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MAJORIZATION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS USING HURWITZ LERCH ZETA FUNCTION

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Abstract: In the present paper, we investigate the majorization problems for functions belonging to the class $J^{p,n}_{s,b}f(z)$ is considered .Moreover we pointout some new or known consequences of our main result.

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1. Introduction

Let $A_p(n)$ be the class of functions which are analytic and p-valent in the unit disk $U = \{z \in C : |z| < 1\}$ of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} (p, n \in N = \{1, 2, ...\} \tag{1}$$

For $g(z) \in A_p(n)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \tag{2}$$

the Hadamard product of $f(z)$ and $g(z)$ is denoted by

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$$(f*g)(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g*f)(z) \tag{3}$$

The following we recall a general Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ defined by (see [16])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{4}$$

$(a \in \mathcal{C}\ \{\mathcal{Z}_0^-\}; s \in \mathcal{C}, \Re(s) > 1\ and\ |z| = 1)$ where, as usual, $\mathcal{Z}_0^- := \mathcal{Z} \setminus \{\mathcal{N}\}$ $(\mathcal{Z} := \{0, \pm 1, \pm 2, \pm 3, ...\}; \mathcal{N} := \{1, 2, 3, ...\})$. Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z,s,a)$ can be found in the recent investigations by Choi and Srivastava [3], Ferreira and Lopez [4], Garg et al. [5], Lin and Srivastava [7], Lin et al. [8], and others.

For the class of analytic functions denote by $\mathcal{A}$ consisting of functions of the form $f(z) = z + \sum\limits_{k=2}^{\infty} a_k z^k$ Srivastava and Attiya [15] (see also Raducanu and Srivastava [13]) introduced and investigated the linear operator:

$$\mathcal{J}_{s,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product (or convolution) by

$$\mathcal{J}_{s,b} f(z) = \mathcal{G}_{s,b} * f(z) \tag{5}$$

$(z \in U; b \in \mathcal{C} \setminus \{\mathcal{Z}_0^-\}; \mu \in \mathcal{C}; f \in \mathcal{A})$, where, for convenience,

$$G_{s,b}(z) := (1+b)^s [\Phi(z,s,b) - b^{-s}] \quad (z \in U). \tag{6}$$

It is easy to observe from (given earlier by [13]) (1), (5) and (6)that

$$\mathcal{J}_{s,b} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+b}{k+b} \right)^s a_k z^k. \tag{7}$$

Motivated essentially by the above-mentioned Srivastava-Attiya operator, we define the operator

$$\mathcal{J}_{s,b}^{p,n}(f) : \mathcal{A}_p(n) \to \mathcal{A}_p(n)$$

which is defined as

$$\mathcal{J}_{s,b}^{p,n} f(z) = z^p + \sum_{k=1}^{\infty} C_b^s(p,n) a_{p+k} z^{p+k} \quad (z \in U; f(z) \in \mathcal{A}_p) \tag{8}$$

where

$$C_b^s(k,p) = \left(\frac{p+b}{k+p+b}\right)^s \tag{9}$$

and (throughout this paper unless otherwise mentioned) the parameters $s, b,$ are constrained as

$$b \in \mathcal{C} \setminus \{\mathcal{Z}_0^-\}; s \in \mathcal{C} \quad and \quad p, \in N.$$

It is easily verified from (8)

$$z(\mathcal{J}_{s+1,b}^{p,n} f)'(z) = (p+b)\mathcal{J}_{s,b}^{p,n} f(z) - b\mathcal{J}_{s+1,b}^{p,n} f(z) \tag{10}$$

For two analytic functions $f, g \in A_p$ we say that f is subordinate to g written $f(z) \prec g(z)$ if there exists a schwarz function $\omega(z)$ which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$,such that $f(z) = g(\omega(z))$,$z \in U$.Furthermore,if the function $g(z)$ is univalent in U,then we have the following equivalence

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) and f(U) \subset g(U) \tag{11}$$

If $f(z)$ and $g(z)$ are analytic functions in U,then due to MacGregor [9]we may say that $f(z)$ is majorized by $g(z)$ in U and written as

$$f(z) << g(z) \ (z \in U) \tag{12}$$

if there exists a function $\phi(z)$,analytic in U, such that

$$|\phi(z)| < 1 \ and \ f(z) = \phi(z)g(z) \ (z \in U) \tag{13}$$

It is noted that the notation of majorization(12) is closely related to the concept of quasi-subordination between analytic functions in U which was considered earlier by [1, 2],
on the other hand,investigated several majorization problems involving a number of subclasses of anlytic functions in U. In present sequel to the work of Altintas etal[2]we propose to investigate the corresponding majorization problem associated with the class of multivalent functions based on Srivastava-Attitya operator as defined below.

Definition 1. A function $f(z) \in A_p(n)$ and suppose that $g(z) \in J_{s,b}^{p,n}(\gamma)$ of p-valent functions of complex order $\gamma \neq 0 in U$ if and only if

$$Re(1 + \frac{1}{\gamma}(z\frac{J_{s+1,b}^{p,n} f^{(q+1)}(z)}{J_{s+1,b}^{p,n} f^{(q)}(z)} - p + q)) > 0 \tag{14}$$

$$(z \in U; p \in N; q \in N_0; \gamma \in C - \{0\}; |2\gamma - (p+q)| \leq (p+q))$$

Motivated by earlier works of [6, 10, 12, 14] in this paper we investigate majorization problems for the function class $J^{p,n}_{s,b}(\gamma)$ of $p-$valently stalike functions of complex order$\gamma \neq 0$ in open unit disc U.

2. Main Result

Theorem 1. *A function $f(z) \in A_p(n)$ and suppose that $g(z) \in J^{p,n}_{s,b}(\gamma)$ if $[J^{p,n}_{s+1,b}f(z)]^{(q)}$ is majorized by $[J^{p,n}_{s+1,b}g(z)]^{(q)}$ in U then*

$$|[J^{p,n}_{s,b}f(z)]^{(q)}| \leq |[J^{p,n}_{s,b}g(z)]^{(q)}| \ (|z| \leq r_0) \tag{15}$$

where r_0 is given by

$$r_0 = r_0(p, b, \gamma) = \frac{L - \sqrt{L^2 - 4(p+q)|2\gamma - (p+q)|}}{2|2\gamma - (p+q)|} \tag{16}$$

where $L = (p+b) + |2\gamma - (p+b)|; p \in N; \gamma \in C - (0)$

Proof. since $g(z) \in J^{p,n}_{s,b}(\gamma)$ we find from ,if

$$h(z) = 1 + \frac{1}{\gamma}(\frac{[J^{p,n}_{s+1,b}g(z)]^{(q+1)}}{[J^{p,n}_{s+1,b}g(z)]^{(q)}} - p + q) \tag{17}$$

then $\Re\{h(z)\} > 0(z \in U)$ and

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)}(\omega \in A) \tag{18}$$

where

$$\omega(z) = c_1 z + c_2 z^2 + ... \tag{19}$$

and A denotes the well known class of bounded analytic functions in U and satisfies the conditions

$$\omega(0) = 0, and |\omega(z)| \leq |z| (z \in U) \tag{20}$$

making use of $(17) and (18)$ we get

$$\frac{[J^{p,n}_{s+1,b}g(z)]^{(q+1)}}{[J^{p,n}_{s+1,b}g(z)]^{(q)}} = \frac{p+q+(2\gamma - p + q)\omega(z)}{1-\omega(z)} \tag{21}$$

In view of equation(10)

$$|[J^{p,n}_{s+1,b}g(z)]^{(q)}| \leq \frac{(1+|z|)(p+b)}{p+b-(2\gamma-p+b)|z|}|[J^{p,n}_{s,b}g(z)]^{(q)}| \tag{22}$$

Since$[J^{p,n}_{s+1,b}f(z)]^{(q)}$is majorized by $[J^{p,n}_{s,b}g(z)]^{(q)}$ in U then we have

$$[J^{p,n}_{s+1,b}f(z)]^{(q)} = \phi(z)[J^{p,n}_{s+1,b}g(z)]^{(q)} \tag{23}$$

Differentiating with respect to z and then multiplying z we get

$$z([J^{p,n}_{s+1,b}f(z)]^{(q+1)} = z\phi^{'}(z)[J^{p,n}_{s+1,b}g(z)]^{(q)} + z\phi(z)[J^{p,n}_{s+1,b}g(z)]^{(q+1)} \tag{24}$$

Using (10) in the above equation we get

$$(p+b)[J^{p,n}_{s,b}f(z)]^{(q)} = (\frac{z}{p+b})(\varphi^{'}(z))[J^{p,n}_{s+1,b}g(z)]^{(q)} + \varphi(z)[J^{p,n}_{s,b}g(z)]^{(q)} \tag{25}$$

Noting that the Schwarz function $\phi(z)$ satisfies

$$|\phi^{'}(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \tag{26}$$

and using (22) and(26) in (25) we have

$$|[J^{p,n}_{s,b}f(z)]^{(q)}|$$
$$\leq \{|\phi(z)| + \frac{|z|(1-|\phi(z)|^2)}{(1-|z|)}\frac{1}{(p+b)-|(2\gamma-(p+b))||z|}\}|[J^{p,n}_{s,b}g(z)]^{(q)}|$$

setting $|z| = r$ and $|\phi(z)| = \rho$ $(0 \leq \rho \leq 1)$

$$|[J^{p,n}_{s,b}f(z)]^{(q)}| \leq \frac{\psi(\rho)}{(1-r)\{(p+b)-|(2\gamma-(p+b))|r\}}|[J^{p,n}_{s,b}g(z)]^{(q)}| \tag{27}$$

where

$$\psi(\rho) = -\rho^2 r + \rho(1-r)\{(p+b)-|2\gamma-(p+b)|r\} + r$$

takes its maximum value at $\rho = 1$ with $r_0 = r_0(p,b,\gamma)$ given by Furthermore,if
$0 \leq \sigma \leq r_0 = r_0(p,b,\gamma)$,the function $\varphi(\rho)$ defined by

$$\varphi(\rho) = -\rho^2 r + \rho(1-r)\{(p+b)-|2\gamma-(p+b)|r\} + r \tag{28}$$

is an increasing function on $(0 \leq \rho \leq 1)$ so that

$$\varphi(\rho) \leq \varphi(1) = (1-r)((p+b)-|(2\gamma-(p+b))|r) \tag{29}$$

$(0 \leq \rho \leq 1)$,$0 \leq \sigma \leq r_0 = r_0(p,b,\gamma)$ then setting $\rho = 1$ in we conclude that holds true for $|z| \leq r_0(p,b,\gamma)$. This completes the proof of Theorem 1. □

Corollary 2. *Let the function $f(z) \in A_p$ and $g(z) \in J^{1,n}_{s,b}(\gamma)$ If $[J^{1,n}_{s+1,b}f(z)]$ is majorized by $[J^{1,n}_{s+1,b}g(z)]$ in U,then*

$$|[J^{1,n}_{s,b}f(z)]| \leq |[J^{1,n}_{s,b}g(z)]| \ (|z| \leq r_1), \tag{30}$$

where

$$r_1 = \frac{Q \pm \sqrt{Q^2 - 4(1+q)|2\gamma - (1+q)|}}{2|2\gamma - (1+q)|} \tag{31}$$

where

$$Q = (1+b) + |2\gamma - (1+b)|$$

Putting $q = 0$ and

$$\gamma = (1 - \frac{\alpha}{p})cos\lambda e^{-i\lambda}(|\lambda| < \frac{\pi}{2}; 0 \leq \alpha \leq p)$$

in Theorem 1 we have the following corollory

Corollary 3. *Let the function $f(z) \in A_p$ and $g(z) \in J^{p,n}_{s,b}(\alpha)$ $(|\lambda| < \frac{\pi}{2})$. If*

$$|J^{p,n}_{s,b}f(z)| \leq |J^{p,n}_{s,b}g(z)| \ (|z| \leq r_2), \tag{32}$$

$r_2 = r_2(p, b, \lambda)$ is given by

$$r_2 = \frac{\delta \pm \sqrt{\delta^2 - 4(p+b)|2(1-\frac{\alpha}{p})cos\lambda e^{-i\lambda} - (b+p)|}}{2|2(1-\frac{\alpha}{p})cos\lambda e^{-i\lambda} - (p+b)|} \tag{33}$$

where

$$\delta = (p+b) + |2(1 - \frac{\alpha}{p})cos\lambda e^{-i\lambda} - (p+b)| \tag{34}$$

putting $m = 0$ in corollary 2 we have the following corollary:

The proof of our next result is essentially based upon the following lemma, for the class of starlike and convex functions of complex order γ considered and studied by Nasar[11].

Lemma 1. *If $f \in C(\gamma)$, the class of convex functions of order γ where $(\gamma \in C \ \{0\})$,then $f \in S(\frac{1}{2}\gamma)$, that is $C(\gamma) \subset S(\frac{1}{2}\gamma)$ the class of starlike functions of order $\frac{\gamma}{2}$.*

Theorem 4. *Let the function $f(z) \in A_p$ and $g(z) \in C^{p,n}_{s,b}$ if $J^{p,n}_{s+1,b}f(z)$ is majorized by $J^{p,n}_{s+1,b}g(z)$ in U then*

$$|[J^{p,n}_{s,b}f(z)]^{(q)}| \leq |[J^{p,n}_{s,b}g(z)]^{(q)}| \ (|z| \leq r_4) \tag{35}$$

where r_4 is given by

$$r_4 = \frac{T - \sqrt{T^2 - 4(p+q)|\gamma - (p+q)|}}{2|\gamma - (p+q)|} \tag{36}$$

where

$$T = (p+b) + |\gamma - (p+b)| \tag{37}$$

The proof of our next result is essentially based upon the following lemmas due to Altintas[1].

Lemma 2. *If the function $h(z)$ of the form $h(z) = 1 - \sum\limits_{k=1}^{\infty} c_k z^k$ be in the class $R(\lambda, \gamma)$ if it satisfies the condition*

$$\Re\left(h'(z) + \lambda z h'(z)\right) > \alpha$$

then

$$\sum_{k=1}^{\infty} c_k \leq \frac{\gamma}{1 + \Re(\lambda)} \tag{38}$$

Lemma 3. *If the function $h(z)$ of the form $h(z) = 1 - \sum\limits_{k=1}^{\infty} c_k z^k$ be in the class $R(\lambda, \gamma)$ if it satisfies the condition*

$$\Re\left(h'(z) + \lambda z h'(z)\right) > \alpha$$

then

$$1 - \frac{|\gamma|}{1 + \Re(\lambda)}|z| \leq |h(z)| \leq 1 + \frac{|\gamma|}{1 + \Re(\lambda)}|z|, (z \in U) \tag{39}$$

Finally we prove

Theorem 5. *Let the function $f(z) \in A_p$ and $g(z) \in R(\lambda, \gamma)$ be analytic in U and suppose that the function $g(z)$is so normalised that it also satisfies the following inclusion property:*

$$\frac{[J_{s,b}^{p,n} g(z)]^{(q)}}{[J_{s+1,b}^{p,n} g(z)]^{(q)}} \in R(\lambda, \gamma)$$

if $[J_{s,b}^{p,n} f(z)]^{(q)}$ is majorized by $[J_{s,b}^{p,n} g(z)]^{(q)} (z \in U)$, then

$$|[J_{s,b}^{p,n} f(z)]^{(q)}| \leq |[J_{s,b}^{p,n} g(z)]^{(q)}| \ (|z| \leq r_5) \tag{40}$$

where r_5 *is given by*

$$r_5 = r_5(\lambda, \gamma) \tag{41}$$

is the smallest root of the equation

$$|\gamma|r^3 - (1 + \Re(\lambda)r^2) - [2 + |\gamma| + 2\Re(\lambda)]r + 1 + \Re(\lambda) = 0 \tag{42}$$

Proof. For an appropriately normalised analytic function $g(z)$ satisfying the inclution property we find from the assertion of Lemma 3 that

$$|\frac{[J^{p,n}_{s,b}g(z)]^{(q)}}{[J^{p,n}_{s+1,b}g(z)]^{(q)}}| \geq 1 - \frac{|\gamma|}{1 + \Re(\lambda)}r \; (|z| = r; 0 < r < 1) \tag{43}$$

or equivalently that

$$|[J^{p,n}_{s+1,b}g(z)]^{(q)}| \leq \frac{1 + \Re(\lambda)}{1 + \Re(\lambda) - |\gamma|r}|[J^{p,n}_{s,b}g(z)]^{(q)}| \, (|z| = r; 0 < r < 1) \tag{44}$$

Since

$$[J^{p,n}_{s+1,b}f(z)]^{(q)} \ll [J^{p,n}_{s+1,b}g(z)]^{(q)} (z \in U)$$

there exists an analytic function ω such that

$$[J^{p,n}_{s+1,b}f(z)]^{(q)} = \omega(z)[J^{p,n}_{s+1,b}g(z)]^{(q)}$$

and $|\omega(z)| < 1$ Thus in view of and just as in the proof of Theorem 1,we have $|\omega(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}$ $(z \in U)$ and

$$[J^{p,n}_{s,b}f(z)]^{(q)} \leq (|\omega(z)| + \frac{1 - |\omega(z)|^2}{1 - r^2}\frac{1 + \Re(\lambda)}{1 + \Re(\lambda) - |\gamma|r})[J^{p,n}_{s,b}g(z)]^{(q)} \tag{45}$$

Where we have set $|\omega(z)| = \rho$ and the function $\theta(\rho)$ defined by

$$\theta(\rho) = \{1 + \Re(\lambda)\} + (1 - r^2)1 + \Re(\lambda) - |\gamma|r\rho - \{1 + \Re(\lambda)\}\rho^2 (0 \leq \rho \leq 1) \tag{46}$$

takes on its maximum value at $\rho = 1$ with $r = r_5(\lambda, \gamma)$ given by Moreover if $0 \leq \eta \leq r_5(\lambda, \gamma)$ where $r_5(\lambda, \gamma)$ is the root of the cubic equation such that $0 < r_5(\lambda, \gamma) < 1$ then the function $\vartheta(\rho)$ defined by

$$\vartheta(\rho) = \{1 + \Re(\lambda)\} + (1 - \eta^2)1 + \Re(\lambda) - |\gamma|\eta\rho - \{1 + \Re(\lambda)\}\rho^2 (0 \leq \rho \leq 1) \tag{47}$$

is seen to be increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\vartheta(\rho) = \theta(1) = (1 - \eta^2)1 + \Re(\lambda) - |\gamma|\eta(0 \leq \rho \leq 1; 0 \leq \eta \leq r_5(\lambda, \gamma) \tag{48}$$

consequently,upon setting $\rho = 1$ in ,we complete the proof of Theorem 3. □

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