \\
&= d_t(x) \wedge (x * d_t(0)) \\
&= (x * d_t(0)) * ((x * d_t(0)) * d_t(x)) \\
&= (x * d_t(0)) * ((x * d_t(x)) * d_t(0) \quad (\because \ (x * y) * z = x * (z * y)) \\
&= x * (x * d_t(x)) \qquad (\because \ (x * z) * (y * z) = x * y) \\
&= d_t(x) \wedge x
\end{aligned}
$$

$\therefore$ d_t is a $(l, r) - t-$derivation of X.

2. Let d_t be a $(r, l) - t-$derivation of X and $d_t(x) = x \wedge d_t(x) \quad \cdots\cdots (1)$.

Put $x = 0$ in (1), we have

$$\begin{aligned} d_t(0) &= 0 \wedge d_t(0) \\ &= d_t(0) * (d_t(0) * 0) \\ &= d_t(0) * d_t(0) \\ &= 0 \end{aligned}$$

$\therefore$ d_t is $t-$regular.

Conversely, suppose that d_t is $t-$regular $(r, l) - t-$derivation of X. Then

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= (x * d_t(0)) \wedge (d_t(x) * 0) \\ &= (x * 0) \wedge d_t(x) \quad (\because \ d_t(0) = 0) \\ &= x \wedge d_t(x) \end{aligned}$$

Hence the proof.

Theorem 3.17. Let d_t be a $(l, r) - t-$derivation of a $TM-$algebra. Then the following hold.

1. $d_t(0) = d_t(x) * x \ \ \forall \ x \in X$.
2. d_t is one -one.
3. d_t is $t-$regular then it is the identity map.
4. If there is an element $x \in X$ such that $d_t(x) = x$, then d_t is the identity map.
5. If $x \leq y$ then $d_t(x) \leq d_t(y)$ for all $x, y \in X$.

Proof.

1. Let d_t be a $(l, r) - t-$derivation of a $TM-$algebra X.

   Then we have $d_t(0) = d_t(x*x) = d_t(x)*x \quad (\because \ d_t$ is a $(l, r)-t-$derivation)

2. Let $d_t(x) = d_t(y)$ for all $x, y \in X$.

   Then $x * t = y * t$ and by applying the right cancellation law we have, $x = y$.

3. Let d_t be a $t-$regular and $x \in X$. Now,

$$x * x = 0 = d_t(0) = d_t(x * x) = d_t(x) * x$$

Hence by right cancellation law, $d_t(x) = x \ \forall \ x \in X$, showing that d_t is the identity map.

4. Let $d_t(x) = x$ for some $x \in X$.

$$0 = x * x = d_t(x) * x = d_t(x * x) = d_t(0).$$

showing that d_t is $t-$regular. Hence by (3) d_t is the identity map

5. Since $x \le y$,

$$d_t(x) * d_t(y) = (x * t) * (y * t) = (x * y) = 0$$

thus proving $d_t(x) \le d_t(y)$.

Theorem 3.18. Let X be a $TM-$algebra and d_t be a $t-$derivation on X. If $x \le y$ and $d_t(x * y) = d_t(x) * d_t(y)$ for all $x, y \in X$. Then $d_t(x) = d_t(y)$.

Proof.

$$\begin{aligned} d_t(x) &= d_t(x * 0) \\ &= d_t(x * (x * y)) \quad (\because \ x \le y) \\ &= d_t(x) * d_t(x * y) \quad (\because \ d_t(x * y) = d_t(x) * d_t(y)) \\ &= d_t(x) * (d_t(x) * d_t(y)) \\ &= d_t(y) \quad (\because \ x * (x * y) = y) \end{aligned}$$

Theorem 3.19. Let d_t be a $t-$regular $(r, l)-t-$derivation of a $TM-$algebra X. Then the following hold.

1. $d_t(x) = x$.
2. $d_t(x) * y = x * d_t(y)$ for all $x, y \in X$.
3. $d_t(x * y) = d_t(x) * y = d_t(x) * d_t(y) = x * d_t(y)$.
4. $Ker(d_t) = \{x \in X : d_t(x) = 0\}$ is a sub algebra of X.

Proof.

1. Since d_t is $t-$regular $(r, l) - t-$derivation of X, for any $x \in X$, we have

$$d_t(x) = d_t(x * 0) = x * d_t(0) = x * 0 = x.$$

2. If d_t is $t-$regular $(r,l)-t-$derivation of X then by (1), $d_t(x) = x$ for all $x \in X$.

   Thus, $d_t(x) * y = x * y = x * d_t(y)$.

3. If d_t is $t-$regular $(r,l)-t-$derivation of X then by (1), $d_t(x) = x \; \forall \; x \in X \; \cdots\cdots (1)$.

   For $x, y \in X$, $d_t(x * y) = x * y = d_t(x) * d_t(y)$ (By (1))

   If d_t is a $(r,l)-t-$derivation of X then $d_t(x * y) = x * d_t(y)$.

   $d_t(x * y) = x * y = d_t(x) * y$. $(\because \; x = d_t(x))$

   Hence $d_t(x * y) = d_t(x) * y = x * d_t(y) = x * y$.

4. Since d_t is $t-$regular, $d_t(0) = 0$. Then $0 \in Ker(d_t)$ showing that $Ker(d_t)$ is a non-empty set.

   Let $x, y \in Ker(d_t)$, then $d_t(x) = 0, \quad d_t(y) = 0$. Now

$$d_t(x * y) = x * y = d_t(x) * d_t(y) = 0 * 0 = 0.$$

   Therefore $(x * y) \in Ker(d_t)$, proving that $Ker(d_t)$ is a sub-algebra of X.

Proposition 3.20. Let X be a $TM-$algebra. Then $Ker(d_t) = \{0\}$ if and only if d_t is $t-$regular.

Proof. Obviously when $Ker(d_t) = \{0\}$ $d_t(0) = 0$, showing that d_t is $t-$regular. On the other hand, if $x \in Ker(d_t)$, d_t is $t-$regular shows that,

$$0 = d_t(0) = d_t(x * x) = d_t(x) * x = 0 * x.$$

Thus, $x = 0$, showing that $Ker(d_t) = \{0\}$.

Definition 3.21. Let X be a $TM-$algebra and let d_t, d_t' be two self maps of X. Then we define $d_t \circ d_t' : X \to X$ by $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$.

Example 3.22. Consider the TM-algebra given in example 3.2. The self-maps $d_t, d_t' : X \to X$ given by
$d_t(0) = 1, d_t(1) = 0, d_t(2) = 3, d_t(3) = 2$
$d_t'(0) = 2, d_t'(1) = 3, d_t'(2) = 0, d_t'(3) = 1$ are $t-$derivations on X.
Now define a self map $(d_t \circ d_t') : X \to X$ by
$(d_t \circ d_t')(0) = 3, (d_t \circ d_t')(1) = 2, (d_t \circ d_t')(2) = 1, (d_t \circ d_t')(3) = 0$.
Then it is easily checked that $(d_t \circ d_t')(x) = d_t(d_t'(x))$ for all $x \in X$ is also a $t-$derivation of X.

Proposition 3.23. Let X be a $TM-$algebra and let d_t, d'_t be a $(l,r)-t-$derivation of X. Then $(d_t \circ d'_t)$ is also a $(l,r)-t-$derivation of X.

Proof. Let X be a $TM-$algebra and let d_t, d'_t be $(l,r)-t-$derivation of X. Then for all $x, y \in X$. We have

$$\begin{aligned} (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t(d'_t(x) * y) \quad (\because d'_t \textit{ is a } (l,r)-t-\textit{derivation of } X\ ) \\ &= (d_t(d'_t(x)) * y \quad (\because d_t \textit{ is a } (l,r)-t-\textit{derivation of } X\ ) \\ &= (d_t \circ d'_t)(x) * y \end{aligned}$$

$\therefore$ $(d_t \circ d'_t)$ is a $(l,r)-t-$derivation of X.

Proposition 3.24. Let X be a $TM-$algebra and let d_t, d'_t be $(r,l)-t-$derivations of X. Then $(d_t \circ d'_t)$ is also a $(r,l)-t-$derivation of X.

Proof. Let X be a $TM-$algebra and let d_t, d'_t be a $(r,l)-t-$derivation of X.

Then for all $x, y \in X$. We have

$$\begin{aligned} (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t(x * d'_t(y)) \quad (\because d'_t \textit{ is a } (r,l)-t-\textit{derivation of } X\ ) \\ &= x * d_t(d'_t(y)) \quad (\because d_t \textit{ is a } (r,l)-t-\textit{derivation of } X\ ) \\ &= x * (d_t \circ d'_t)(y) \end{aligned}$$

$\therefore$ $(d_t \circ d'_t)$ is a $(r,l)-t-$derivation of X.

Combining the above two propositions we get the following theorem.

Theorem 3.25. Let X be a $TM-$algebra and let d_t, d'_t be $t-$derivation of X. Then $(d_t \circ d'_t)$ is also a $t-$derivation of X.

Theorem 3.26. Let X be a $TM-$algebra. Let d_t be a $(r,l)-t-$derivation of X and d'_t be a $(l,r)-t-$derivation of X. Then $d_t \circ d'_t = d'_t \circ d_t$.

Proof. Let d'_t be a $(l,r)-t-$derivation of X. Then we have
$d'_t(x * y) = d'_t(x) * y$.

$$\begin{aligned} \textit{Now } (d_t \circ d'_t)(x * y) &= d_t(d'_t(x * y)) \\ &= d_t(d'_t(x) * y) \\ &= d'_t(x) * d_t(y) \quad \cdots\cdots (1) \\ &\quad (\because d_t \textit{ is a } (r,l)-t-\textit{derivation of } X\ ) \end{aligned}$$

$$\textit{Again } (d'_t \circ d_t)(x * y) = d'_t(d_t(x * y))$$

$$\begin{aligned} &= d_t'(x * d_t(y)) \\ &\quad (\because d_t \textit{ is a } (r,l)-t-\textit{derivation of } X\ ) \\ &= d_t'(x) * d_t(y) \quad \cdots\cdots (2) \\ &\quad (\because\ d_t' \textit{ is a } (l,r)-t-\textit{derivation of } X\ ) \end{aligned}$$

From (1) and (2), $(d_t \circ d_t'(x * y) = (d_t' \circ d_t)(x * y)$.

This is true for all $x, y \in X$. In particular this true for all x and $y = 0$.

Put $y = 0$, $(d_t \circ d_t')(x * 0) = (d_t' \circ d_t)(x * 0)$

$(d_t \circ d_t')(x) = (d_t' \circ d_t)(x)$ for all $x \in X$.

Hence $d_t \circ d_t' = d_t' \circ d_t$.

The following theorem can be easily obtained by above theorem 3.26.

Theorem 3.27. Let X be a $TM-$algebra and let d_t, d_t' be two $t-$derivations of X, then $d_t \circ d_t' = d_t' \circ d_t$.

Definition 3.28. Let X be a $TM-$algebra and let d_t, d_t' be two self maps of X. Then we define $d_t * d_t' : X \to X$ defined by $(d_t * d_t')(x) = d_t(x) * d_t'(x)$ for all $x \in X$.

Example 3.29. Consider the TM-algebra $(X, *, 0)$ given in 3.2. Define $d_t : X \to X$ by

$d_t(0) = 1,\ d_t(1) = 0,\ d_t(2) = 3,\ d_t(3) = 2$ be a $t-$derivation of X.

Define $d_t' : X \to X$ by $d_t'(0) = 2,\ d_t'(1) = 3,\ d_t'(2) = 0,\ d_t'(3) = 1$ be a $t-$derivation of X.

Now $(d_t * d_t')(0) = 3 = d_t(0) * d_t'(0)$.

$(d_t * d_t')(1) = 3 = d_t(1) * d_t'(1)$ $(d_t * d_t')(2) = 3 = d_t(2) * d_t'(2)$ $(d_t * d_t')(3) = 3 = d_t(3) * d_t'(3)$.

Theorem 3.30. Let X be a $TM-$algebra and let d_t, d_t' be two $t-$derivations of X. Then $d_t * d_t' = d_t' * d_t$.

Proof. Let X be a $TM-$algebra and let d_t, d_t' be $t-$derivation of X. Then we have

$$\begin{aligned} (d_t \circ d_t')(x * y) &= d_t(d_t'(x * y)) \\ &= d_t(d_t'(x) * y) \\ &= d_t(d_t'(x) * y) \quad (\because\ d_t' \textit{ is a } (l,r)-t-\textit{derivation of } X\ ) \\ &= d_t'(x) * d_t(y) \quad \cdots\cdots (1) \\ &\quad (\because\ d_t \textit{ is a } (r,l)-t-\textit{derivation of } X\ ) \end{aligned}$$

$$\textit{Again } (d_t \circ d_t')(x * y) = d_t(d_t'(x * y))$$

$$\begin{aligned} &= d_t(x * d_t'(y)) \\ &\quad (\because\ d_t' \ is\ a\ (r,l) - t - derivation\ of\ X\ ) \\ &= d_t(x) * d_t'(y) \quad \cdots\cdots (2) \\ &\quad (\because\ d_t \ is\ a\ (l,r) - t - derivation\ of\ X\ ) \end{aligned}$$

From (1) and (2), $d_t(x) * d_t'(y) = d_t'(x) * d_t(y) \quad \cdots\cdots (3)$.

Put $x = y$ in (3), $d_t(x) * d_t'(x) = d_t'(x) * d_t(x)$.

$(d_t * d_t')(x) = (d_t' * d_t)(x)$ (By definition 3.28)

Hence $d_t * d_t' = d_t' * d_t$.

Definition 3.31. Let $L_tDer(X)$ denote the set of all $(l,r)-t-$derivations of X. Define the binary operation $\wedge$ on $L_tDer(X)$ as follows: For $d_t, d_t' \in L_tDer(X)$, define $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x)\ \forall\ x \in X$.

Lemma 3.32. If d_t and d_t' are $(l,r)-t-$derivations on X. Then $(d_t \wedge d_t')$ is also a $(l,r)-t-$derivation on X.

Proof. Let d_t, d_t' be $(l,r)-t-$derivation on X. Then we have

$$\begin{aligned} (d_t \wedge d_t')(x * y) &= d_t(x * y) \wedge d_t'(x * y) \quad (\ By\ definition\ ) \\ &= (d_t(x) * y) \wedge (d_t'(x) * y) \\ &\quad (\because\ d_t, d_t' \ are\ (l,r) - t - derivations\ ) \\ &= (d_t'(x) * y) * ((d_t'(x) * y) * (d_t(x) * y)) \\ &= d_t(x) * y \quad \cdots\cdots (1). \end{aligned}$$

Again,

$$\begin{aligned} (d_t \wedge d_t')(x) * y &= (d_t(x) \wedge d_t'(x)) * y \\ &= (d_t'(x) * (d_t'(x) * d_t(x))) * y \\ &= d_t(x) * y \quad \cdots\cdots (2) \end{aligned}$$

From (1) and (2), $(d_t \wedge d_t')(x * y) = (d_t \wedge d_t')(x) * y$.

Hence $(d_t \wedge d_t')$ is a $(l,r)-t-$derivation of X.

Lemma 3.33. The binary composition $\wedge$ defined on $L_tDer(X)$ is associative.

Proof. Let X be a $TM-$algebra. Let d_t, d_t', d_t'' be $(l,r)-t-$derivations on X. Now,

$$\begin{aligned} ((d_t \wedge d_t') \wedge d_t'')(x * y) &= (d_t \wedge d_t')(x * y) \wedge d_t''(x * y) \\ &= (d_t(x) * y) \wedge (d_t''(x) * y) \quad (\ by\ lemma\ 3.32\ ) \end{aligned}$$

$$\begin{aligned} &= (d_t''(x) * y) * ((d_t''(x) * y) * (d_t(x) * y)) \\ &= d_t(x) * y \quad \cdots\cdots (1) \quad (\because\ y * (y * x) = x) \end{aligned}$$

Again

$$\begin{aligned} (d_t \wedge (d_t' \wedge d_t'')(x * y) &= d_t(x * y) \wedge (d_t' \wedge d_t'')(x * y) \\ &= d_t(x * y) \wedge (d_t'(x) * y) \quad (\ By\ lemma 3.32\ ) \\ &= (d_t(x) * y) \wedge (d_t'(x) * y) \\ &\quad (\because\ d_t\ is\ a\ (l, r) - t - derivation\ of\ X\ ) \\ &= (d_t'(x) * y) * ((d_t'(x) * y) * (d_t(x) * y)) \\ &= d_t(x) * y \quad \cdots\cdots (2) \end{aligned}$$

From (1) and (2), $((d_t \wedge d_t') \wedge (x * y) = (d_t \wedge (d_t' \wedge d_t'')(x * y)$.

Put $y = 0$, we get $((d_t \wedge (d_t' \wedge d_t'')(x) = (d_t \wedge (d_t' \wedge d_t''))(x)$ for all $x \in X$.

Hence $(d_t \wedge d_t') \wedge d_t'' = d_t \wedge (d_t' \wedge d_t'')$.

This prove that the binary operation $\wedge$ is associative.

Combining the above two lemmas we get the following theorem.

Theorem 3.34. $L_tDer(X)$ is a semi-group under the binary operation $\wedge$ defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t''(x)$ for all $x \in X$, and $d_t, d_t' \in L_tDer(X)$.

Definition 3.35. Let $R_tDer(X)$ denote the set of all $(r, l) - t-$derivations on X. Define the binary operation $\wedge$ on $R_tDer(X)$ as follows: For $d_t, d_t' \in R_tDer(X)$. Define $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x)$.

Analogously we prove the following theorem.

Theorem 3.36. $R_tDer(X)$ is a semi-group under the binary operation $\wedge$defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x)$ for all $x \in X$ and $d_t, d_t' \in R_tDer(X)$.

Combining the above two theorems we get the following theorem

Theorem 3.37. If ${}_tDer(X)$ denotes the set of all $t-$derivations on X then it is a semi-group under the binary operation $\wedge$ defined by $(d_t \wedge d_t')(x) = d_t(x) \wedge d_t'(x)$ for all $x \in X$ and $d_t, d_t' \in {}_tDer(X)$,

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