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POSITIVE SOLUTIONS OF SUMMATION BOUNDARY VALUE PROBLEM FOR A GENERALIZED SECOND-ORDER DIFFERENCE EQUATION

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Abstract: In this paper, by using Krasnoselskii's fixed point theorem, we study the existence of positive solutions to the three-point summation boundary value problem

$$\Delta^2 y(t-1) + a(t)f(y(t)) = 0, \qquad t \in \{1, 2, ..., T\},$$
$$y(0) = \beta \sum_{s=1}^{\eta} y(s), \quad y(T+1) = \alpha \sum_{s=1}^{\eta} y(s),$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$, $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$, $0 < \beta < \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}$ and $\Delta y(t-1) = y(t) - y(t-1)$ is the forward difference operator. We show the existence of at least one positive solution if f is neither superlinear and sublinear by applying the fixed point theorem in cones.

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1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4] and the papers [6-11]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$\begin{aligned} u(0) &= 0, & u(T+1) &= 0 \\ u(0) &= 0, & au(s) &= u(T+1), \\ u(0) &= 0, & u(T+1) - au(s) &= b. \\ u(0) - \alpha\Delta u(0) &= 0, & u(T+1) &= \beta u(s). \\ u(0) - \alpha\Delta u(0) &= 0, & \Delta u(T+1) &= 0 \end{aligned}$$

and so forth.

We refer the reader to [12-16] for some recent results of the existence of positive solutions of boundary value problem for dynamic equations.

We are interested in the existence of positive solutions of the following second order difference equation with three-point summation boundary value problem (BVP):

$$\Delta^2 y(t-1) + a(t)f(y(t)) = 0, \qquad t \in \{1, 2, ..., T\}, \tag{1.1}$$

with the three-point summation boundary condition

$$y(0) = \beta \sum_{s=1}^{\eta} y(s), \qquad y(T+1) = \alpha \sum_{s=1}^{\eta} y(s), \tag{1.2}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$.

Throughout this paper, we suppose the following conditions hold:

$(A1)$ $f \in C([0,\infty), [0,\infty))$;

$(A2)$ $a \in C(\mathbb{N}_{T+1}, [0,\infty))$ and there exists $t_0 \in \mathbb{N}_{\eta,T+1}$ such that $a(t_0) > 0$.

The aim of this paper is to give some results for existence of positive solutions to (1.1)-(1.2), assuming that $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$, $0 < \beta < \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}$ and f is neither superlinear and sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \ \ i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1.1)-(1.2) we mean that a function $u(t) : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (1.1)-(1.2).

For the existence problems of positive solution of the BVP (1.1),(1.2), Sitthiwirattham [17] used the Krasnoselskii's fixed-point theorem to prove the following result.

Theorem 1.1. *(see [17].) The BVP (1.1),(1.2) has at least one positive solution in the case*
(i) $f_0 = 0$ *and* $f_\infty = \infty$*; or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$.

From Theorem 1.1, the following two problems are natural.
PROBLEM 1. Whether or not we can obtain a similar conclusion, if $f_0 = f_\infty = 0$ or $f_0 = f_\infty = \infty$.
PROBLEM 2. Whether or not we can get a similar conclusion, if $f_0, f_\infty \notin \{0, \infty\}$.
Motivated by the results of [17], the aim of this paper is to establish some simple criteria for the existence of positive solutions of the BVP (1.1),(1.2), which gives a positive answer to the questions stated above. The key tool in our approach is the follwing the Krasnoselskii's fixed-point theorem [4].

Theorem 1.2. *(See[4]). Let* E *be a Banach space, and let* $K \subset E$ *be a cone. Assume* Ω_1, Ω_2 *are open subsets of* E *with* $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, *and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2$*; or*
(ii) $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_2$.
Then A *has a fixed point in* $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The paper is organized as follows. In Section 2, we establish conditions for the existence of two positive solutions of the BVP (1.1),(1.2) under $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$. In Section 3, we obtain some existence results for positive solutions of the BVP (1.1),(1.2) under $f_0, f_\infty \notin \{0, \infty\}$.

2. The Existence Results of the BVP (1.1), (1.2) for the Case $f_0 = f_\infty = \infty$ OR $f_0 = f_\infty = 0$

In this section, we establish conditions for the existence of two positive solutions for the BVP (1.1),(1.2) under $f_0 = f_\infty = \infty$ or $f_0 = f_\infty = 0$.

Lemma 2.1. *Let $\beta \neq \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}$. Then for $y \in C(\mathbb{N}_{T+1}, [0,\infty))$, the problem*

$$\Delta y^2(t-1) + h(t) = 0, \quad t \in N_{1,T}, \tag{2.1}$$

$$y(0) = \beta \sum_{s=1}^{\eta} y(s), \qquad y(T+1) = \alpha \sum_{s=1}^{\eta} y(s), \tag{2.2}$$

has a unique solution

$$\begin{aligned} y(t) =& \frac{\beta\eta(\eta+1) + 2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)u(s) \\ &- \frac{\beta(T+1) - (\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s-1)u(s) \\ &- \sum_{s=1}^{t-1}(t-s)u(s), \quad t \in \mathbb{N}_{T+1} \end{aligned} \tag{2.3}$$

Lemma 2.2. *Let $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$, $0 < \beta < \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}$. If $h \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $h(t) \geqslant 0$ for $t \in \mathbb{N}_{1,T}$, then the unique solution y of* (2.1)-(2.2) *satisfies $u(t) \geqslant 0$ for $t \in \mathbb{N}_{T+1}$.*

Lemma 2.3. *Let $\alpha > \frac{2T+2}{\eta(\eta+1)}$, $\beta > max\left\{\frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}, 0\right\}$. If $h \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $h(t) \geqslant 0$ for $t \in \mathbb{N}_{1,T}$, then problem* (2.1)-(2.2) *has no positive solutions.*

Lemma 2.4. *Let $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$, $0 < \beta < \frac{2T+2-\alpha\eta(\eta+1)}{\eta(2T-\eta+1)}$. If $h \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $h(t) \geqslant 0$, then the unique solution to problem* (2.1)-(2.2) *satisfies*

$$\inf_{t \in \mathbb{N}_{T+1}} y(t) \geqslant \gamma \|y\|,$$

where

$$\gamma := \min\left\{\frac{\alpha(\eta+1)(T+1-\eta)}{(T+1)(2-\beta(\eta-1)) - \alpha\eta(\eta+1)}, \frac{\alpha\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)},\right.$$

$$\left.\frac{\beta(\eta+1)(T+1-\eta)}{(2-\beta(\eta-1))(T+1)},\ \frac{\beta\eta(\eta+1)}{(2-\beta(\eta-1))(T+1)}\right\}. \tag{2.4}$$

Let $E = C(\mathbb{N}_{T+1}, [0,\infty))$, and only the the sup norm is used. It is easy to see that the BVP (1.1)-(1.2) has a solution $y = y(t)$ if and only if u is a solution of the operator equation

$$y = Ay,$$

where

$$\begin{aligned} Au(t) \triangleq & \frac{\beta\eta(\eta+1)+2t(1-\beta\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s)) \\ & -\frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \\ & \times\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(y(s)) \\ & -\sum_{s=1}^{t-1}(t-s)a(s)f(y(s)) \end{aligned} \tag{2.5}$$

Denote

$$K = \Big\{ y\in E\ :\ y\geqslant 0,\ \ \min_{t\in\mathbb{N}_{T+1}} y(t)\geqslant\gamma\|y\|\Big\},$$

where γ is defined in (2.4).

It is obvious that K is a cone in E. Moreover, by Lemma 2.3, $A(K)\subset K$. It is also easy to see that $A : K\to K$ is completely continuous.
In what follow, for the sake of convenience, set

$$\begin{aligned} \Lambda_1 =& \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \\ \Lambda_2 =& \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s). \end{aligned}$$

Theorem 2.5. *Suppose that the following assumptions are satisfied:*
(H_1) $f_0 = f_\infty = \infty$,
(H_2) *There exists a constant* $\rho_1 > 0$, *such that*

$$f(u)\leqslant \Lambda_1^{-1}\rho_1 \quad for\ \ u\in[0,\rho_1].$$

Then BVP (1.1), (1.2) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0<\|y_1\|<\rho_1<\|y_2\|.$$

Proof. At first, in view of $f_0 = \lim_{u\to 0^+}(f(u)/u) = \infty$, for any $M_* \in [\Lambda_2^{-1}, \infty)$, there exists $\rho_* \in (0, \rho_1)$ such that

$$f(u) \geqslant M_* u, \quad \text{for } \ 0 \leqslant u \leqslant \rho_*. \tag{2.6}$$

Set $\Omega_{\rho_*} = \{y \in E : \|y\| < \rho_*\}$ for $y \in K \cap \partial\Omega_{\rho_*}$. Since $y \in K$, then $\min_{t\in\mathbb{N}_{T+1}} y(t) \geqslant \gamma\|y\| = \gamma\rho_*$. Thus from (2.5) and (2.6), we get

$$\begin{aligned}
Au(\eta) =& \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
&-\frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\\
&\times\sum_{s=1}^{\eta}(\eta-s)(\eta-s+1)a(s)f(y(s))-\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(y(s))\\
=& \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
&+\frac{1}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\\
&\sum_{s=1}^{\eta-1}(\eta-s)\Big[-(2-\beta\eta+\beta)T+\big(\beta(T-\eta)+\alpha\eta+1\big)s+(\eta-1)\beta\Big]\\
&\times a(s)f(y(s))\\
\geqslant& \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
&+\frac{-T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(y(s))\\
&+\frac{\beta(t-\eta)+\alpha\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta s-s^2)a(s)f(y(s))\\
&+\frac{(\eta-1)\beta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(y(s))\\
\geqslant& \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))
\end{aligned}$$

$$
\begin{aligned}
&+\frac{-T(2-\beta\eta+\beta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(\eta-s)a(s)f(y(s))\\
=&\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}sa(s)f(y(s))\\
\geqslant&\gamma\rho_* M_*\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T-\alpha\eta(\eta+1))-\beta\eta(2T-\eta-1)}\sum_{s=1}^{T}sa(s)\\
=&M_*\Lambda_2\rho_*\geqslant\rho_*=\|y\|,
\end{aligned}
$$

which implies

$$\|Ay\|\geqslant\|y\| \quad \text{for} \ \ y\in K\cap\partial\Omega_{\rho_*}. \tag{2.7}$$

Next, since $f_\infty=\lim_{y\to\infty}(f(u)/u)=\infty$, then for any $M^*\in[\Lambda_2^{-1},\infty)$, there exists $\rho^*>\rho_1$ such that

$$f(u)\geqslant M^*u, \quad \text{for} \ \ u\geqslant\gamma\rho^*. \tag{2.8}$$

Set $\Omega_{\rho^*}=\{u\in E:\|u\|<\rho^*\}$ for $u\in K\cap\partial\Omega_{\rho^*}$. Since $u\in K$, then $\min_{t\in\mathbb{N}_{T+1}}u(t)\geqslant\gamma\|u\|=\gamma\rho^*$. Thus from (2.8) for any $u\in K\cap\partial\Omega_{\rho^*}$, we have

$$
\begin{aligned}
Au(\eta)\geqslant&\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{a}(s)f(y(s))\\
\geqslant&\gamma\rho^* M^*\frac{(2-\beta\eta+\beta)(T-\eta)}{(2T-\alpha\eta(\eta+1))-\beta\eta(2T-\eta-1)}\sum_{s=1}^{T}sa(s)\\
=&M^*\Lambda_2\rho^*\geqslant\rho^*=\|y\|,
\end{aligned}
$$

which implies

$$\|Ay\|\geqslant\|y\| \quad \text{for} \ \ y\in K\cap\partial\Omega_{\rho^*}. \tag{2.9}$$

Finally, let $\Omega_{\rho_1}=\{u\in E:\|u\|<\rho_1\}$ for any $u\in K\cap\partial\Omega_{\rho_1}$. Then from (2.5) and (H_2) we obtain

$$
\begin{aligned}
Au(t)=&\frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
&-\frac{\beta(T+1)-(\beta-\alpha)t}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\\
&\times\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(y(s))
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{s=1}^{t-1}(t-s)a(s)f(y(s))\\
\leqslant&\frac{2t(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
\leqslant&\frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\
\leqslant&\Lambda_1^{-1}\rho_1\frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)\\
\leqslant&\rho_1=\|y\|,
\end{aligned}
$$

which yields

$$\|Ay\|\leqslant\|y\| \quad \text{for} \;\; y\in K\cap\partial\Omega_{\rho_*}. \tag{2.10}$$

Hence, since $\rho_*<\rho_1<\rho^*$ and from (2.7), (2.9) and (2.10), it follows from Theorem 1.2 that A has a fixed point y_1 in $K\cap(\overline{\Omega}_{\rho_1}\setminus\Omega_{\rho_*})$, and a fixed point y_2 in $K\cap(\overline{\Omega}_{\rho^*}\setminus\Omega_{\rho_1})$. Both are positive solutions of the BVP (1.1), (1.2) and

$$0<\|y_1\|<\rho_1<\|y_2\|.$$

The proof is complete. □

Theorem 2.6. *Suppose that the following assumptions are satisfied:*
(H_3) $f_0=f_\infty=0$,
(H_4) *There exists a constant* $\rho_2>0$, *such that*

$$f(u)\geqslant\Lambda_2^{-1}\rho_2 \quad \text{for} \;\; u\in[\gamma\rho_2,\rho_2].$$

Then BVP (1.1), (1.2) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0<\|y_1\|<\rho_2<\|y_2\|.$$

Proof. Firstly, since $f_0=\lim_{u\to 0^+}(f(u)/u)=0$, for any $\varepsilon\in(0,\Lambda_1^{-1}]$, there exists $\rho_*\in(0,\rho_2)$ such that

$$f(u)\leqslant\varepsilon u, \quad \text{for} \;\; u\in[0,\rho_*]. \tag{2.11}$$

Let $\Omega_{\rho_*}=\{y\in E:\|y\|<\rho_*\}$ for any $y\in K\cap\partial\Omega_{\rho_*}$. Then from (2.5) and (2.11), we get

$$Ay(t)\leqslant\frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(y(s))$$

$$\begin{aligned}&\leqslant \varepsilon\rho_* \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)\\ &\leqslant \varepsilon\Lambda_1\rho_* \leqslant \rho_* = \|y\|,\end{aligned}$$

which implies

$$\|Ay\| \leqslant \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{\rho_*}. \tag{2.12}$$

Secondly, in view of $f_\infty = \lim_{u\to\infty}(f(u)/u) = 0$, for any $\varepsilon_1 \in (0, \Lambda_1^{-1}]$ there exists $\rho_0 > \rho_2$, such that

$$f(u) \leqslant \varepsilon_1 u, \quad \text{for} \quad u \in [\rho_0, \infty). \tag{2.13}$$

We consider the next two cases.

Case (i) Suppose that $f(u)$ is unbounded. Then from $f \in C([0,\infty),[0,\infty))$, there exists $\rho^* > \rho_0$ such that

$$f(u) \leqslant f(\rho^*), \quad \text{for} \quad u \in [0, \rho^*]. \tag{2.14}$$

Since $\rho^* > \rho_0$, then from (2.13) and (2.14) one has

$$f(u) \leqslant f(\rho^*) \leqslant \varepsilon_1\rho^*, \quad \text{for} \quad u \in [0, \rho^*]. \tag{2.15}$$

For $y \in K$, and $\|y\| = \rho^*$, from (2,5) and (2.15), we obtain

$$\begin{aligned}Ay(t) \leqslant& \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\ \leqslant& \varepsilon_1\rho^* \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)\\ \leqslant& \varepsilon_1\Lambda_1\rho^* \leqslant \rho^* = \|y\|.\end{aligned}$$

Case (ii) Suppose that $f(u)$ is bounded, say $f(u) \leqslant L$ for all $u \in [0,\infty)$. Taking $\rho^* \geqslant \max\{L/\varepsilon_1, \rho_0\}$, for $u \in K$ with $\|u\| = \rho^*$, from (2,5) we get

$$\begin{aligned}Ay(t) \leqslant& \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)f(y(s))\\ \leqslant& L\frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)\\ \leqslant& \varepsilon_1\rho^* \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)\end{aligned}$$

$$\leqslant \varepsilon_1 \Lambda_1 \rho^* \leqslant \rho^* = \|y\|.$$

Hence, in either case, we always may set $\Omega_{\rho^*} = \{y \in E : \|y\| < \rho^*\}$ such that

$$\|Ay\| \leqslant \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{\rho^*}. \tag{2.16}$$

Finally, set $\Omega_{\rho_2} = \{y \in E : \|y\| < \rho_2\}$ for $y \in K \cap \partial\Omega_{\rho_2}$, since $\min_{t\in\mathbb{N}_{T+1}} y(t) \geqslant \gamma\|y\| = \gamma\rho_2$.
Hence, from (2.5) and (H_4), we have

$$\begin{aligned}
Ay(\eta) =& \frac{(2-\beta\eta+\beta)\eta}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T} (T-s)a(s)f(y(s)) \\
&- \frac{\beta(T+1)-(\beta-\alpha)\eta}{(2T+2-\alpha\eta^2)-\beta\eta(2T-\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)a(s)f(y(s)) \\
&- \sum_{s=1}^{\eta} (\eta-s)a(s)f(y(s)) \\
\geqslant& \gamma\rho_2\Lambda_2^{-1} \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T} sa(s) \\
\geqslant& \rho_2 = \|y\|,
\end{aligned}$$

which yields

$$\|Ay\| \geqslant \|y\| \quad \text{for} \quad y \in K \cap \partial\Omega_{\rho_2}. \tag{2.17}$$

Thus, since $\rho_* < \rho < \rho^*$ and from (2.12), (2.16) and (2.17), it follows from Theorem 1.1 that A has a fixed point u_1 in $K \cap (\overline{\Omega}_{\rho_2} \setminus \Omega_{\rho_*})$, and a fixed point u_2 in $K \cap (\overline{\Omega}_{\rho^*} \setminus \Omega_{\rho_2})$. Both are positive solutions of the BVP (1.1), (1.2) and

$$0 < \|y_1\| < \rho_2 < \|y_2\|.$$

The proof is complete. □

3. The Existence Results of the BVP (1.1), (1.2) for the Case $f_0, f_\infty \notin \{0, \infty\}$

In this section, we discuss the existence for the positive solution of the BVP (1.1),(1.2) assuming $f_0, f_\infty \notin \{0, \infty\}$.

Now, we shall state and prove the following main result.

Theorem 3.1. *Suppose* (H_2) *and* (H_4) *hold, furthermore* $\rho_1 \neq \rho_2$. *Then the BVP* (1.1)-(1.2) *has at least one positive solution* y *satisfying* $\rho_1 < \|y\| < \rho_2$ *or* $\rho_2 < \|y\| < \rho_1$.

Proof. Without loss of generality, we may assume that $\rho_1 < \rho_2$. Let $\Omega_{\rho_1} = \{y \in E : \|y\| < \rho_1\}$, for any $y \in K \cap \partial\Omega_{\rho_1}$ with $\|y\| = \rho_1$, and thus, from (2.5) and (H_2), one has

$$\begin{aligned} Ay(t) \leq & \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ \leq & \Lambda_1^{-1}\rho_1 \frac{2(T+1)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s) \\ \leq & \rho_1 = \|y\|, \end{aligned}$$

which yields

$$\|Ay\| < \|y\|, \qquad \text{for } \; y \in K \cap \partial\Omega_{\rho_*}. \tag{3.1}$$

Now, set $\Omega_{\rho_2} = \{y \in E : \|y\| < \rho_2\}$, for any $y \in K \cap \partial\Omega_{\rho_2}$, we have, $min_{\mathbb{N}_{T+1}} y(t) \geq \gamma\|y\| = \gamma\rho_2$, from (2.5) and (H_4), we can get

$$\begin{aligned} Ay(\eta) \geqslant & \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s)a(s)f(y(s)) \\ \geqslant & \gamma\rho_2\Lambda_2^{-1} \frac{(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T} sa(s) \\ \geqslant & \rho_2 = \|y\|, \end{aligned}$$

which implies

$$\|Ay\| > \|y\|, \qquad \text{for } \; y \in K \cap \partial\Omega_{\rho_2}. \tag{3.2}$$

Hence, since $\rho_1 < \rho_2$ and from (3.1) and (3.2), it follows from Theorem 1.2 that A has a fixed point y in $K \cap (\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1})$. Moreover, it is a positive solutions of the BVP (1.1),(1.2) and

$$\rho_1 < \|y\| < \rho_2.$$

The proof is therefore complete. □

Corollary 3.2. *Assume that the following assumptions hold.*
(H_5) $f_0 = \alpha_1 \in [0, \theta_1\Lambda_1^{-1})$, *where the constant* $\theta_1 \in [0,1)$.
(H_6) $f_\infty = \beta_1 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty]$, *where the constant* $\theta_2 > 1$.
Then, the BVP (1.1), (1.2) *has at least one positive solutions.*

Proof. In view of $f_0 = \alpha_1 \in [0, \theta_1\Lambda_1^{-1})$, for $\varepsilon = \theta_1\Lambda_1^{-1} - \alpha_1 > 0$, there exists a sufficiently small $\rho_1 > 0$ such that

$$f(u) \leq (\alpha_1 + \varepsilon)u = \theta_1\Lambda_1^{-1}u \leq \theta_1\Lambda_1^{-1}\rho_1, \qquad \text{for } \ u \in [0, \rho_1].$$

Since $\theta_1 \in [0, 1)$, then $\theta_1\Lambda_1^{-1} \in [0, \Lambda_1^{-1})$, By the inequality above, (H_2) is satisfied.

Since $f_\infty = \beta_1 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty]$, for $\varepsilon = \beta_1 - (\theta_2/\gamma)\Lambda_2^{-1} > 0$, there exists a sufficiently large $\rho_2 (> \rho_1)$ such that

$$\frac{f(u)}{u} \geq \beta_1 - \varepsilon = \frac{\theta_2}{\gamma}\Lambda_2^{-1}, \qquad \text{for } \ u \in [\mu\rho_2, \infty);$$

thus, when $u \in [\gamma\rho_2, \rho_2]$, one has

$$f(u) \geq \frac{\theta_2}{\gamma}\Lambda_2^{-1}u \geq \theta_2\Lambda_2^{-1}\rho_2.$$

Since $\theta_2 > 1$, $\theta_2\Lambda_2^{-1} \in (\Lambda_2^{-1}, \infty)$, then from the above inequality, Condition (H_4) of Theorem 2.6 is satisfied.

Hence, from Theorem 3.1, the desired result holds.

Corollary 3.3. *Assume that the following assumptions hold.*
(H_7) $f_0 = \alpha_2 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty]$, *where the constant* $\theta_2 > 1$.
(H_8) $f_\infty = \beta_2 \in [0, \theta_1\Lambda_1^{-1})$, *where*$\theta_1 \in [0, 1)$.
Then, the BVP (1.1), (1.2) *has at least one positive solutions.*

Proof. Since $f_0 = \alpha_2 \in ((\theta_2/\gamma)\Lambda_2^{-1}, \infty]$, for $\varepsilon = \alpha_2 - (\theta_2/\mu)\Lambda_2^{-1} > 0$, there exists a sufficiently small $\rho_2 > 0$ such that

$$\frac{f(u)}{u} \geq \alpha_2 - \varepsilon = \frac{\theta_2}{\gamma}\Lambda_2^{-1}, \qquad \text{for } \ u \in (0, \rho_2).$$

Thus, when $u \in [\gamma\rho_2, \rho_2]$, one has

$$f(u) \geq \frac{\theta_2}{\gamma}\Lambda_2^{-1}u \geq \theta_2\Lambda_2^{-1}\rho_2,$$

which implies (H_4) hold.

In view of $f_\infty = \beta_2 \in [0, \theta_1\Lambda_1^{-1})$, for $\varepsilon = \theta_1\Lambda_1^{-1} - \beta_1 > 0$, there exists a sufficiently large $\rho_0 (> \rho_2)$ such that

$$\frac{f(u)}{u} \leq \beta_2 + \varepsilon = \theta_1\Lambda_1^{-1}u, \qquad \text{for } \ u \in [\rho_0, \infty].$$

We consider the following two cases.
CASE (i). Suppose that $f(u)$ is unbounded. Because $f \in C([0,\infty),[0,\infty))$, we know there is a $\rho_1 > \rho_0$ such that

$$f(u) \leq f(\rho_1), \qquad \text{for } \ u \in [0,\rho_1].$$

Since $\rho_1 > \rho_0$, then from (3.3),(3.4), one has

$$f(u) \leq f(\rho_1) \leq \theta_1 \Lambda_1^{-1} \rho_1, \qquad \text{for } \ u \in [0,\rho_1].$$

Since $\theta_1 \in [0,1)$, then $\theta_1\Lambda_1^{-1} \in [0,\Lambda_1^{-1})$, By the inequality above, (H_2) is satisfied. CASE (ii). Suppose that $f(u)$ is bounded, say

$$f(u) \leq M, \qquad \text{for } \ u \in [0,\infty].$$

In this case, taking sufficiently large $\rho_1 > M/\theta_1\Lambda_1^{-1}$, then from (3.5), we know

$$f(u) \leq M \leq \theta_1 \Lambda_1^{-1} \rho_1, \qquad \text{for } \ u \in [0,\rho_1].$$

Since $\theta_1 \in [0,1)$, then $\theta_1\Lambda_1^{-1} \in [0,\Lambda_1^{-1})$, By the inequality above, (H_2) is satisfied.

Hence, from Theorem 3.1, we get the conclusion of Corollary 3.2.

Corollary 3.4. *Assume Conditions* (H_2), (H_6) *and* (H_7) *hold. Then, the BVP* (1.1), (1.2) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| < \rho_1 < \|y_2\|.$$

Proof. From (H_6) and the proof of Corollary 3.2, we know that there exists a sufficiently large $\rho_2 > \rho_1$ such that

$$f(u) \geq \theta_2 \Lambda_2^{-1} \rho_2 = M_2 \rho_2, \qquad \text{for } \ u \in [\gamma\rho_2, \rho_2],$$

where $M_2 = \theta_2\Lambda_2^{-1} \in (\Lambda_2^{-1},\infty)$.

In view of (H_7) and the proof of Corollary 3.3, we see that there exists a sufficiently small $\rho_2^* \in (0,\rho_1)$ such that

$$f(u) \geq \theta_2 \Lambda_2^{-1} \rho_2 = M_2 \rho_2^*, \qquad \text{for } \ u \in [\gamma\rho_2^*, \rho_2^*],$$

where $M_2 = \theta_2\Lambda_2^{-1} \in (\Lambda_2^{-1},\infty)$.

Using this and (H_2), we know by Theorem 3.1 that the BVP (1.1), (1.2) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| < \rho_1 < \|y_2\|.$$

Thus, the proof is complete

Corollary 3.5. *Assume Conditions* (H_4), (H_5) *and* (H_8) *hold. Then, the BVP* (1.1), (1.2) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| < \rho_2 < \|y_2\|.$$

Proof. By (H_5) and the proof of Corollary 3.2, we obtain that there exists a sufficiently small $\rho_1 \in (0, \rho_2)$ such that

$$f(u) \leq \theta_1 \Lambda_1^{-1} \rho_1 = M_1 \rho_1, \qquad \text{for } \ u \in [0, \rho_1],$$

where $M_1 = \theta_1 \Lambda_1^{-1} \in (0, \Lambda_1^{-1})$.

In view of (H_8) and the proof of Corollary 3.3, there exists a sufficiently large $\rho_1^* > \rho_2$ such that

$$f(u) \leq \theta_1 \Lambda_1^{-1} \rho_1^* = M_1 \rho_1^*, \qquad \text{for } \ u \in [0, \rho_1^*],$$

where $M_1 = \theta_1 \Lambda_1^{-1} \in (0, \Lambda_1^{-1})$.

Using this and (H_4), we see by Theorem 3.1 that the BVP (1.1), (1.2) has two positive solutions y_1 and y_2 such that

$$\rho_1 < \|y_1\| < \rho_2 < \|y_2\| < \rho_1^*.$$

Thus, the proof is complete

4. Some Examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1. Consider the BVP

$$\Delta^2 y(t-1) + \frac{1}{\pi^{t+1}}(y^{8/9} + \frac{1}{y^3}) = 0, \qquad t \in \mathbb{N}_{1,5}, \tag{4.1}$$

$$y(0) = \frac{1}{50}\sum_{s=1}^{3} y(s), \quad y(T+1) = \frac{1}{100}\sum_{s=1}^{3} y(s). \tag{4.2}$$

Set $\alpha = \frac{1}{100}$, $\beta = \frac{1}{50}$, $\eta = 3$, $a(t) = \frac{1}{\pi^{t+1}}$, $f(y) = y^{8/9} + \frac{1}{y^3}$. Since $f_0 = f_\infty = \infty$, then (H_1) holds. Again $\Lambda_1 = \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1))-\beta\eta(2T-\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s) \approx 0.675$, because $f(u)$ is monotone increasing function for $u \geq 0$, taking $\rho_1 = 4$, then when $u \in [0, \rho_1]$, we get

$$f(y) \leq f(4) \approx 4.766 \ < \ 5.93 \ = \ \Lambda_1^{-1}\rho_1,$$

which implies (H_2) holds. Hence, by Theorem 2.5, the BVP (4.1), (4.2) has at least two positive solution y_1 and y_2 such that $0 < \|y_1\| < 4 < \|y_2\|$.

Example 4.2. Consider the BVP

$$\Delta^2 y(t-1) + e^{10} y^2 e^{-y} = 0, \qquad t \in \mathbb{N}_{1,5}, \tag{4.3}$$

$$y(0) = \frac{1}{2}\sum_{s=1}^{2} y(s), \quad y(T+1) = \frac{1}{3}\sum_{s=1}^{2} y(s). \tag{4.4}$$

Set $\alpha = \frac{1}{3}$, $\beta = \frac{1}{2}$, $\eta = 2$, $a(t) \equiv e^{10}$, $f(y) = y^2 e^{-y}$. Since $f_0 = f_\infty = 0$, then (H_3) holds. Again

$$\gamma = \frac{\alpha\eta^2}{(2-\beta\eta)(T+1)} = \frac{2}{9},$$

$$\Lambda_2 = \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)}\sum_{s=1}^{T} sa(s) = 15e^{10},$$

because $f(u)$ is monotone decreasing function for $u \geq 2$, taking $\rho_2 = 15$, then when $u \in [\gamma\rho_2, \rho_2] = [\frac{10}{3}, 15]$, we get

$$f(u) \geq f(10) = 225e^{-10} > e^{-10} = \Lambda_2^{-1}\rho_2,$$

which implies (H_4) holds. Hence, by theorem 3.2, the BVP (4.3), (4.4) has at least two positive solution y_1 and y_2 such that $0 < \|y_1\| < 15 < \|y_2\|$.

Example 4.3. Consider the BVP

$$\Delta^2 y(t-1) + (1-t)^2 \left(\frac{aye^{2y}}{b+e^y+e^{2y}}\right) = 0, \qquad t \in \mathbb{N}_{1,4}, \tag{4.5}$$

$$y(0) = \frac{1}{4}\sum_{s=1}^{3} y(s), \quad y(T+1) = \frac{1}{5}\sum_{s=1}^{3} y(s). \tag{4.6}$$

where $a = 3$, $b > 98$. Set $\alpha = \frac{1}{5}$, $\beta = \frac{1}{4}$, $\eta = 3$, $a(t) \equiv (1-t)^2$, $f(y) = \frac{aye^{2y}}{b+e^y+e^{2y}}$. Then

$$\gamma = \frac{\alpha\eta(T-\eta)}{(T+1)(2-\beta\eta) - \alpha\eta^2} = \frac{12}{89},$$

$$\Lambda_1 = \frac{2T+2}{2T+2-\alpha\eta(\eta+1)} \sum_{s=\eta}^{T} (T-s+1)a(s) = 50,$$

$$\Lambda_2 = \frac{2\eta\gamma}{2T+2-\alpha\eta(\eta+1)} \sum_{s=\eta}^{T} (T-s+1)a(s) \approx 5.88,$$

and
$$f_0 = \frac{a}{2+b} = \frac{2}{2+b} \in (0, \Lambda_1^{-1}) = (0, 0.02),$$

$$f_\infty = a = 2 \in (\frac{1}{\mu}\Lambda_2^{-1}, \infty) = (0.74, \infty).$$

Thus, the conditions (H_5) and (H_6) hold. Therefore by corollary 3.2, the BVP (4.5), (4.6) has at least two positive solution.

Example 4.4. Consider the BVP

$$\Delta^2 y(t-1) + 3ty(0.01 + \frac{c}{1+y^2}) = 0, \qquad t \in \mathbb{N}_{1,4}, \tag{4.7}$$

$$y(0) = 0, \quad y(T+1) = \frac{1}{4}\sum_{s=1}^{2} y(s). \tag{4.8}$$

where $c > 0.049$. Set $\alpha = \frac{1}{4}$, $\eta = 2$, $a(t) \equiv 3t$, $f(y) = y(0.01 + \frac{c}{1+y^2})$.

Then

$$\gamma = \frac{\alpha\eta(\eta+1)}{2T+} = \frac{3}{20},$$

$$\Lambda_1 = \frac{(2T+2)(1-\beta\eta)+\beta\eta(\eta+1)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{T} (T-s+1)a(s)$$
$$= 360,$$

$$\Lambda_2 = \frac{\gamma(2-\beta\eta+\beta)(T-\eta)}{(2T+2-\alpha\eta(\eta+1)) - \beta\eta(2T-\eta+1)} \sum_{s=1}^{T} sa(s) = 108,$$

and $f_0 = 0.001 + c \in (\frac{1}{\mu}\Lambda_2^{-1}, \infty) = (0.05, \infty)$,

$$f_\infty = 0.001 \in (0, \Lambda_1^{-1}) = (0, 0.003).$$

Thus, the conditions (H_7) and (H_8) hold. Therefore by corollary 3.3, the BVP (4.7), (4.8) has at least two positive solution.

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References

[1] V. A. Il'in and E. I. Moiseev, Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, Differential Equations, 23(1987) 803-810.

[2] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, Journal of Mathematical Analysis and Applications, 168(1992) no.2 540-551.

[3] R.P. Agarwal, Focal Boundary Value Problems for Differential and Difference Equations, Kluwer Academic Publishers, Dordrecht, 1998.

[4] R.P. Agarwal, D. O'Regan, P.J.Y. Wong, Positive Solutions of Differential, Difference and Integral Equations, Kluwer Academic Publishers, Dordrecht, 1999.

[5] M.A. Krasnoselskii, Positive Solutions of Operator Equations, Noordhoof, Gronignen, 1964.

[6] R.W.Leggett, L.R.Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math.J., 28(1979) 673-688

[7] Z.Bai,X. Liang,Z. Du, Triple positive solutions for some second-order boundary value problem on a measure chain. Comput. Math. Appl., 53(2007) 1832-1839

[8] X.He, W. Ge, Existence of three solutions for a quasilinear two-point boundary value problem. Comput. Math. Appl., 45(2003) 765-769

[9] X. Lin, W. Lin, Three positive solutions of a secound order difference Equations with Three-Point Boundary Value Problem, J.Appl. Math. Comut. Vol.31(2009) 279-288.

[10] G. Zhang, R. Medina, Three-point boundary value problems for difference equations, Comp. Math. Appl. 48(2004) 1791-1799.

[11] J. Henderson, H.B. Thompson, Existence of multiple solutions for second order discrete boundary value problems, Comput. Math. Appl. 43 (2002) 1239-1248.

[12] B. Karna and B. Lawrence, An existence result for a multipoint boundary value problem on a time scale, Advances in Difference Equations (2006), Article ID 63208, 8 pages.

[13] C. S. Goodrich, On positive solutions to nonlocal fractional and integer-order difference equations, Appl. Anal. Discrete Math. 5 (2011), 122-132.

[14] C. S. Goodrich, Existence of a positive solution to a system of discrete fractional boundary value problems, Appl. Math. Comput. 217 (2011), 4740-4753.

[15] C. S. Goodrich, Positive solutions to boundary value problems with non-linear boundary conditions, Nonlinear Anal. 75 (2012), 417-432.

[16] C. S. Goodrich, The existence of a positive solution to a second-order p-Laplacian BVP on a time scale, Appl. Math. Lett. 25 (2012), 157-162.

[17] T. Sitthiwirattham, J. Tariboon, Positive solutions of a generalized second order difference equation with three-point summation boundary value problem, Journal of Applied MathematicsVol.2012, Article ID 569313, 15 pages.