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THE SLANT HELICES ACCORDING TO TYPE-2 BISHOP FRAME IN EUCLIDEAN 3-SPACE

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Abstract: In this study, we have defined slant helix according to type-2 Bishop frame in Euclidean 3-Space. Furthermore, we have given some necassary and sufficient conditons for the slant helix.

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1. Introduction

In differential geometry, a curve of constant slope or general helix in Euclidean 3-space E^3 is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by M. A. Lancret in 1802[8] and first proved by B. de Saint Venant in 1845[3]. A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of κ and τ are non-zero constant it is, of course, a general helix. We call it a circular helix. It's known that

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straight line and circle are degenerate-helix examples ($\kappa = 0$, if the curve is straight line and $\tau = 0$, if the curve is a circle). The study of these curves in E^3 as spherical curves is given by Monterde. Natural scientists have long held a fascination, sometimes bordering on mystical obsession for helical structures in nature. Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. This fact was published for the first time by Watson and Crick in 1953. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals fores forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actinomycete, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures). Helix is one of the most fascinating curves in science and nature. Also we can see the helix curve or helical structures in fractal geometry, for instance hyperhelices. In the field of computer aided design and computer graphics, helices can be used for the tool path description, the simulation of kinematic motion or the design of highways, etc. From the view of differential geometry, a helix is a geometric curve with non-vanishing constant curvature and non-vanishing constant torsion. The helix may be called a circular helix or W-curve. Besides, a slant helix in Euclidean space E^3 was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi[13] showed that α is a slant helix in E^3 if and only if the geodesic curvature of the principal normal of a space curve α is a constant function. Kula and Yayli[6] have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix.

In this paper,we consider slant helices with type-2 Bishop frame and obtain the differential equations which are characterizations of a slant helix. We hope that these results will be helpful to mathematicians who are specialized on this area.

2. Preliminaries

Let M be an $n-$dimensional smooth manifold equipped with a metric $\langle , \rangle$. A tangent space $T_P(M)$ at a point $P \in M$ is furnished with the canonical inner product. If $\langle , \rangle$ is positive definite, then M is a Riemannian manifold. A curve on an Riemannian manifold M is a smooth mapping $\alpha : I \to M$, where I is an open interval in the real line R^1. As an open submanifold of R^1, I has a coordinate system consisting of the identity map u of I. The velocity vector of α at $s \in I$

$$\alpha^{'}(s) = \frac{d\alpha(u)}{du} \in T_{\alpha(s)}(M).$$

A curve $\alpha(s)$ is said to be regular if $\alpha^{'}(s)$ is not zero for any s. Let $\alpha(s)$ be a curve on M, denote by $\{T, N, B\}$ the moving Frenet frame along the curve α. Then T, N and B are respectively the tangent, the principal normal and binormal vectors of the curve α. If α is a space curve, then this set of orthogonal unit vectors, known as the Frenet-Serret frame[1], has the following properties,

$$\begin{aligned} \alpha^{'}(s) &= T \\ D_T T &= \kappa N \\ D_T N &= -\kappa T + \tau B \\ D_T B &= -\tau N \end{aligned}$$

where D denotes the covariant differentiation in M and κ, τ are respectively curvature and torsion of the curve α. A curve α on M is described by the Frenet formula. For example, if all curvatures of a curve are identically zero, then the curve is a geodesic. If only the curvature κ is a non-zero constant and the torsion τ is all identically zero, then the curve is called a circle. If the curvature κ and the torsion τ are non-zero constants, then the curve is called a helix. If the curvature κ and the torsion τ are not constant but $\frac{\kappa}{\tau}$ is constant, then the curve is a called a general helix.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame.

Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3. The type-2 Bishop frame of the $\alpha(s)$ is denoted by $\{N_1, N_2, B\}$ and defined by[7]

$$\begin{bmatrix} N_1^{'} \\ N_2^{'} \\ B^{'} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & -k_2 \\ k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}$$

The relation matrix between Frenet-Serret and type-2 Bishop frames can be expressed by

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin\theta(s) & -\cos\theta(s) & 0 \\ \cos\theta(s) & \sin\theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}$$

Here, the type-2 Bishop curvatures are defined by

$$k_1(s) = -\tau\cos\theta(s), \quad k_2(s) = -\tau\sin\theta(s)$$

It can be also deduced as

$$\theta' = \kappa = \frac{\left(\frac{k_2}{k_1}\right)'}{1 + \left(\frac{k_2}{k_1}\right)^2}$$

The frame $\{N_1, N_2, B\}$ is properly oriented, τ and $\theta(s) = \int_0^s \kappa(s)\,ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{N_1, N_2, B, k_1, k_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$.

3. The Slant Helices According to Type-2 Bishop Frame in E^3

Definition 1. A regular curve $\alpha : I \to E^3$ is called a slant helix if the unit vector $N_2(s)$ of α has constant angle θ with some fixed unit vector u; that is,

$$\langle N_2(s), u\rangle = \cos\theta \quad \text{for all } s \in I$$

The condition is not altered by reparametrization, so without loss of generality we may assume that slant helices have unit speed. Slant helices can be identified by a simple condition on natural curvatures.

Theorem 2. *Let $\alpha : I \to E^3$ be a unit speed curve with nonzero natural curvatures. Then α is a slant helix if and only if*

$$\frac{k_1}{k_2} = -\cot\theta$$

Proof. Let α be a slant helix in E^3 and $\langle N_2, u\rangle = \cos\theta$. Then α is slant helix; from the definiton, we have

$$\langle N_2, u\rangle = consnt.$$

where u is a unit vector, called the axis of slant helix. By differentiation we get

$$\left\langle N_2^{'}, u\right\rangle = \langle -k_2 B, u\rangle = -k_2 \langle B, u\rangle = 0.$$

Hence

$$\langle B, u\rangle = 0.$$

Again differentiating from the last equalitiy, we can write as follows

$$\begin{aligned} \left\langle B^{'}, u\right\rangle &= \langle k_1 N_1 + k_2 N_2, u\rangle \\ &= k_1 \langle N_1, u\rangle + k_2 \langle N_2, u\rangle \\ &= k_1 \sin\theta + k_2 \cos\theta = 0 \end{aligned}$$

Therefore we obtain

$$\frac{k_1}{k_2} = -\cot\theta$$

Suppose that $\frac{k_1}{k_2} = -\cot\theta$. Then we can write $u \in Sp\{N_1, N_2\}$, i.e.,

$$u = N_1 \sin\theta + N_2 \cos\theta$$

Differentiating the last equality,

$$\begin{aligned} u^{'} &= -k_1 B \sin\theta - k_2 B \cos\theta = 0 \\ u^{'} &= (-k_1 \sin\theta - k_2 \cos\theta) B = 0 \end{aligned}$$

So u is a constant vector. Thus, the proof is completed. □

Theorem 3. *Let $\alpha = \alpha(s)$ be a unit speed curve in E^3. Then α is a slant helix if and only if*

$$\det(N_1^{'}, N_1^{''}, N_1^{'''}) = 0$$

Proof. Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned} N_1^{'} &= -k_1 B \\ N_1^{''} &= -k_1^2 N_1 - k_1 k_2 N_2 - k_1^{'} B \\ N_1^{'''} &= (-3k_1 k_1^{'}) N_1 + (-k_2^{'} k_1 - 2k_1^{'} k_2) N_2 \\ &\quad + (-k_1^{''} + k_1^3 + k_1 k_2^2) B \end{aligned}$$

So we get

$$\det(N_1^{'}, N_1^{''}, N_1^{'''})$$

$$
\begin{aligned}
&= \begin{vmatrix} 0 & 0 & -k_1 \\ -k_1^2 & -k_1 k_2 & -k_1' \\ -3k_1 k_1' & -k_2' k_1 - 2k_1' k_2 & -k_1'' + k_1^3 + k_1 k_2^2 \end{vmatrix} \\
&= k_1^3 k_2^2 \left(\frac{k_1}{k_2} \right)'
\end{aligned}
$$

Since α is a slant helix, $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_1', N_1'', N_1''') = 0$$

Suppose that $\det(N_1', N_1'', N_1''') = 0$. Then it is clear that $\frac{k_1}{k_2} = const$. For being

$$\left(\frac{k_1}{k_2} \right)' = 0$$

Thus, the proof is completed. □

Theorem 4. *Let $\alpha = \alpha(s)$ be a unit speed curve in E^3. Then α is a slant helix if and only if*

$$\det(N_2', N_2'', N_2''') = 0$$

Proof. Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$
\begin{aligned}
N_2' &= -k_2 B \\
N_2'' &= -k_1 k_2 N_1 - k_2^2 N_2 - k_2' B \\
N_2''' &= (-k_1' k_2 - 2k_2' k_1) N_1 + (-3k_2 k_2') N_2 \\
&\quad + (-k_2'' + k_2^3 + k_2 k_1^2) B
\end{aligned}
$$

So we get

$$
\begin{aligned}
&\det(N_2', N_2'', N_2''') \\
&= \begin{vmatrix} 0 & 0 & -k_2 \\ -k_1 k_2 & -k_2^2 & -k_2' \\ -k_1' k_2 - 2k_2' k_1 & -3k_2 k_2' & \begin{matrix} -k_2'' + k_2^3 \\ +k_2 k_1^2 \end{matrix} \end{vmatrix} \\
&= k_2^5 \left(\frac{k_1}{k_2} \right)'
\end{aligned}
$$

Since α is a slant helix, $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(N_2', N_2'', N_2''') = 0$$

Suppose that $\det(N_2^{'}, N_2^{''}, N_2^{'''}) = 0$. Then it is clear that the $\frac{k_1}{k_2} = const$. For being

$$\left(\frac{k_1}{k_2}\right)^{'} = 0$$

Thus, the proof is completed. □

Theorem 5. *Let $\alpha = \alpha(s)$ be a unit speed curve in E^3. Then α is a slant helix if and only if*

$$\det(B^{'}, B^{''}, B^{'''}) = 0.$$

Proof. Suppose that $\frac{k_1}{k_2}$ be constant. We have equalities as

$$\begin{aligned} B^{'} &= k_1 N_1 + k_2 N_2 \\ B^{''} &= k_1^{'} N_1 + k_2^{'} N_2 - \left(k_1^2 + k_2^2\right) B \\ B^{'''} &= \left(k_1^{''} - k_1^3 - k_1 k_2^2\right) N_1 + \left(-k_1^2 k_2 + k_2^{''} - k_2^3\right) N_2 \\ &\quad + \left(-3k_1 k_1^{'} - 3k_2 k_2^{'}\right) \end{aligned}$$

So we get

$$\begin{aligned} &\det(B^{'}, B^{''}, B^{'''}) \\ &= \begin{vmatrix} k_1 & k_2 & 0 \\ k_1^{'} & k_2^{'} & -\left(k_1^2 + k_2^2\right) \\ \begin{matrix} k_1^{''} - k_1^3 \\ -k_1 k_2^2 \end{matrix} & \begin{matrix} -k_1^2 k_2 + k_2^{''} \\ -k_2^3 \end{matrix} & -3(k_1 k_1^{'} + k_2 k_2^{'}) \end{vmatrix} \\ &= 3k_1 k_1^{'} (k_1^{'} k_2 - k_1 k_2^{'}) + 3k_2 k_2^{'} (k_1^{'} k_2 - k_1 k_2^{'}) \\ &\quad - \left(k_1^2 + k_2^2\right) (k_1^{''} k_2 - k_1 k_2^{''}) \\ &= 3k_1 k_1^{'} k_2^2 \left(\frac{k_1}{k_2}\right)^{'} + 3k_2^3 k_2^{'} \left(\frac{k_1}{k_2}\right)^{'} \\ &\quad - \left(k_1^2 + k_2^2\right) \left[\left(\frac{k_1}{k_2}\right)^{''} k_2^4 + 2k_2^3 k_2^{'} \left(\frac{k_1}{k_2}\right)^{'}\right] \end{aligned}$$

Since α is a slant helix, $\frac{k_1}{k_2}$ is constant. Hence, we have

$$\det(B^{'}, B^{''}, B^{'''}) = 0$$

Suppose that $\det(B^{'}, B^{''}, B^{'''}) = 0$. Then it is clear that the $\frac{k_1}{k_2} = const$. For being

$$\left(\frac{k_1}{k_2}\right)^{'} = 0$$

Thus, the proof is completed. □

Next we consider general slant helices in a Euclidean manifold M. Then we have equalities

$$\begin{aligned} \alpha'(s) &= T \\ D_T N_1 &= -k_1 B \\ D_T N_2 &= -k_2 B \\ D_T B &= k_1 N_1 + k_2 N_2 \end{aligned} \tag{1}$$

for any $s \in I$, where $N_1(s)$ and $N_2(s)$ are vector fields and $k_1$and k_2 are functions of parameter s.

Theorem 6. *A unit speed curve α on M is a general slant helix if and only if*

$$D_T(D_T D_T N_1) = D_T N_1 \left(\frac{k_1''}{k_1} - k_2^2 - k_1^2 \right) - 3k_1' D_T B \tag{2}$$

Proof. Suppose that α is general slant helix. Then, from eq.1, we have

$$\begin{aligned} D_T\left(D_T N_1\right) &= D_T\left(-k_1 B\right) = -k_1' B - k_1 D_T B \\ &= -k_1^2 N_1 - k_1 k_2 N_2 - k_1' B \end{aligned} \tag{3}$$

and

$$\begin{aligned} D_T(D_T D_T N_1) &= -k_1'' B - k_1' D_T B - 2k_1' k_1 N_1 - k_1^2 D_T N_1 \\ &\quad - \left(k_2' k_1 + k_1' k_2\right) N_2 - k_1 k_2 D_T N_2 \end{aligned} \tag{4}$$

by considering 1 we get

$$\begin{aligned} D_T(D_T D_T N_1) &= \left(-2k_1' k_1\right) N_1 - \left(k_2' k_1 + k_1' k_2\right) N_2 \\ &\quad + \left(k_1 k_2^2 - k_1''\right) B - k_1' D_T B - k_1^2 D_T N_1 \end{aligned}$$

and since α is general slant helix, we have $\frac{k_1}{k_2} = const$. If we substitute the values,

$$B = \frac{-1}{k_1} D_T N_1 \tag{5}$$

we get as desired

$$D_T(D_T D_T N_1) = D_T N_1 \left(\frac{k_1''}{k_1} - k_2^2 - k_1^2 \right) - 3k_1' D_T B$$

Conversely, let us assume that equation 2 holds. We show that the curve α is general slant helix. If we take derivative of eq. 5, we obtain

$$\begin{aligned} D_T B &= D_T\left(\frac{-1}{k_1} D_T N_1\right) \\ &= \frac{k_1'}{k_1^2} D_T N_1 - \frac{1}{k_1} D_T D_T N_1 \end{aligned}$$

and so,

$$\begin{aligned} D_T D_T B &= \left(\frac{k_1'}{k_1^2}\right)' D_T N_1 + \frac{k_1'}{k_1^2} D_T D_T N_1 \\ &\quad + \frac{k_1'}{k_1^2} D_T D_T N_1 - \frac{1}{k_1} D_T D_T D_T N_1 \end{aligned} \tag{6}$$

From eq. 2 and 6,

$$\begin{aligned} D_T D_T B &= \left[\left(\frac{k_1'}{k_1^2}\right)' - \frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right] D_T N_1 \\ &\quad + \frac{2k_1'}{k_1^2} D_T D_T N_1 + \frac{3k_1'}{k_1} D_T B \end{aligned}$$

Substituting eq. 3 and 4 in this last equality we obtain

$$\begin{aligned} D_T D_T B &= \left[\left(\frac{k_1'}{k_1^2}\right)' - \frac{k_1''}{k_1^2} + \frac{k_2^2}{k_1} + k_1\right] D_T N_1 \\ &\quad + k_1' N_1 + \frac{k_1' k_2}{k_1} N_2 - \frac{2\left(k_1'\right)^2}{k_1^2} B \end{aligned} \tag{7}$$

On the other hand we can write $D_T D_T B$ as follows

$$D_T D_T B = k_1 D_T N_1 + k_1' N_1 + k_2' N_2 - k_2^2 B \tag{8}$$

From comparision the eq. 7 and 8 we obtain equalities below

$$\frac{k_1' k_2}{k_1} = k_2'$$

and so

$$\frac{k_1'}{k_1} = \frac{k_2'}{k_2} \tag{9}$$

Integrating eq. 9 we get

$$\frac{k_1}{k_2} = const.$$

Thus α is a general slant helix. □

Theorem 7. *A unit speed curve α on M is a general slant helix if and only if*

$$D_T(D_T D_T N_2) = D_T N_2 \left(\frac{-k_2''}{k_2} - k_2^2 - k_1^2 \right) - 3k_2' D_T B \tag{10}$$

Proof. Suppose that α is general slant helix. Then from eq.1, we have

$$\begin{aligned} D_T\left(D_T N_2\right) &= D_T\left(-k_2 B\right) = -k_2' B - k_2 D_T B \\ &= -k_2' B - k_2^2 N_2 - k_1 k_2 N_1 \end{aligned}$$

and

$$\begin{aligned} D_T(D_T D_T N_2) &= -k_2'' B - k_2' D_T B - 2k_2' k_2 N_2 - k_2^2 D_T N_2 \\ &\quad - \left(k_2' k_1 - k_1' k_2\right) N_1 - k_1 k_2 D_T N_1 \end{aligned}$$

by considering eq.1 we get

$$\begin{aligned} D_T(D_T D_T N_2) &= B\left(-k_2'' + k_1^2 k_2\right) - k_2' D_T B - 2k_2' k_2 N_2 \\ &\quad - k_2^2 D_T N_2 - \left(k_2' k_1 - k_1' k_2\right) N_2 \end{aligned} \tag{11}$$

Since α is general slant helix, we have $\frac{k_1}{k_2} = const.$And this upon derivation give rise to

$$k_2' k_1 - k_1' k_2 = 0 \tag{12}$$

From 11 and 12 we have

$$\begin{aligned} D_T(D_T D_T N_2) &= B\left(-k_2'' + k_1^2 k_2\right) - k_2' D_T B \\ &\quad - 2k_2' k_2 N_2 - k_2^2 D_T N_2 \end{aligned}$$

If we substitute the values

$$B = \frac{-1}{k_2} D_T N_2$$

So we get as desired

$$D_T(D_T D_T N_2) = D_T N_2 \left(\frac{-k_2''}{k_2} - k_2^2 - k_1^2 \right) - 3k_2' D_T B$$

Conversely, it's appear with similar calculations that in case eq.10 is holds, α is a general slant helix. □

Theorem 8. *A unit speed curve α on M is a general slant helix if and only if*

$$D_T(D_T D_T B) = -D_T B\left(\frac{k_1''}{k_1}+\frac{k_2''}{k_2}+k_2^2+k_1^2\right) + 3k_1' D_T N_1 + 3k_2' D_T N_2 \tag{13}$$

Proof. Suppose that α is general slant helix. Then, from eq.1, we have

$$\begin{aligned} D_T\left(D_T B\right) &= D_T\left(k_1 N_1 + k_2 N_2\right) \\ &= k_1' N_1 + k_1 D_T N_1 + k_2' N_2 + k_2 D_T N_2 \\ &= k_1' N_1 + k_2' N_2 - k_1^2 B - k_2^2 B \end{aligned}$$

and

$$\begin{aligned} D_T(D_T D_T B) &= k_1'' N_1 + k_1' D_T N_1 + k_2'' N_2 + k_2' D_T N_2 \\ &\quad -2k_1' k_1 B - k_1^2 D_T B - 2k_2' k_2 B - k_2^2 D_T B \end{aligned}$$

Similarly calculations we have

$$D_T(D_T D_T B) = -D_T B\left(\frac{k_1''}{k_1}+\frac{k_2''}{k_2}+k_2^2+k_1^2\right) + 3k_1' D_T N_1 + 3k_2' D_T N_2$$

Conversely, it's appear with similar calculations that in case eq.13 is holds, α is a general slant helix. □

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