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AN EFFICIENT NUMERICAL METHOD FOR SOLVING FREDHOLM INTEGRAL EQUATIONS OVER $(0, +\infty)$

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Abstract: In this paper, two types of integral equations of the second kind over semiaxis are presented. Then we introduce a numerical method by using the product rule and Nyström method. The method reduces the integral equation to a system of linear algebraic equations. Some numerical examples are presented to show the efficiency and accuracy of the method.

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1. Introduction

Let us consider the following Fredholm integral equation

$$f(y) - \int_0^{+\infty} k(x, y) f(x) u(x) dx = g(y), \quad y \in (0, +\infty), \tag{1}$$

where $k(x, y)$ and $g(y)$ are known functions, $u(x) = (1 + x)^\lambda x^\gamma e^{\frac{-x^\beta}{2}}$, is a generalized Laguerre weight, where $x \in \mathbb{R}^+$, $\alpha > -1$, $\beta > \frac{1}{2}$, $\gamma \geq -1$, $\lambda \geq 0$ and f is the unknown function which should be determined.

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2. Preliminaries

2.1. Functional Spaces

In the sequel we consider the following set of continuous functions on $\mathbb{R}^+ = (0, +\infty), (f \in C^0(\mathbb{R}^+))$ and

$$C_u := \left\{ f \in C^0(\mathbb{R}^+) : \lim_{x\to 0^+} f(x)u(x) = 0 = \lim_{x\to +\infty} f(x)u(x) \right\}$$

with the norm

$$\|f\|_{C_u} := \|fu\| := \max_{x\geq 0} |(fu)(x)|.$$

Moreover, we denote by $W_r = W_r(u), \quad r \geq 1$, the Sobolev-type space defined by

$$W_r = W_r(u) = \{f \in C_u : f^{(r-1)} \in AC(\mathbb{R}^+) \text{ and } \|f^{(r)}\varphi^r u\| < +\infty\},$$

with the norm

$$\|f\|_{w_r} := \|fu\| + \|f^{(r)}\varphi^r u\|,$$

(see [8] and [5]) AC stand for absolutly continuous.

2.2. Extended Lagrange Interpolation

We denote by $\mathcal{P}_n$ the set of all polynomials of degree at most n. For sufficiently large n (say $n > n_0$) we introduce the quantity $a_n = a_n(w) = 4\ \frac{\Gamma(\beta)^{\frac{2}{\beta}}}{\Gamma(2\beta)^{\frac{1}{\beta}}}\ m^{\frac{1}{\beta}}$ (Maskhar-Rachmanove-saff numbers, see [7]). Here after C denotes a positive constant which may be different in different formulas. We write $C \neq C(a, b, \ldots)$ to say that C is a constant independent of the parameters $a, b, \ldots$ and we write $C = C(a, b, \ldots)$ to say that C depends on $a, b, \ldots$. Let $x_k = x_{n,k},\ k = 1, 2, \ldots, n$ be zeros of the n-th Laguerre orthogonal polynomial ordered in increasing order. We recall that all of these zeros are contained in the interval $(0, 4m)$ and that $x_m \sim 4m - m^{\frac{1}{3}}$ (see [9]). Let $w(x) = x^\alpha e^{-x^\beta},\ \alpha > -1,\ \beta > \frac{1}{2}$. The polynomial $L^*_{n+1}(w, f; x)$ denotes the Lagrange polynomial which interpolates a given function at the zeros of $p_n(w)$ and at a_n, that is

$$L^*_{n+1}(w, f; x_k) = \sum_{k=1}^{n+1} l^*_k(x) f(x_k), \quad x_{n+1} = a_n,$$

where

$$l_k^*(x) = \frac{a_n - x}{a_n - x_k} \frac{p_n(w;x)}{p_n'(w;x_k)(x-x_k)}, \quad k \le n, \quad l_{n+1}^*(x) = \frac{p_n(w;x)}{p_n(w;a_n)}.$$

For any fixed $0 < \theta < 1$, define

$$x_j = x_j(w) = \min\left\{x_k : x_k \ge \theta a_n, \ \ k = 1, 2, \dots, n\right\}.$$

We define

$$L_{n+1}^{**}(w, f; x) = \sum_{k=1}^{j} l_k^*(x) f(x_k),$$

the polynomial $L_{n+1}^{**}(w, f; x)$ belongs to a subspace of $\mathcal{P}_n$, namely $\mathcal{P}_n^*$ with

$$\mathcal{P}_n^* = \{p \in \mathcal{P}_n : p(x_i) = p(a_n) = 0, x_i > x_j\} \subset \mathcal{P}_n.$$

In other words, $L_{n+1}^{**} : C_u \to \mathcal{P}_n^*$ is a projection from C_u onto $\mathcal{P}_n^*$ (see [5] and [8]).

2.3. Product Rule

A Gaussian-type formula is the following so called "product rule"

$$\int_0^{+\infty} f(x)k(x,y)u(x)dx = \sum_{k=1}^{j} f(x_k)\lambda_k(y) + e_n^*(f,y)$$
$$=: I_n(f,y) + e_n^*(f,y), \quad (2)$$

where $k : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$, $\lambda_k(y) = \int_0^{+\infty} l_k^*(x)k(x,y)u(x)dx$, and $e_m^*(f)$ is the error of the quadrature rule [3].

3. Numerical Results

Now we return to the equation

$$f(y) - \int_0^{+\infty} k(x,y)f(x)u(x)dx = g(y), \quad y \in (0,+\infty),$$

we rewrite this equation in the operator form

$$(I - \mathcal{K})f = g \quad (3)$$

where I is the identity symbol. By multiplying both sides of the equation by the weight function $u(y)$, we have

$$(fu)(y) - u(y)\int_0^{+\infty} k(x,y)f(x)u(x)dx = (gu)(y), \quad y \in [0,+\infty), \tag{4}$$

approximating the integral in (4) by product rule, we get

$$(fu)(y) - u(y)\sum_{k=1}^{j} f(x_k)\lambda_k(y) = (gu)(y)$$

where

$$\lambda_k(y) = \int_0^{+\infty} l_k^*(x)k(x,y)u(x)dx.$$

Now, we apply the Nyström method, and get

$$\sum_{k=1}^{j}\left[\delta_{ik} - \frac{u(x_i)}{u(x_k)}\lambda_k(x_i)\right]\alpha_k = (gu)(x_i), \quad i = 1,\ldots,j \tag{5}$$

from (5), we obtain $f(x_k)$ for $k = 1,\ldots,j$. Then the approximate solution obtains by $\bar{f}_n(y) = \sum_{i=1}^{j}\varphi_i(y)\alpha_i \in \mathcal{P}_n^*$ with $\varphi_i(z) = \frac{l_i^*(z)}{u(x_i)}$ and $\alpha_i = fu(x_i)$.

4. Numerical Solution of Another Kind of Fredholm Integral Equation

In this section, we consider the integral equation of the form

$$f(y) - \int_0^{+\infty} k(x,y)f(x)w(x)dx = g(y), \tag{6}$$

where $w(x) = x^{\alpha}e^{-x}$, $\alpha > -1$ and $u(x) = \sqrt{w(x)}(\frac{x}{1+x})^a(1+x)^b$. Moreover, let the real constants a, b satisfy the conditions $a > -1 + \max(\frac{-\alpha}{2}, \frac{1}{4})$, $b < \frac{-1}{4}$. We now offer a way to solve the equation (6). First, we introduce the following product rule

$$\int_0^{+\infty} k(x,y)f(x)w(x)dx = \sum_{i=1}^{n}\lambda_i(y)f(x_i) + e_n^*(f,y)$$

with

$$\lambda_i(y) = \int_0^{+\infty} k(x,y) l_i(x) w(x) dx,$$

(see [9]). Now, we are going to represent a method which is based upon the product rule and the Nyström method for solving this kind of integral equation. To do this, we multiply both sides of equation (6) by $u(y)$, then by applying product rule we have

$$(fu)(y) - u(y) \sum_{k=1}^{j} f(x_k) \lambda_k(y) = (gu)(y).$$

Now, we can apply the Nyström method, and get

$$\sum_{k=1}^{j} \left[\delta_{ik} - \frac{u(x_i)}{u(x_k)} \lambda_k(x_i) \right] \alpha_k = (gu)(x_i), \quad i = 1, \ldots, j. \tag{7}$$

By solving this system we can find, $\alpha_i,\ i = 1, \ldots, j$ and so the approximate solution of integral equation will be obtained.

5. Numerical Examples

Example 1. Consider the equation

$$f(y) - \int_0^{+\infty} xyf(x) e^{\frac{-x}{2}} dx = e^{-y} - 0.444444y,$$

with $k(x,y) = xy$ and $u(x) = e^{\frac{-x}{2}}$ and $\theta = 0.5$ and consider the equation in C_u, with exact solution $f(x) = e^{-x}$. We can construct and solve the system (5). In Table 1, we report the values of the weighted error $u(y)|e^{-y} - \bar{f}_n(y)|$ at some points.

Example 2. Let

$$f(y) - \int_0^{+\infty} 2xyf(x) e^{-x} dx = e^{-y} - 0.5y.$$

Here $k(x,y) = 2xy$ and $w(x) = e^{-x}$. Taking $u(x) = e^{\frac{-x}{2}}$ and $\theta = 0.5$, we can construct and solve the system (7). In Table 2, we report the values of the weighted error $u(y)|e^{-y} - \bar{f}_n(y)|$ at some points.

n	$y=1$	$y=10$	$y=100$	$y=1000$	$y=10000$
16	2.17603 (-1)	5.35497 (-4)	2.831 (-23)	3.97283 (-216)	2.057608 (-2168)
32	1.43843 (-1)	5.11362 (-4)	1.80273 (-25)	1.07388 (-213)	7.800829 (-2163)
64	2.2313 (-1)	3.05902 (-7)	5.27528 (-29)	7.80687 (-235)	8.9784356 (-2199)

Table 1: Weighted errors $|[e^{-y} - \bar{f}_n(y)]u(y)|$ at the points $y = 1, 10, 100, 1000, 10000.$ for Example 1.

n	$y=1$	$y=10$	$y=100$	$y=1000$	$y=10000$
16	7.187331 (-2)	3.01172 (-4)	1.59291 (-23)	2.23538 (-216)	3.01172 (-2168)
32	6.31523 (-2)	1.03209 (-3)	3.6374 (-25)	2.16679 (-213)	3.081485 (-2161)
64	2.2313 (-1)	3.05901 (-7)	1.05438 (-28)	1.56023 (-234)	1.794537 (-2198)

Table 2: Weighted errors $|[e^{-y} - \bar{f}_n(y)]u(y)|$ at the points $y = 1, 10, 100, 1000, 10000.$ for Example 2.

n	y = 1	y = 10	y = 100	y = 1000	y = 10000
16	7.09255 (-2)	7.67593 (-14)	1.01124 (-222)	7.2671 (-3216)	1.75964(-42170)
32	6.98888 (-2)	1.68034 (-4)	3.83948 (-29)	1.40611 (-217)	9.183649 (-2169)
64	1.11565 (-1)	1.17938 (-17)	3.36694 (-229)	1.040014 (-3235)	9.12940735(-42201)

Table 3: Weighted errors $|[e^{-y} - \bar{f}_n(y)]u(y)|$ at the points $y = 1, 10, 100, 1000, 10000.$ for example 3.

Example 3. Here we consider the equation

$$f(y) - \int_0^{+\infty} 2xy^3 f(x) x e^{-x} dx = e^{-y} - 1.18519 y^3,$$

with $k(x,y) = 2xy^3$ and $w(x) = xe^{-x}$. Taking $u(x) = x^{\frac{1}{2}}(1+x)^{-1}e^{\frac{-x}{2}}$ and $\theta = 0.5$, we obtain the results presented in Table 3.

Example 4. Consider the equation

$$f(y) - \int_0^{+\infty} (2x+y)e^{-x+y} f(x) x^4 e^{-x} dx = e^y(-2.02752 - 0.94464y) + 3\sin(x).$$

Where $k(x,y) = (2x+y)e^{-x+y}$ and $w(x) = x^4 e^{-x}$ with the exact solution $f(x) = 3\sin(x)$. Taking $u(x) = x^2(\frac{x}{1+x})(1+x)^{-2}e^{\frac{-x}{2}}$ and $\theta = 0.4$, we obtain the results presented in Table 4.

n	$y = 1$	$y = 10$	$y = 100$	$y = 1000$	$y = 10000$
16	2.13305 (-2)	3.91871 (-2)	6.0038 (-21)	1.13907 (-213)	1.13931 (-213)
32	8.19234 (-2)	8.28765 (-3)	2.98861 (-22)	1.00302 (-209)	7.556268(-2158)
64	8.61788 (-2)	3.96789 (-2)	1.87203 (-22)	3.22162 (-214)	3.815379 (-2172)

Table 4: Weighted errors $|[3\sin(x) - \bar{f}_n(y)]u(y)|$ at the points $y = 1, 10, 100, 1000, 10000.$ for example 4.

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