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AN APPLICATION OF MULTIPLIER TRANSFORMATIONS FOR CERTAIN SUBCLASSES OF MEROMORPHICALLY P-VALENT FUNCTIONS

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Abstract: In this paper, we introduce the class $L(p,m,n,\beta,\lambda,\ell,A,B)$ of meromorphically p-valent functions and investigate some inclusion properties, coefficient bounds, distortions theorems, $\delta-$neighborhoods and partial sums.We also obtain linear combination, weighted mean, arithmetic mean and convolution properties.

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1. Introduction

Let $L(p,m)$ denote the class of meromorphically multivalent functions $f(z)$ of the form:

$$f(z) = z^{-p} + \sum_{k=m}^{\infty} a_k z^k \text{ for any } \quad m \geq p,\ p \in N = \{1,2,...\}, a_k \geq 0 \qquad (1.1)$$

which are analytic and p-valent in the punctured unit disc $U^{\square} = \{z : z \in \mathbb{C} \text{and } 0 < |z| < 1\} = U \backslash \{0\}$.

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Definition 1. For two functions f and g analytic in U, we say that $f$is subordinate to g, and write $f \prec g$ in U or $f(z) \prec g(z)(z \in U)$. If there exists a Schwarz function $w(z)$, which is analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \qquad (z \in U),$$

such that

$$f(z) = g(w(z)) \quad (z \in U),$$

it is known that

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U),$$

(see [15], p. 4),

Definition 2. For functions $f(z) \in L(p,m)$ given by (1.1) and $g(z) \in L(p,m)$ defined by

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \ \left(b_k \geq 0, p \in N, m \geq p\right),$$

we define the convolution (or Hadamard product) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=m}^{\infty} a_k b_k z^k, \quad (p \in N, m \geq p; z \in U^{\square}). \tag{1.2}$$

Definition 3. (see [9]) For functions $f(z) \in L(p,m)$,EL-Ashwah Define the following differential operators as follows:

$$I_p^n(\lambda,\ell) f(z) = z^{-p} + \sum_{k=m}^{\infty} \left[\frac{\ell + \lambda(k+p)}{\ell}\right]^n a_k z^k, \tag{1.3}$$

where

$I_p^n(\lambda,\ell)(\lambda \geq 0, \ell > 0, n \in N_0 = N \cup \{0\})$.

We can write (1.3) in the form:

$$I_p^n(\lambda,\ell) f(z) = \left(\Phi_{\lambda,\ell}^{p,n} * f\right)(z),$$

where

$$\Phi_{\lambda,\ell}^{p,n} = z^{-p} + \sum_{k=m}^{\infty} \left[\frac{\ell + \lambda(k+p)}{\ell}\right]^n z^k. \tag{1.4}$$

It is easily verified from (1.3) that

$$\lambda z(I_p^n(\lambda,\ell)f(z))' = \ell I_p^{n+1}(\lambda,\ell)f(z) - (\lambda p+\ell)I_p^n(\lambda,\ell)f(z)(\lambda > 0).$$

We note that: $I_p^0(\lambda,\ell)f(z) = f(z)$ and

$$I_p^1(1,1)f(z) = \frac{(z^{p+1}f(z))'}{z^p} = (p+1)f(z) + zf'(z).$$

Also by specializing the parameters λ,ℓ, and p, we obtain the following operators studied by various authors:

(i) $I_1^n(1,\ell)f(z) = I(n,\ell)f(z)$ (see Cho et al. [7,8]);

(ii) $I_1^n(\lambda,1)f(z) = D_{\lambda,p}^n f(z)$(see Al-Oboudi and Al-Zkeri [1]);

(iii) $I_1^n(1,1)f(z) = I^n f(z)$ (see Uralegaddi and Somantha [19]);

$(v) I_p^n(1,1)f(z) = D_p^n f(z)$ (see Aouf and Hossen [5], Liu and Srivastava[14], and Srivastava and Patel, [17]).

Definition 4. For fixed parameter λ,ℓ,A,B $(\lambda \geq 0, \ell > 0, -1 \leq B < A \leq 1)$ a functions $f(z) \in L(p,m)$ is said to be in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$ if

$$-\frac{z^{p+1}}{p}\left\{(1-\beta)\left(I_p^n(\lambda,\ell)f(z)\right)' + \beta\left(I_p^{n+1}(\lambda,\ell)f(z)\right)'\right\} \prec \frac{1+A(z)}{1+B(z)} \tag{1.5}$$

where $\prec$ denotes subordination , $p \in N, \beta \geq 0, n \in N_0$ and $z \in U$.

By definition of subordination, the condition (1.5) is equivalent to

$$-z^{p+1}\left\{(1-\beta)\left(I_p^n(\lambda,\ell)f(z)\right)' + \beta\left(I_p^{n+1}(\lambda,\ell)f(z)\right)'\right\} = p\frac{1+Aw(z)}{1+Bw(z)} \tag{1.6}$$

where $w(z) \in H = \{w \text{ regular}, w(0) = 0 \text{ and } |w(z)| < 1, z \in U\}$.

It easy to see that the condition (1.6) is equivalent to

$$\left|\frac{z^{p+1}\left\{(1-\beta)\left(I_p^n(\lambda,\ell)f(z)\right)' + \beta\left(I_p^{n+1}(\lambda,\ell)f(z)\right)'\right\} + p}{Bz^{p+1}\left\{(1-\beta)\left(I_p^n(\lambda,\ell)f(z)\right)' + \beta\left(I_p^{n+1}(\lambda,\ell)f(z)\right)'\right\} + Ap}\right| < 1$$

$$(z \in U). \tag{1.7}$$

Remark. (i) $L(p,m,n,0,1,1,A,B) = L(p,m,n,A,B)$ (see Atshan and Kulkarni [6]).

(ii) $L(p, m, 0, \frac{\ell\sigma}{\lambda(1-\sigma(p+1))}, \lambda, \ell, A, B) = L(p, m, n, \sigma, \lambda, \ell, A, B)$ (see Lashin [11]), where $L(p, m, n, \sigma, \lambda, \ell, A, B)$ denote the class of functions in $L(p, m)$ satisfying the following condition:

$$\frac{-z^{p+1}\left\{(I_p^n(\lambda,\ell)f(z))' + \sigma z(I_p^n(\lambda,\ell)f(z))''\right\}}{1-\sigma(p+1)} \prec p\frac{1+Az}{1+Bz}.$$

We note that $L(p, m, n, 0, \lambda, \ell, 1-\frac{2\gamma}{p}, -1) = L(p, m, n, \lambda, \ell, \gamma)$, where $L(p, m, n, \lambda, \ell, \gamma)$ denote the class of functions in $L(p, m)$ satisfying the following condition:

$$Re\left\{-z^{p+1}(I_p^n(\lambda,\ell)f(z))'\right\} > \gamma \quad (0 \le \gamma < p\,; z \in U).$$

2. Coefficient Bounds

Theorem 1. *Let the function $f(z)$ of the from (1.1), be in $L(p, m)$.*
Then the function $f(z)$ belongs to the class $L(p, m, n, \beta, \lambda, \ell, A, B)$ if and only if

$$\sum_{k=m}^{\infty} k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n\left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]a_k < p(A-B), \tag{2.1}$$

where $-1 \le B < A \le 1, \ell > 0, \lambda \ge 0, \beta \ge 0, m \ge p, n \in N_0, p \in N$.

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{p(A-B)}{k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n\left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]}z^m \quad (m \ge p).$$

Proof. Assume that the condition (2.1) is true. We must show that $f \in L(p, m, n, \beta, \lambda, \ell, A, B)$, or equivalently prove that

$$\left|\frac{z^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + p}{Bz^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + Ap}\right| < 1, \tag{2.2}$$

we have

$$\left|\frac{z^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + p}{Bz^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + Ap}\right|$$

$$= \left| \frac{\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}}{p(A-B) + B\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}} \right|$$

$$\leq \left\{ \frac{\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k}{p(A-B) + B\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k} \right\} < 1.$$

The last inequality by (2.1) is true.
Conversely suppose that $f(z) \in L(p, m, n, \beta, \lambda, \ell, A, B)$. We must show that the condition (2.1) holds true. We have

$$\left| \frac{z^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + p}{Bz^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + Ap} \right| < 1,$$

hence we get

$$\left| \frac{\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}}{p(A-B) + B\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}} \right| < 1.$$

Since $\mathrm{Re}(z) < |z|$, so we have

$$\mathrm{Re} \left\{ \frac{\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}}{p(A-B) + B\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k z^{k+p}} \right\} < 1.$$

We choose the values of z on the real axis and letting $z \to 1^-$, then we obtain

$$\left\{ \frac{\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k}{p(A-B) + B\sum\limits_{k=m}^{\infty} k \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k} \right\} < 1,$$

then

$$\sum_{k=m}^{\infty} k(1-B) \left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right] a_k < p(A-B)$$

and the proof is complete.

Corollary 1. Let $f(z) \in L(p,m,n,\beta,\lambda,\ell,A,B)$, then we have

$$a_k \leq \frac{p(A-B)}{k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]} \quad , \ (k \geq m).$$

where $-1 \leq B \leq 0, \quad B < A \leq 1, \quad \lambda \geq 0 \ ; \ \ell > 0, \quad \beta \geq 0 \ , \ p \in N \ , \quad n \in N_0, \ z \in U^{\square}$.

Equality holds for the functions of the form

$$f_k(z) = z^{-p} + \frac{p(A-B)}{k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]} z^k, (k \geq m, m \geq p).$$

3. Some Properties of the Class $L(p,m,n,\beta,\lambda,\ell,A,B)$

Theorem 2. *Let* $0 \leq n_2 < n_1, 0 \leq \beta_2 < \beta_1 < \beta$ *then*

$$L(p,m,n_2,\beta_2,\lambda,\ell,A,B) \subset L(p,m,n_1,\beta_1,\lambda,\ell,A,B).$$

Proof. From Theorem 1, we have

$$\begin{aligned} &\sum_{k=m}^{\infty} k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^{n_2}\left[\frac{\ell+\lambda\beta_2(k+p)}{\ell}\right] a_k \\ &< \sum_{k=m}^{\infty} k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^{n_1}\left[\frac{\ell+\lambda\beta_1(k+p)}{\ell}\right] a_k \\ &\leq p(A-B). \end{aligned}$$

for $f(z) \in L(p,m,n_1,\beta_1,\lambda,\ell,A,B). \Longrightarrow f(z) \in L(p,m,n_2,\beta_2,\lambda,\ell,A,B)$.

Corollary 2. $L(p,m,n,A,B) \subset L(p,m,n,\beta,\lambda,\ell,A,B)$.

The proof is now immediate if $\lambda = 1, \ell = 1$ and $\beta = 0$.

4. Distortion Theorems

Theorem 3. *Let the function* $f(z)$ *be defined by* (1.1). be in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$,then for $0<|z|=r<1$,

$$\frac{1}{r^p}-\frac{(A-B)}{(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n\left[\frac{\ell+2\lambda\beta p}{\ell}\right]}r^p\le|f(z)|$$
$$\le\frac{1}{r^p}+\frac{(A-B)}{(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n\left[\frac{\ell+2\lambda\beta p}{\ell}\right]}r^p. \quad (4.1)$$

where$(-1\le B\le 0, B<A\le 1, \ell>0, \lambda\ge 0, \beta\ge 0, n\in N_0, m\ge p, p\in N)$.

Proof. That $f(z)$ in the class $L(p,m,n,\lambda,\ell,A,B)$. In view of Theorem 1, we have

$$p(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n\left[\frac{\ell+2\lambda p\beta}{\ell}\right]\sum_{k=m}^{\infty}a_k$$
$$\le\sum_{k=m}^{\infty}k(1-B)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n\left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]a_k\le p(A-B),$$

which yields

$$\sum_{k=m}^{\infty}a_k\le\frac{(A-B)}{(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n\left[\frac{\ell+2\lambda p\beta}{\ell}\right]}. \quad (4.2)$$

Consequently, we obtain

$$|f(z)|\le\frac{1}{r^p}+r^k\sum_{k=m}^{\infty}a_k\le\frac{1}{r^p}+r^p\sum_{k=m}^{\infty}a_k$$
$$\le\frac{1}{r^p}+\frac{(A-B)}{(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n\left[\frac{\ell+2\lambda p\beta}{\ell}\right]}r^p,$$

by (4.2) this gives the right hand inequality of (4.1). Also

$$|f(z)|\ge\frac{1}{r^p}-r^k\sum_{k=m}^{\infty}a_k\ge\frac{1}{r^p}-r^p\sum_{k=m}^{\infty}a_k$$

$$\frac{1}{r^p} - \frac{(A-B)}{(1-B)\left[\frac{\ell+2\lambda p}{\ell}\right]^n \left[\frac{\ell+2\lambda p\beta}{\ell}\right]} r^p.$$

5. Neighborhood and Partial Sums

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [10], and Ruscheweyh [16], and Stankiewicz [18], and(more recently) by Altintas et al.([2] , [3]and[4]), Liu [12] and Liu and Srivastava [13], we being by introducing here the $\delta-$neighborhood of a function $f(z) \in L(p,m)$ of the form (1.1) by means of the following definition.

Definition 5. Let $-1 \leq B \leq A \leq 1, \quad \ell > 0, \lambda \geq 0, \beta \geq 0, m \geq p, p \in N, n \in N_0$ and $\delta \geq 0$. We define $\delta-$neighborhoods of a function $f(z) \in L(p,m)$ and denote $N_\delta(f)$ such that

$$N_\delta(f) = \left\{ g \in L(p,m) : g(z) = z^{-p} + \sum_{k=m}^{\infty} b_m z^k \quad \text{and} \right.$$

$$\left. \sum_{k=m}^{\infty} \frac{k(1+|B|)\left[\frac{\ell+\lambda(k+p)}{\ell}\right]^n \left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]}{p(A-B)} |b_k - a_k| \leq \delta \right\}. \tag{5.1}$$

Theorem 4. *Let* $\delta > 0$ and $f(z) \in L(p,m)$ given by (1.1) satisfies the inclusion property:

$$\frac{f(z)+\epsilon z^{-p}}{1+\epsilon} \in L(p,m,n,\beta,\lambda,\ell,A,B), \tag{5.2}$$

for any complex number $\in$ such that $|\in| < \delta$, then $N_\delta(f) \subset L(p,m,n,\beta,\lambda,\ell,A,B)$.

Proof. It is easily to seen from (1.7) that $f(z) \in L(p,m,n,\beta,\lambda,\ell,A,B)$ if and only if for any complex number σ with $|\sigma| = 1$, we have

$$\left[\frac{z^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + p}{Bz^{p+1}\left\{(1-\beta)(I_p^n(\lambda,\ell)f(z))' + \beta(I_p^{n+1}(\lambda,\ell)f(z))'\right\} + Ap}\right] \neq \sigma \qquad (z \in U).$$

which is equivalent to

$$\frac{(f * Q)(z)}{z^{-p}} \neq 0 \qquad (z \in U^*), \tag{5.3}$$

where

$$Q(z) = z^{-p} + \sum_{k=m}^{\infty} c_k z^k,$$

$$c_k := \frac{k(1-\sigma B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n \left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{\sigma p(B-A)}. \quad (5.4)$$

It follows from (5.4) that

$$|c_k| = \left| \frac{k(1-\sigma B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n \left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{\sigma p(B-A)} \right|$$

$$\leq \frac{k(1+|B|)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n \left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}$$

if $f(z) \in L(p,m)$ given by (1.1), satisfies the inclusion property (5.2), then (5.3) yields

$$\left|\frac{f(z) * Q(z)}{z^{-p}}\right| \geq \delta \qquad (z \in U^{*}). \quad (5.5)$$

Now, if we suppose that

$$g(z) = z^{-p} + \sum_{k=m}^{\infty} b_k z^k \in N_\delta(f), \quad (5.6)$$

we easily see that

$$\begin{aligned} \left|\frac{(g-f)(z) * Q(z)}{z^{-p}}\right| &= \left|\sum_{k=m}^{\infty} (b_k - a_k) c_k z^{k+p}\right| \\ &\leq \sum_{k=m}^{\infty} |b_k - a_k|\, |c_k|\, |z|^{k+p} \\ &\leq |z| \sum_{k=m}^{\infty} \frac{k(1+|B|)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n \left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)} \\ &\quad \times |b_k - a_k| \leq \delta \end{aligned}$$

then

$$\left|\frac{(g)(z) * Q(z)}{z^{-p}}\right| = \left|\frac{(f + (g - f)(z) * Q(z)}{z^{-p}}\right|$$
$$\geq \left|\frac{f(z) * Q(z)}{z^{-p}}\right| - \left|\frac{(g - f)(z) * Q(z)}{z^{-p}}\right| > 0$$

thus, for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{(g)(z) * Q(z)}{z^{-p}} \neq 0 \quad (z \in U^{\square})$$

which implies that $g(z) \in L(p, m, n, \beta, \lambda, \ell, A, B)$.

Theorem 5. *Let* $-1 \leq B < A \leq 1, B \geq 0, \ell > 0, \lambda \geq 0$, $n \in N_0$ *and let* $f(z)$ *be defined by (1.1) and the partial sums* $S_1(z)$ *and* $S_q(z)$ *be defined* $S_1(z) = z^{-p}$ *and*

$$S_q(z) = z^{-p} + \sum_{k=m}^{q} a_k z^k, \qquad (q > m, m \geq p, p \in N).$$

Also suppose that $\sum_{k=m}^{\infty} c_k a_k \leq 1$, *where*

$$c_k = \frac{k(1 + |B|)\left[\frac{\ell + \lambda(k + p)}{\ell}\right]^n \left[\frac{\ell + \lambda\beta(k + p)}{\ell}\right]}{p(A - B)}.$$

Then

(i) $f \in L(p, m, n, \beta, \lambda, \ell, A, B)$.

$$(ii)\mathrm{Re}\left\{\frac{f(z)}{S_q(z)}\right\} > 1 - \frac{1}{c_{q+1}}. \tag{5.7}$$

$$(iii)\mathrm{Re}\left\{\frac{S_q(z)}{f(z)}\right\} > 1 - \frac{1}{1 + c_{q+1}} \quad ,(z \in U, q > m). \tag{5.8}$$

The estimates in (5.7) and (5.8) are sharp.

Proof. (i) Since $\frac{z^{-p} + \epsilon z^{-p}}{1 + \epsilon} = z^{-p} \in L(p, m, n, \beta, \lambda, \ell, A, B), |\epsilon| < 1$, then by Theorem 4, we have $N_1(z^{-p}) \subset L(p, m, n, \beta, \lambda, \ell, A, B), p \in N (N_1(z^{-p})$ denoting the 1- neighborhood). Now since $\sum_{k=m}^{\infty} c_k a_k \leq 1$, then $f \in N_1(z^{-p})$ and $f \in L(p, m, n, \beta, \lambda, \ell, A, B)$.

(ii)Since $\{c_k\}$ is an increasing sequence, we obtain

$$\sum_{k=m}^{q} a_k + c_{q+1} \sum_{k=q+1}^{\infty} a_k \leq \sum_{k=m}^{\infty} c_k a_k \leq 1 \tag{5.9}$$

by setting

$$\begin{aligned} G_1(z) &= c_{q+1}\left\{\frac{f(z)}{s_q(z)} - (1 - \frac{1}{c_{q+1}})\right\} \\ &= 1 + \frac{c_{q+1} \sum\limits_{k=q+1}^{\infty} a_k z^{k+p}}{1 + \sum\limits_{k=m}^{q} a_k z^{k+p}} \end{aligned}$$

from (5.9) we get

$$\begin{aligned} \left|\frac{G_1(z)-1}{G_1(z)+1}\right| &= \left|\frac{c_{q+1} \sum\limits_{k=q+1}^{\infty} a_k z^{k+p}}{2 + 2\sum\limits_{k=m}^{q} a_k z^{k+p} + c_{q+1} \sum\limits_{k=q+1}^{\infty} a_k z^{k+p}}\right| \\ &\leq \frac{c_{q+1} \sum\limits_{k=q+1}^{\infty} a_k}{2 - 2\sum\limits_{k=m}^{q} a_k - c_{q+1} \sum\limits_{k=q+1}^{\infty} a_k} \leq 1 \qquad (z \in U). \end{aligned}$$

which readily yields the assertion (5.7) of Theorem 5,if we teak

$$f(z) = z^{-p} + \frac{z^{q+1}}{c_{q+1}} \tag{5.10}$$

with $z = re^{\frac{i\pi}{q+p+1}}$ and let $r \to 1^-$, we obtain

$$\frac{f(z)}{s_{q(z)}} = 1 + \frac{z^{q+p+1}}{c_{q+1}} \to 1 - \frac{1}{c_{q+1}},$$

which shows that the bound in (5.7) is best possible for each $q \in N, q > m, m \geq p$.

(iii) Similarly, if we put

$$\begin{aligned} G_2(z) &= (1+c_{q+1})\left(\frac{S_q(z)}{f(z)} - \frac{c_{q+1}}{1+c_{q+1}}\right) \\ &= 1 - \frac{(1+c_{q+1})\sum\limits_{k=q+1}^{\infty} a_k z^{k+p}}{1+\sum\limits_{k=m}^{\infty} a_k z^{k+p}}. \end{aligned}$$

and make use of (5.9), we can deduce that

$$\left|\frac{G_2(z)-1}{G_2(z)+1}\right| \le \frac{(1+c_{q+1})\sum\limits_{k=q+1}^{\infty} a_k}{2-2\sum\limits_{k=m}^{q} a_k + (1-c_{q+1})\sum\limits_{k=q+1}^{\infty} a_k} \le 1.$$

which yields inequality (5.8) of Theorem 5. The bound in (5.8) is sharp for each $q \in N, q > m$, with the extremal function $f(z)$ given by (5.10).The proof of Theorem 5, is now complete.

6. Linear Combination

In the theorem below, we prove a linear combination for the class $L(p, m, n, \beta, \lambda, \ell, A, B)$.

Theorem 6. *Let* $f_i(z) = z^{-p} + \sum\limits_{k=m}^{\infty} a_{k,i} z^k (a_{k,i} \ge 0, i = 1, 2, ..., j, k \ge m, m \ge p)$, *belong to* $L(p, m, n, \beta, \lambda, \ell, A, B)$, *then*

$$F(z) = \sum_{i=1}^{j} c_i f_i(z) \in L(p, m, n, \beta, \lambda, \ell, A, B),$$

where $\sum\limits_{i=1}^{j} c_i = 1$.

Proof. By Theorem 1, we can write for every $i \in \{1, 2..., j\}$

$$\sum_{k=m}^{\infty} \frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n \left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)} a_{k,i} < 1,$$

therefore

$$F(z)=\sum_{i=1}^{j}c_i\left(z^{-p}+\sum_{k=m}^{\infty}a_{k,i}z^k\right)=z^{-p}+\sum_{k=m}^{\infty}\left(\sum_{i=1}^{j}c_ia_{k,i}\right)z^k.$$

However

$$\sum_{k=m}^{\infty}\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\left(\sum_{i=1}^{j}c_ia_{k,i}\right)$$
$$=\sum_{i=1}^{j}\left[\sum_{k=m}^{\infty}\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)a_{k,i}}{p(A-B)}\right]c_i\leq 1,$$

then $F(z)\in L(p,m,n,\beta,\lambda,\ell,A,B)$, so that the proof is complete.

7. Weighted Mean and Arithmetic Mean

Definition 6. Let $f(z)$ and $g(z)$ belong to $L(p,m)$, then the weighted mean $h_j(z)$ of $f(z)$ and $g(z)$ is given by

$$h_j(z)=\frac{1}{2}\left[(1-j)f(z)+(1+j)g(z)\right].$$

In theorem below we will show the weighted mean for this class.

Theorem 7. *If $f(z)$ and $g(z)$ are in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$, then the weighted mean of $f(z)$ and $g(z)$ is also in $L(p,m,n,\beta,\lambda,\ell,A,B)$.*

Proof. We have for $h_j(z)$ by definition 6,

$$\begin{aligned}h_j(z)&=\frac{1}{2}\left[(1-j)\left(z^{-p}+\sum_{k=m}^{\infty}a_kz^k\right)+(1+j)\left(z^{-p}+\sum_{k=m}^{\infty}b_kz^k\right)\right]\\&=z^{-p}+\sum_{k=m}^{\infty}\frac{1}{2}\left((1-j)a_k+(1+j)b_k\right)z^k.\end{aligned}$$

Since $f(z)$ and $g(z)$ are in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$ so by theorem 1, we must prove that

$$\sum_{k=m}^{\infty} k(1-B)\left(\tfrac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\tfrac{\ell+\lambda\beta(k+p)}{\ell}\right)\left[\tfrac{1}{2}(1-j)a_k+\tfrac{1}{2}(1+j)b_k\right]$$

$$\begin{aligned} &= \frac{1}{2}(1-j)\sum_{k=m}^{\infty} k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)a_k \\ &\quad +\frac{1}{2}(1+j)\sum_{k=m}^{\infty} k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)b_k \\ &\leq \frac{1}{2}(1-j)(A-B)p+\frac{1}{2}(1+j)(A-B)p=p(A-B). \end{aligned}$$

The proof is complete.

Theorem 8. *Let* $f_1(z), f_2(z), ..., f_j(z)$ *defined by*

$$f_i(z)=z^{-p}+\sum_{k=m}^{\infty} a_{k,i}z^k \quad ,(a_{k,i}\geq 0, i=1,2,...,j \ ,k\geq m, m\geq p) \tag{7.1}$$

be in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$*, then the arithmetic mean of* $f_i(z)(i=1,2,...,j)$ *defined by*

$$h(z)=\frac{1}{j}\sum_{i=1}^{j} f_i(z) \tag{7.2}$$

is also in the class $L(p,m,n,\beta,\lambda,\ell,A,B)$.

Proof. By (7.1), (7.2) we can write

$$h(z)=\frac{1}{j}\sum_{i=1}^{j}\left(z^{-p}+\sum_{k=m}^{\infty} a_{k,i}z^k\right)=z^{-p}+\sum_{k=m}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{j} a_{k,i}\right)z^k.$$

Since $f_i(z)\in L(p,m,n,\beta,\lambda,\ell,A,B)$ for every $i=1,2,...,j$, so by using Theorem 1, we prove that

$$\begin{aligned} &\sum_{k=m}^{\infty} k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)\left(\frac{1}{j}\sum_{i=1}^{j} a_{k,i}\right) \\ &= \frac{1}{j}\sum_{i=1}^{j}\left[\sum_{k=m}^{\infty} k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\frac{\ell+\lambda\beta(k+p)}{\ell}a_{k,i}\right] \\ &\leq \frac{1}{j}\sum_{i=1}^{j} p(A-B)=p(A-B). \end{aligned}$$

The proof is complete.

8. Convolution Properties

Theorem 9. *If* $f(z)$ *and* $g(z)$ *belong to* $L(p,m,n,\beta,\lambda,\ell,A,B)$ *such that*

$$f(z)=z^{-p}+\sum_{k=m}^{\infty}a_kz^k \quad , \quad g(z)=z^{-p}+\sum_{k=m}^{\infty}b_kz^k, \tag{8.1}$$

then

$$T(z)=z^{-p}+\sum_{k=m}^{\infty}(a_k^2+b_k^2)z^k$$

is in the class $L(p,m,n,\beta,\lambda,\ell,A_1,B_1)$ *such that* $A_1\geq(1-B_1)\mu^2+B_1$,*where*

$$\mu=\frac{\sqrt{2}(A-B)}{\sqrt{m\left(\frac{\ell+\lambda(m+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(m+1)}{\ell}\right)}(1-B)}.$$

Proof. Since $f,g\in L(p,m,n,\beta,\lambda,\ell,A,B)$, Theorem 1, yields

$$\sum_{k=m}^{\infty}\left(\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right]}{p(A-B)}a_k\right)^2\leq 1$$

and

$$\sum_{k=m}^{\infty}\left(\left[\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\right]b_k\right)^2\leq 1$$

we obtain from the last two inequalities

$$\sum_{k=m}^{\infty}\frac{1}{2}\left[\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\right]^2\left(a_2^2+b_k^2\right)\leq 1. \tag{8.2}$$

However, $T(z)\in L(p,m,n,\beta,\lambda,\ell,A_1,B_1)$ if and only if

$$\sum_{k=m}^{\infty}\left[\frac{k(1-B_1)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A_1-B_1)}\right]\left(a_2^2+b_k^2\right)\le 1 \tag{8.3}$$

where $-1\le B_1<A_1\le 1$, but (8.2) implies (8.3) if

$$\frac{k(1-B_1)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A_1-B_1)}<\frac{1}{2}\left[\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\right]^2$$

Hence, if

$$\frac{1-B_1}{A_1-B_1}<\frac{k\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{2p}\alpha^2,\ \text{where } \alpha=\frac{1-B}{A-B}.$$

In other words,

$$\frac{1-B_1}{A_1-B_1}<\frac{k\left(\frac{\ell+\lambda(k+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+1)}{\ell}\right)\alpha^2}{2}.$$

This is equivalent to

$$\frac{A_1-B_1}{1-B_1}>\frac{2}{k\left(\frac{\ell+\lambda(k+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+1)}{\ell}\right)\alpha^2}.$$

So we can write

$$\frac{A_1-B_1}{1-B_1}>\frac{2(A-B)^2}{m\left(\frac{\ell+\lambda(m+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(m+1)}{\ell}\right)(1-B)^2}=\mu^2.$$

Hence we get $A_1\ge(1-B_1)\mu^2+B_1$.

Theorem 10. *Let $f(z)$ and $g(z)$ of the form (8.1) belong to $L(p,m,n,\beta,\lambda,\ell,A,B)$, then the convolution (or Hadamard product) of two functions f and g belong to the class, that is, $(f*g)(z)\in L(p,m,n,\beta,\lambda,\ell,A_1,B_1)$, where $A_1\ge(1-B_1)v+B_1$ and*

$$v=\frac{(A-B)^2}{m(1-B)^2\left(\frac{\ell+\lambda(m+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(m+1)}{\ell}\right)}.$$

Proof. Since $f, g \in L(p, m, n, \beta, \lambda, \ell, A, B)$, by using the Cauchy - Schwarz inequality and Theorem 1, we obtain

$$\left(\sum_{k=m}^{\infty} \frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\sqrt{a_k b_k}\right)$$

$$\leq \left(\sum_{k=m}^{\infty} \frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)} a_k\right)^{1/2}$$

$$\times \left(\sum_{k=m}^{\infty} \frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)} b_k\right)^{1/2} \leq 1. \quad (8.4)$$

We must find the values of A_1, B_1 so that

$$\sum_{k=m}^{\infty} \frac{k(1-B_1)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A_1-B_1)} a_k b_k < 1. \quad (8.5)$$

Therefore, by (8.4) , (8.5) holds true if

$$\sqrt{a_k b_k} \leq \frac{(1-B)(A_1-B_1)}{(1-B_1)(A-B)} \quad , (k \geq m, m \geq p, a_k \neq 0, b_k \neq 0) \quad (8.6)$$

By (8.4), we have $\sqrt{a_k b_k} < \dfrac{p(A-B)}{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right]^n\left[\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}$, therefore (8.6) holds true if

$$\frac{k(1-B_1)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A_1-B_1)}$$

$$\leq \left[\frac{k(1-B)\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)}\right]^2,$$

which is equivalent to

$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2\left(\frac{\ell+\lambda(k+p)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+p)}{\ell}\right)}{p(A-B)^2}.$$

Alternatively, we can write

$$\frac{(1-B_1)}{(A_1-B_1)} < \frac{k(1-B)^2\left(\frac{\ell+\lambda(k+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(k+1)}{\ell}\right)}{(A-B)^2},$$

to obtain

$$\frac{A_1-B_1}{1-B_1} > \frac{(A-B)^2}{m(1-B)^2\left(\frac{\ell+\lambda(m+1)}{\ell}\right)^n\left(\frac{\ell+\lambda\beta(m+1)}{\ell}\right)} = v.$$

Hence we get $A_1 > v(1-B_1)+B_1$.

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