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SEMI-CLASSICAL ASYMPTOTICS IN AN ABSTRACT BOSE FIELD MODEL

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1. Introduction

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ (h : the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta\hbar)$ of the heat semigroup of a perturbed second quantization operator were derived by Arai [1]. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. The purpose of this paper is to derive an asymptotic formula for $Z(\beta\hbar)$.

The outline of this paper is as follows. In Section 2, following [1], we review a classical limit in the abstract boson Fock space $L^2(E, d\mu)$ over a real separable Hilbert space $\mathscr{H}$. In Section 3, we introduce a class of locally convex spaces. This gives a general framework for the semi-classical analysis discussed in this paper. In the last section, we derive a semi-classical asymptotic formula for $Z(\beta\hbar)$ mentioned above.

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2. A Classical Limit in The Abstract Boson Fock Space

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it, following the work of Arai [1].

Let $\mathscr{H}$ be a real separable Hilbert space, and A be a strictly positive self-adjoint operator acting in $\mathscr{H}$. We denote by $\{\mathscr{H}_s(A)\}_{s\in\mathbb{R}}$ the Hilbert scale associated with A [1]. For all $s \in \mathbb{R}$, the dual space of $\mathscr{H}_s(A)$ can be naturally identified with $\mathscr{H}_{-s}(A)$.

For all $s \geq 0$, the continuous bilinear form $(\phi, f) \longmapsto \phi(f)$ on $\mathscr{H}_{-s}(A) \times \mathscr{H}_s(A)$ extends to the continuous bilinear form on $\mathscr{H}_{-s}(A) \times \mathscr{H}$, by the density of $\mathscr{H}_s(A)$ in $\mathscr{H}$.

We denote by $\mathscr{I}_1(\mathscr{H})$ the ideal of the trace class operators on $\mathscr{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

Assumption 1. $A^{9-\gamma} \in \mathscr{I}_1(\mathscr{H})$.

Under Assumption I, the embedding mapping of $\mathscr{H}$ into

$$E := \mathscr{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure μ on $(E, \mathscr{B})$ such that the Borel field $\mathscr{B}$ is generated by $\{\phi(f) | f \in \mathscr{H}\}$ and

$$\int_E e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|^2_{\mathscr{H}}/2}, \quad f \in \mathscr{H},$$

where $\|\cdot\|_{\mathscr{H}}$ denotes the norm of $\mathscr{H}$.

The complex Hilbert space $L^2(E, d\mu)$ is canonically isomorphic to the boson Fock space over $\mathscr{H}$, which is called the Q-space representation of it. We denote by $d\Gamma(A)$ the second quantization of A and set

$$H_0 = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_0} \in \mathscr{I}_1(L^2(E, d\mu))$.

Definition 2.1. A mapping V of a Banach space X into a Banach space Y is said to be polynomially continuous if there exists a polynomial P of two real variables with positive coefficients such that

$$\|V(\phi) - V(\psi)\| \leq P(\|\phi\|, \|\psi\|)\|\phi - \psi\|, \quad \phi, \psi \in X.$$

Let V be a real valued function on E. Throughout this paper, we assume the following.

Assumption 2. *The function V is bounded from below, 3-times Fréchet differentiable, and V, V', V'', V''' are polynomially continuous.*

For $\hbar > 0$, we define $V_\hbar$ by

$$V_\hbar(\phi) := V(\sqrt{\hbar}\,\phi), \qquad \phi \in E.$$

and set

$$H_\hbar := H_0 \dot{+} \frac{1}{\hbar} V_\hbar,$$

where $\dot{+}$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_\hbar} \in \mathscr{I}_1(L^2(E, d\mu))$ [1]. For all $s \in \mathbb{R}$, $A^{s/2}$ is a continuous linear operator from $\mathscr{H}_s(A)$ to E and it extends to a continuous linear operator from $\mathscr{H}_{-\gamma+s}(A)$ to E.

Theorem 2.2. [1]. *Let $\beta > 0$. Then*

$$\lim_{\hbar \to 0} \frac{\mathrm{Tr}\, e^{-\beta\hbar H_\hbar}}{\mathrm{Tr}\, e^{-\beta\hbar H_0}} = \int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) d\mu(\phi).$$

We set

$$\Omega = E^3, \ \ \nu = \mu \otimes \mu \otimes \mu.$$

Then ν is a probability measure on Ω.

Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of A, and $\{e_n\}_{n=1}^\infty$ be the complete orthonormal system (CONS) of $\mathscr{H}$ with $Ae_n = \lambda_n e_n$, and

$$\sum_{n=1}^\infty \frac{1}{\lambda_n^{\gamma-9}} < \infty \tag{2.1}$$

Let φ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then $\{f_{n,m}\}_{n,m=1}^\infty$ is a CONS of $\mathscr{H}$. For all $\phi \in E$, we define

$$\phi_n := \phi(e_n), \ \ \phi_{n,m} := \phi(f_{n,m}).$$

Then $\{\phi_n\}_n$ and $\{\phi_{n,m}\}_{n,m}$ are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N}$,

$$\int_E \phi_n d\mu(\phi) = 0, \ \ \int_E \phi_n \phi_m d\mu(\phi) = \delta_{nm} \tag{2.2}$$

$$\int_E \phi_{n,m}\phi_{n',m'} d\mu(\phi) = \delta_{nn'}\delta_{mm'}. \tag{2.3}$$

For all $m_1, \cdots, m_p \in \mathbb{N}$, we have

$$\sup_{n_1,\cdots,n_p\in\mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty. \tag{2.4}$$

For all $N, M \in \mathbb{N}$, we set

$$\begin{aligned} F_{N,M}(\varepsilon,\omega,s) &= \sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \\ &\quad + \sum_{n=1}^{N} \sum_{m=1}^{M} \sqrt{\frac{4\varepsilon^2\lambda_n}{\beta(\varepsilon^2\lambda_n^2 \;+\; (2\pi m)^2)}} (\psi_{n,m}\cos(2\pi m s) \\ &\quad + \theta_{n,m}\sin(2\pi m s))e_n, \\ &\quad \varepsilon \ge 0,\ \omega = (\phi,\psi,\theta) \in \Omega,\ 0 \le s \le 1. \end{aligned} \tag{2.5}$$

Then we have

$$\frac{\mathrm{Tr} e^{-\beta\hbar H_\hbar}}{\mathrm{Tr} e^{-\beta\hbar H_0}} = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 V\left(F_{N,M}(\varepsilon,\omega,s)\right) ds\right) d\nu(\omega), \tag{2.6}$$

where $\varepsilon = \beta\hbar$ (See [1], Lemma 5.2, Lemma 5.3.).

3. A Class of Locally Convex Spaces

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

Definition 3.1. A mapping f from $\mathbb{R}_+$ to a locally convex space X is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm p on X,

$$p_\delta(f) := \sup_{0\le\varepsilon\le\delta} p(f(\varepsilon)) < \infty.$$

We denote by $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to X. The topology defined by the seminorms $\{p_\delta\}_{p,\delta}$ turns $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ into a locally convex space. If X is a Fréchet space, $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ is a Fréchet space.

Let $\{E_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces with the property that

$$E_{n+1} \subset E_n,\ \|\phi\|_n \le \|\phi\|_{n+1}, \quad \phi \in E_{n+1},$$

for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of E_n. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ turns $\bigcap_{n\in\mathbb{N}} E_n$ into a Fréchet space.

Let (X, P) be a probability space and Y be a Banach space. We denote by $L^p(X, dP; Y)$ the Banach space of the Y-valued L^p-functions on (X, P). Then $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$ can be provided with the structure of Fréchet space.

Definition 3.2. Let f be a mapping from $\mathbb{R}_+$ to $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$. We say that f is in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0\le\varepsilon\le\delta} \|f(\varepsilon)(x)\|_Y \le g(x),$$

P-a.e.x.

The set $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$ is a linear subspace of

$$(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{l.b.}}.$$

In what follows, we omit x in $f(\varepsilon)(x)$.

Lemma 3.3. *Let $\{f_\lambda\}_{\lambda\in\Lambda}$ and $\{g_\lambda\}_{\lambda\in\Lambda}$ be nets in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$. Suppose that*

$$\overline{\lim_{\lambda}} \sup_{0\le\varepsilon\le\delta} \int_X |f_\lambda(\varepsilon)|^p dP < \infty \text{ and } g_\lambda \longrightarrow 0$$

in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$, for all $p \in \mathbb{N}$ and $\delta > 0$. Then $f_\lambda g_\lambda \longrightarrow 0$ in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP))^{\mathbb{R}_+}_{\text{l.b.}}$.

Proof. Let $p \in \mathbb{N}$ and $\delta > 0$. For each $\varepsilon \ge 0$, by the Schwarz inequality, we have

$$\int_X |f_\lambda(\varepsilon) g_\lambda(\varepsilon)|^p dP \le \left(\int_X |f_\lambda(\varepsilon)|^{2p} dP\right)^{1/2} \left(\int_X |g_\lambda(\varepsilon)|^{2p} dP\right)^{1/2}.$$

Hence we have

$$\sup_{0\le\varepsilon\le\delta} \int_X |f_\lambda(\varepsilon) g_\lambda(\varepsilon)|^p dP$$
$$\le \left(\sup_{0\le\varepsilon\le\delta} \int_X |f_\lambda(\varepsilon)|^{2p} dP\right)^{1/2} \left(\sup_{0\le\varepsilon\le\delta} \int_X |g_\lambda(\varepsilon)|^{2p} dP\right)^{1/2}.$$

Then, by the assumption on f_λ and g_λ, we have $f_\lambda g_\lambda \longrightarrow 0$. □

Let $X_1, \cdots, X_n$ and Z be non-empty sets and G be a real-valued function on $X_1 \times \cdots \times X_n$ and F_j be a mapping from Z to X_j, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on Z, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$

Lemma 3.4. *Let Q be a polynomial of n real variables and*

$$F_j \in \left(\bigcap_{p\in\mathbb{N}} L^p(X,dP;Y)\right)^{\mathbb{R}_+}_{\text{l.b.}},\ j=1,\cdots,n.$$

Then, for all $\delta>0$,

$$\overline{\lim_{G_1\to F_1,\cdots,G_n\to F_n}}\sup_{0\le\varepsilon\le\delta}\int_X |Q(\|G_1(\varepsilon)\|,\cdots,\|G_n(\varepsilon)\|)|dP<\infty.$$

Proof. It is sufficient to consider the case where

$$Q(x_1,\cdots,x_n)=x_1^{p_1}\cdots x_n^{p_n},\ x_1,\cdots,x_n\in\mathbb{R},\ p_1,\cdots,p_n\in\mathbb{N}.$$

Let $G_j \in \left(\bigcap_{p\in\mathbb{N}} L^p(X,dP;Y)\right)^{\mathbb{R}_+}_{\text{l.b.}}$, $j=1,\cdots,n$. By the Schwarz inequality, we have

$$\begin{aligned}&\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n}dP\\ &\qquad\le\left(\int_X\|G_1(\varepsilon)\|^{2p_1}dP\right)^{1/2}\left(\int_X\|G_2(\varepsilon)\|^{2p_n}\cdots\|G_n(\varepsilon)\|^{2p_n}dP\right)^{1/2}.\end{aligned}$$

Then, for all $\delta>0$,

$$\sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n}dP$$

$$\le\left(\sup_{0\le\varepsilon\le\delta}\int_X\|G_1(\varepsilon)\|^{2p_1}dP\right)^{1/2}\left(\sup_{0\le\varepsilon\le\delta}\int_X\|G_2(\varepsilon)\|^{2p_n}\cdots\|G_n(\varepsilon)\|^{2p_n}dP\right)^{1/2}.$$

By

$$\left(\sup_{0\le\varepsilon\le\delta}\int_X\|G_1(\varepsilon)\|^{2p_1}dP\right)^{1/2}\longrightarrow\left(\sup_{0\le\varepsilon\le\delta}\int_X\|F_1(\varepsilon)\|^{2p_1}dP\right)^{1/2},$$

as $G_1\to F_1$, we inductively have

$$\overline{\lim_{G_1\to F_1,\cdots,G_n\to F_n}}\sup_{0\le\varepsilon\le\delta}\int_X \|G_1(\varepsilon)\|^{p_1}\cdots\|G_n(\varepsilon)\|^{p_n}dP<\infty.$$

□

Proposition 3.5. *Let Q be a polynomial of n real valuables. Then the mapping $(F_1, \cdots, F_n) \longmapsto Q(\|F_1\|, \cdots, \|F_n\|)$ from $\left((\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}} \right)^n$ to $\left(\bigcap_{p\in\mathbb{N}} L^p(X, dP) \right)^{\mathbb{R}_+}_{\text{u.i.}}$ is continuous.*

Proof. We first show that the mapping in the Proposition 3.5 is well defined. Let $\delta > 0$, $p_1, \cdots, p_n \in \mathbb{N}$, and $F_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$, $j = 1, \cdots, n$. We assume that there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0\le\varepsilon\le\delta} (\|F_2(\varepsilon)\|^{p_2} \cdots \|F_n(\varepsilon)\|^{p_n}) \le g,$$

P-a.e.. By the assumption on F_1, there exists a nonnegative function $h \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that

$$\sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \le h,$$

P-a.e.. Then, we have

$$\begin{aligned} \sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \cdots \|F_n(\varepsilon)\|^{p_n} &\le \sup_{0\le\varepsilon\le\delta} \|F_1(\varepsilon)\|^{p_1} \sup_{0\le\varepsilon\le\delta} \|F_2(\varepsilon)\|^{p_2} \cdots \|F_n(\varepsilon)\|^{p_n} \\ &\le hg, \end{aligned}$$

P-a.e.

By the Schwarz inequality, we have $hg \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$. Hence, we inductively have

$$\|F_1\|^{p_1} \cdots \|F_n\|^{p_n} \in \left(\bigcap_{p\in\mathbb{N}} L^p(X, dP) \right)^{\mathbb{R}_+}_{\text{u.i.}}.$$

Let $G_j \in (\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\text{u.i.}}$, $j = 1, \cdots, n$. Then

$$\begin{aligned} &| \, \|F_1\|^{p_1} \cdots \|F_n\|^{p_n} - \|G_1\|^{p_1} \cdots \|G_n\|^{p_n} | \\ &\le \sum_{j=1}^{n} \|G_1\|^{p_1} \cdots \|G_{j-1}\|^{p_{j-1}} \, | \, \|F_j\|^{p_j} - \|G_j\|^{p_j} \, | \, \|F_{j+1}\|^{p_{j+1}} \cdots \|F_n\|^{p_n}. \end{aligned}$$

Then, there exist polynomials $\{Q_j\}_{j=1}^n$ of $2n$ variables with positive coefficients such that

$$| \, \|F_1\|^{p_1} \cdots \|F_n\|^{p_n} - \|G_1\|^{p_1} \cdots \|G_n\|^{p_n} |$$

$$\leq \sum_{j=1}^{n} Q_j(\|F_1\|,\cdots,\|F_n\|,\|G_1\|,\cdots,\|G_n\|)\,\|F_j-G_j\|.$$

Applying Lemma 3.3 and Lemma 3.4, we have

$$\sum_{j=1}^{n} Q_j(\|F_1\|,\cdots,\|F_n\|,\|G_1\|,\cdots,\|G_n\|)\,\|F_j-G_j\| \longrightarrow 0,$$

as $F_1 \to G_1,\cdots,F_n \to G_n$. Hence the mapping in the Proposition 3.5 is continuous. □

Proposition 3.6. *Let Z_j be a Banach space ($j=1,\cdots,n$), L be a continuous multilinear form on $Z_1\times\cdots\times Z_n$, and V_j be a polynomially continuous mapping from Y to $Z_j (j=1,\cdots,n)$. Then the mapping $(F_1,\cdots,F_n)\longmapsto L(V_1\circ F_1,\cdots,V_n\circ F_n)$ from $\left((\bigcap_{p\in\mathbb{N}} L^p(X,dP;Y))^{\mathbb{R}_+}_{\mathrm{u.i.}}\right)^n$ to $\left(\bigcap_{p\in\mathbb{N}} L^p(X,dP)\right)^{\mathbb{R}_+}_{\mathrm{u.i.}}$ is continuous.*

Proof. We first show that the mapping in the Proposition 3.6 is well defined. Let $F_j \in (\bigcap_{p\in\mathbb{N}} L^p(X,dP;Y))^{\mathbb{R}_+}_{u.i.}$, $j=1,\cdots,n$. Then

$$|L(V_1\circ F_1,\cdots,V_n\circ F_n)| \leq \|L\|\|V_1\circ F_1\|\cdots\|V_n\circ F_n\|.$$

Since V_j is polynomially bounded, there exists a polynomial Q of n real variables with positive coefficients such that

$$|L(V_1\circ F_1,\cdots,V_n\circ F_n)| \leq Q(\|F_1\|,\cdots,\|F_n\|).$$

By Proposition 3.5, $Q(\|F_1\|,\cdots,\|F_n\|) \in \left(\bigcap_{p\in\mathbb{N}} L^p(X,dP)\right)^{\mathbb{R}_+}_{\mathrm{u.i.}}$. Hence we have

$$L(V_1\circ F_1,\cdots,V_n\circ F_n) \in \left(\bigcap_{p\in\mathbb{N}} L^p(X,dP)\right)^{\mathbb{R}_+}_{\mathrm{u.i.}}.$$

Let $G_j \in (\bigcap_{p\in\mathbb{N}} L^p(X,dP;Y))^{\mathbb{R}_+}_{u.i.}$, $j=1,\cdots,n$. Then

$$|L(V_1\circ F_1,\cdots,V_n\circ F_n) - L(V_1\circ G_1,\cdots,V_n\circ G_n)|$$

$$\leq \|L\|\sum_{j=1}^{n}\|V_1\circ G_1\|\cdots\|V_{j-1}\circ G_{j-1}\|\|V_j\circ F_j - V_j\circ G_j\|\|V_{j+1}\circ F_{j+1}\|\cdots\|V_n\circ F_n\|.$$

Since V_j is polynomially continuous, there exist polynomials $\{Q_j\}_{j=1}^n$ of $2n$ real variables with positive coefficients such that

$$|L(V_1 \circ F_1, \cdots, V_n \circ F_n) - L(V_1 \circ G_1, \cdots, V_n \circ G_n)|$$

$$\leq \sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\|.$$

Applying Lemma 3.3 and Proposition 3.5, we have

$$\sum_{j=1}^{n} Q_j(\|F_1\|, \cdots, \|F_n\|, \|G_1\|, \cdots, \|G_n\|) \, \|F_j - G_j\| \longrightarrow 0,$$

as $F_1 \to G_1, \cdots, F_n \to G_n$. Hence the mapping in the Proposition 3.6 is continuous. □

Let P_j be a probability measure on a set X_j, $j = 1, 2$. For $F \in (\bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$, we define a mapping $\int_{X_2} F dP_2$ from $\mathbb{R}_+$ to the set of functions on X_1 by

$$\left(\int_{X_2} F dP_2\right)(\varepsilon) = \int_{X_2} F(\varepsilon) dP_2, \qquad \varepsilon \geq 0.$$

By the property

$$\int_{X_1} \int_{X_2} |F(\varepsilon)|^p dP_2 dP_1 < \infty$$

for all $\varepsilon \geq 0$ and $p \in \mathbb{N}$, we have

$$\int_{X_2} |F(\varepsilon)|^p dP_2 < \infty,$$

P_1-a.e.. Hence $\int_{X_2} F dP_2$ is well defined.

Let $\delta > 0$. Then, by the assumption on F, there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2))$ such that

$$\sup_{0\leq\varepsilon\leq\delta} |F(\varepsilon)| \leq g, \qquad P_1 \otimes P_2 - \text{a.e.}$$

Then, we have

$$\sup_{0\leq\varepsilon\leq\delta} \left| \int_{X_2} F(\varepsilon) dP_2 \right| \quad \leq \quad \sup_{0\leq\varepsilon\leq\delta} \int_{X_2} |F(\varepsilon)| dP_2$$

$$\leq \int_{X_2} g \, dP_2,$$

P_1-a.e.. For all $p \in \mathbb{N}$, by Jensen's inequality,

$$\begin{aligned} \int_{X_1} \left| \int_{X_2} g \, dP_2 \right|^p dP_1 &\leq \int_{X_1} \int_{X_2} g^p \, dP_2 dP_1 \\ &< \infty. \end{aligned}$$

Hence we have $\int_{X_2} F dP_2 \in \left(\bigcap_{p \in \mathbb{N}} L^p(X_1, dP_1) \right)^{\mathbb{R}_+}_{u.i.}$.

Proposition 3.7. *The mapping* $F \longmapsto \int_{X_2} F dP_2$ *from*

$$(\bigcap_{p \in \mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$$

to

$$\left(\bigcap_{p \in \mathbb{N}} L^p(X_1, dP_1) \right)^{\mathbb{R}_+}_{\text{u.i.}}$$

is continuous linear.

Proof. Let $F \in (\bigcap_{p \in \mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{u.i.}}$. Then, by Jensen's inequality,

$$\left| \int_{X_2} F(\varepsilon) dP_2 \right|^p \leq \int_{X_2} |F(\varepsilon)|^p \, dP_2.$$

Hence, for all $\delta > 0$, we have

$$\begin{aligned} \sup_{0 \leq \varepsilon \leq \delta} \int_{X_1} \left| \int_{X_2} F(\varepsilon) dP_2 \right|^p dP_1 &\leq \sup_{0 \leq \varepsilon \leq \delta} \int_{X_1} \int_{X_2} |F(\varepsilon)|^p dP_2 dP_1 \\ &\longrightarrow 0 \end{aligned}$$

as $F \to 0$ in $(\bigcap_{p \in \mathbb{N}} L^p(X_1 \times X_2, d(P_1 \otimes P_2)))^{\mathbb{R}_+}_{\text{l.b.}}$. Hence the mapping is continuous. □

4. An Asymptotic Formula

We set

$$Z(\varepsilon) = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 F_{N,M}(\varepsilon,\omega,s)ds\right) d\nu(\omega), \qquad \varepsilon \geq 0, \tag{4.7}$$

(See (2.5) and (2.6)). In this section, we examine the differentiability of Z.

For all $n, m \in \mathbb{N}$, we set

$$\alpha_{n,m}(\varepsilon) = \sqrt{\frac{4\varepsilon^2\lambda_n}{\beta(\varepsilon^2\lambda_n^2 \;+\; (2\pi m)^2)}}, \qquad \varepsilon \geq 0.$$

Then, for all $\delta > 0$, there exists a constant $C > 0$ such that

$$|\alpha_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \qquad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.8}$$

$$|\alpha'_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \qquad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.9}$$

$$|\alpha''_{n,m}(\varepsilon)| \leq \frac{C\lambda_n^{5/2}}{m}, \qquad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.10}$$

$$|\alpha'''_{n,m}(\varepsilon)| \leq \frac{C(\lambda_n^{5/2} \;+\; \lambda_n^{9/2})}{m}, \qquad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.11}$$

For $n, m \in \mathbb{N}$, we set

$$\beta_{n,m}(\omega, s) = \psi_{n,m}\cos(2\pi ms) \;+\; \theta_{n,m}\sin(2\pi ms), \qquad \omega \in \Omega,\ s \in \mathbb{R}.$$

We denote by $\mu^{(L)}_{[\,0,1\,]}$ the Lebesgue measure on $[\,0,1\,]$.

Lemma 4.1. *$\{F_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F''_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes\mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$.*

Proof. By (4.2), (4.3), (4.4), (4.5), $F_{N,M}, F'_{N,M}, F''_{N,M}, F'''_{N,M} \in (\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes\mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$.

The set $\{\lambda_n^{\gamma/2}e_n\}_{n=1}^\infty$ is a CONS of E. Then, for $N, N' \in \mathbb{N}$ with $N > N'$,

$$\left\|\sum_{n=N'+1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n\right\|_E^2 = \sum_{n=N'+1}^{N} \frac{\phi_n^2}{\lambda_n^{\gamma+1}}.$$

Then, for all $p \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{n=N'+1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right\|_E^{2p} &= \left(\sum_{n=N'+1}^{N}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \\ &= \sum_{n_1,\cdots,n_p=N'+1}^{N}\frac{1}{\lambda_{n_1}^{\gamma+1}}\cdots\frac{1}{\lambda_{n_p}^{\gamma+1}}\phi_{n_1}^2\cdots\phi_{n_p}^2.\end{aligned}$$

By (2.4) and the fact that

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma+1}} < \infty,$$

we have

$$\int_\Omega \left\|\sum_{n=N'+1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right\|_E^{2p} d\nu(\phi) \longrightarrow 0,$$

as $N, N' \to 0$.

Let Λ_1, Λ_2 be finite subsets of $\mathbb{N}$. Then,

$$\left\|\sum_{n\in\Lambda_1}\left(\sum_{m\in\Lambda_2}\alpha_{n,m}(\varepsilon)\beta_{n,m}\right)e_n\right\|_E^2 = \sum_{n\in\Lambda_1}\frac{1}{\lambda_n^\gamma}\left(\sum_{m\in\Lambda_2}\alpha_{n,m}(\varepsilon)\beta_{n,m}\right)^2.$$

For all $p \in \mathbb{N}$,

$$\begin{aligned}&\left\|\sum_{n\in\Lambda_1}\left(\sum_{m\in\Lambda_2}\alpha_{n,m}(\varepsilon)\beta_{n,m}\right)e_n\right\|_E^{2p} \\ &= \sum_{n_1,\cdots,n_p\in\Lambda_1}\sum_{m_1,\cdots,m_p\in\Lambda_2}\sum_{l_1,\cdots,l_p\in\Lambda_2}\frac{1}{\lambda_{n_1}^\gamma}\cdots\frac{1}{\lambda_{n_p}^\gamma}\alpha_{n_1,m_1}(\varepsilon)\alpha_{n_1,l_1}(\varepsilon)\cdots\alpha_{n_p,m_p}(\varepsilon) \\ &\quad\times\alpha_{n_p,l_p}(\varepsilon)\beta_{n_1,m_1}\beta_{n_1,l_1}\cdots\beta_{n_p,m_p}\beta_{n_p,l_p}.\end{aligned}$$

By (2.3), (4.2), and the fact that

$$\sum_{m=1}^{\infty}\frac{1}{m^2} < \infty, \quad \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma-1}} < \infty,$$

we have

$$\left\|F_{N,M} - F_{N',M'}\right\|_{\left(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu_{[\,0,1\,]}^{(L)});E)\right)_{\text{l.b.}}^{\mathbb{R}_+}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$.

Similarly, by

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} < \infty,$$

and (4.3), we have

$$\left\| F'_{N,M} - F'_{N',M'} \right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$. By

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-5}} < \infty,$$

and (4.4), we have

$$\left\| F''_{N,M} - F''_{N',M'} \right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$. By

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-9}} < \infty,$$

and (4.5), we have

$$\left\| F'''_{N,M} - F'''_{N',M'} \right\|_{(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,],d(\nu\otimes\mu^{(L)}_{[\,0,1\,]});E))^{\mathbb{R}_+}_{\text{l.b.}}} \longrightarrow 0,$$

as $N, N', M, M' \to \infty$. □

Lemma 4.2. *The mapping* $F \longmapsto \exp\left(-\beta \int_0^1 V \circ F ds\right)$ *from* $(\bigcap_{p\in\mathbb{N}} L^p(\Omega\times [\,0,1\,], d(\nu\otimes\mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$ *to* $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$ *is continuous.*

Proof. Let $F, G \in (\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes\mu^{(L)}_{[\,0,1\,]}; E))^{\mathbb{R}_+}_{\text{l.b.}}$. Since V is bounded from below, by the inequality

$$|e^x - e^y| \le (e^x + e^y)|x - y|, \qquad x, y \in \mathbb{R},$$

there exists a constant $C \ge 0$ such that

$$\left|\exp\left(-\beta\int_0^1 V\circ F ds\right) - \exp\left(-\beta\int_0^1 V\circ G ds\right)\right| \le C\left|\int_0^1 V\circ F ds - \int_0^1 V\circ G ds\right|.$$

Hence, by Proposition 3.7,

$$\left|\exp\left(-\beta\int_0^1 V\circ F ds\right)-\exp\left(-\beta\int_0^1 V\circ G ds\right)\right|\longrightarrow 0,$$

as $F\to G$. □

For all $N, M\in\mathbb{N}$, we set

$$G_{N,M}(\varepsilon,\omega)=\exp\left(-\beta\int_0^1 V\left(F_{N,M}(\varepsilon,\omega,s)\right)ds\right)\qquad \varepsilon\geq 0,\ \omega\in\Omega.$$

Lemma 4.3. *$\{G_{N,M}\}_{N,M\in\mathbb{N}}, \{G'_{N,M}\}_{N,M\in\mathbb{N}}, \{G''_{N,M}\}_{N,M\in\mathbb{N}}, \{G'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega,d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.*

Proof. By Lemma 4.1, Lemma 4.2 and the completeness of $((\bigcap_{p\in\mathbb{N}} L^p(\Omega\times[\,0,1\,], d(\nu\otimes\mu^{(L)}_{[\,0,1\,]};E))^{\mathbb{R}_+}_{\text{l.b.}}$, $\{G_{N,M}\}_{N,M\in\mathbb{N}}$ is a Cauchy net in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega,d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.

For all $\varepsilon\geq 0$,

$$G'_{N,M}(\varepsilon,\omega)=-\beta G_{N,M}(\varepsilon,\omega)\int_0^1 V'(F_{N,M}(\varepsilon,\omega,s))(F'_{N,M}(\varepsilon,\omega,s))ds.$$

In general, for each n-times continuously Fréchet differentiable mapping F from a Banach space X to a Banach space Y, $\phi\in X$, and $n\in\mathbb{N}$, $F^{(n)}(\phi)$ is identified with an element in $\mathscr{L}^{(n)}(X^n,Y)$(the Banach space of continuous multilinear mapping from X^n to Y). The mapping $(L,x_1,\cdots,x_n)\longmapsto L(x_1,\cdots,x_n)$ from $\mathscr{L}^{(n)}(X^n,Y)\times X^n$ to Y is continuous multilinear. Then, by Proposition 3.6, 3.7, and Lemma 4.1, $\{G'_{N,M}\}_{N,M\in\mathbb{N}}$ is a Cauchy net in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega,d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.

Similarly and inductively, $\{G''_{N,M}\}_{N,M\in\mathbb{N}}$ and $\{G'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega,d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$. □

Now we have the following theorem.

Theorem 4.4. *The function Z defined by* (4.1) is 3-times continuously differentiable with the following properties :

$$Z(0)=\int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\right)d\mu(\phi) \tag{4.12}$$

$$Z'(0)=0 \tag{4.13}$$

$$Z''(0)=\sum_{m=1}^{\infty}\int_{E^2} d\mu(\phi)d\mu(\psi)\,(-\beta)\exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\right)$$

$$\times V''\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\left(A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right),\right.$$
$$\left.A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right)\right) \tag{4.14}$$

Proof. By Lemma 4.1, Lemma 4.3, and the fact that $\alpha_{n,m}$ is infinitely differentiable for all $n,m\in\mathbb{N}$, $\int_\Omega H_{N,M}(\varepsilon,\omega)\,d\nu(\omega)$ with $H_{N,M}=G_{N,M}, G'_{N,M}, G''_{N,M}, G'''_{N,M}$ uniformly converges in ε. Hence one can interchange the limit $\lim_{N,M\to\infty}$ with differentiations in ε. Hence Z is 3-times continuously differentiable in $\mathbb{R}_+$.

By Theorem 2.2, we obtain (4.6).

For $\varepsilon\geq 0$

$$Z'(\varepsilon)=\lim_{N,M\to\infty}\int_\Omega G'_{N,M}(\varepsilon,\omega)d\nu(\omega)\,.$$

In particular

$$Z'(0)=\lim_{N,M\to\infty}\int_\Omega G'_{N,M}(0,\omega)d\nu(\omega),$$

and

$$\begin{aligned} G'_{N,M}(0,\omega) &= -\beta\, G_{N,M}(0,\omega)\int_0^1 V'\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}\right) \\ &\quad\times\left(\sum_{n=1}^{N}\alpha'_{n,m}(0)\beta_{n,m}(\omega,s)e_n\right)ds \\ &= -\beta\, G_{N,M}(0,\omega)V'\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}\right) \\ &\quad\times\left(\int_0^1\sum_{n=1}^{N}\alpha'_{n,m}(0)\beta_{n,m}(\omega,s)e_n ds\right). \end{aligned}$$

By the fact that

$$\int_0^1\cos(2\pi ms)ds=\int_0^1\sin(2\pi ms)ds=0,\qquad m\in\mathbb{N},$$

we obtain (4.7). For $\varepsilon\geq 0$

$$G''_{N,M}(\varepsilon,\omega) = -\beta G'_{N,M}(\varepsilon,\omega)\int_0^1 V'(F_{N,M}(\varepsilon,\omega,s))(F'_{N,M}(\varepsilon,\omega,s))ds$$

$$-\beta G_{N,M}(\varepsilon,\omega)\int_0^1 V''(F_{N,M}(\varepsilon,\omega,s))(F'_{N,M}(\varepsilon,\omega,s),$$
$$F'_{N,M}(\varepsilon,\omega,s))+V'(F_{N,M}(\varepsilon,\omega,s))(F''_{N,M}(\varepsilon,\omega,s))ds.$$

In particular,

$$\begin{aligned} & Z''(0) \\ = & \lim_{N,M\to\infty}\sum_{m=1}^{M}\int_{E^2} d\mu(\phi)d\mu(\psi)\,(-\beta)\exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right)\right) \\ \times & V''\left(\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right)\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n,\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{N}\sqrt{\lambda_n}\psi_{n,m}e_n\right), \end{aligned}$$

where we have used the fact that

$$\int_0^1 \cos(2\pi ms)\sin(2\pi ns)ds=0,$$

$$\int_0^1 \cos(2\pi ms)\cos(2\pi ns)ds=\int_0^1 \sin(2\pi ms)\sin(2\pi ns)ds=\frac{\delta_{mn}}{2},\qquad n,m\in\mathbb{N}.$$

By (2.2), we have

$$\begin{aligned} \int_E\sum_{n=1}^{\infty}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}d\mu(\phi) &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma+1}}\int_E\phi_n^2 d\mu(\phi) \\ &= \sum_{n=1}^{\infty}\frac{1}{\lambda_n^{\gamma+1}} \\ &< \infty. \end{aligned}$$

Hence we have

$$A^{-1/2}\phi=\sum_{n=1}^{\infty}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\in E,\qquad \mu-a.e.\ \phi\in E.$$

Then, for all $p,N\in\mathbb{N}$,

$$\begin{aligned} \left\|\sqrt{\frac{2}{\beta}}\sum_{n=1}^{N}\frac{\phi_n}{\sqrt{\lambda_n}}e_n\right\|_E^{2p} &= \left(\frac{2}{\beta}\sum_{n=1}^{N}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \\ &\le \left(\frac{2}{\beta}\right)^p\left(\sum_{n=1}^{\infty}\frac{\phi_n^2}{\lambda_n^{\gamma+1}}\right)^p \end{aligned}$$

$$= \left(\frac{2}{\beta}\right)^p \sum_{n_1,\cdots,n_p=1}^{\infty} \frac{1}{\lambda_{n_1}^{\gamma+1}} \cdots \frac{1}{\lambda_{n_p}^{\gamma+1}} \phi_{n_1}^2 \cdots \phi_{n_p}^2,$$

$\mu-$a.e. $\phi \in E$. Then, by (2.4), we have

$$\int_E \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right) \right\|^2_{\mathscr{L}^{(2)}(E\times E,\mathbb{R})} d\mu(\phi) < \infty.$$

For all $m \in \mathbb{N}$,

$$\begin{aligned} \sum_{n=1}^{\infty} \int_E \frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}} d\mu(\psi) &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \int_E \psi_{n,m}^2 d\mu(\psi) \\ &= \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \\ &< \infty. \end{aligned}$$

Then we have

$$A^{1/2} \left(\sum_{n=1}^{\infty} \psi_{n,m} e_n \right) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \psi_{n,m} e_n \in E,$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.

For all $N, m \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right\|^2 &= \sum_{n=1}^{N} \frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}} \\ &\leq \sum_{n=1}^{\infty} \frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}}, \end{aligned}$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.

Then, for all $N, m \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n} \psi_{n,m} e_n \right\|^4 &\leq \left(\sum_{n=1}^{\infty} \frac{\psi_{n,m}^2}{\lambda_n^{\gamma-1}} \right)^2 \\ &= \sum_{n_1,n_2=1}^{\infty} \frac{1}{\lambda_{n_1}^{\gamma-1}} \frac{1}{\lambda_{n_2}^{\gamma-1}} \psi_{n_1,m}^2 \psi_{n_2,m}^2, \end{aligned}$$

$\mu-$a.e. $\psi \in E$, $m \in \mathbb{N}$.

Then, by (2.4), we have

$$\int_E \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n \right\|^4 d\mu(\psi) < \infty.$$

Since V is bounded from below, there exists a constant $C \geq 0$ such that

$$\begin{aligned} &\left| -\beta \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right) \right. \\ &\left. \times V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \left(\sum_{n=1}^{N} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} \sqrt{\lambda_n} e_n, \sum_{n=1}^{N} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} \sqrt{\lambda_n} e_n \right) \right|^2 \\ &\leq C \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n \right\|^4 . \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\int_{E^2} \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n \right\|^4 d\mu(\phi) d\mu(\psi) \\ = &\int_E \sup_{N\in\mathbb{N}} \left\| V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} \right) \right\|^2 d\mu(\phi) \int_E \sup_{N\in\mathbb{N}} \left\| \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n \right\|^4 d\mu(\psi) \\ < &\ \infty. \end{aligned}$$

Hence, by the dominated convergence theorem, we have for all $M \in \mathbb{N}$,

$$\begin{aligned} &\sum_{m=1}^{M} \int_{E^2} d\mu(\phi) d\mu(\psi) (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right) \right) \\ &\times V'' \left(\sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n \right) \left(\frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n, \frac{1}{\sqrt{\beta}\pi m} \sum_{n=1}^{N} \sqrt{\lambda_n}\psi_{n,m} e_n \right) \\ \longrightarrow &\sum_{m=1}^{M} \int_{E^2} d\mu(\phi) d\mu(\psi) (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi \right) \right) \\ &\times V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi \right) \\ &\times \left(A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right), A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta}\pi m} \psi_{n,m} e_n \right) \right), \end{aligned}$$

as $N \to \infty$. There exists a constant $C \geq 0$ such that

$$\begin{aligned}
&\left| \int_{E^2} d\mu(\phi) d\mu(\psi) \, (-\beta) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) \right. \\
&\times V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \left(A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta} \pi m} \psi_{n,m} e_n \right), A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta} \pi m} \psi_{n,m} e_n \right) \right) \Bigg| \\
\leq & C \int_{E^2} \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| \left\| A^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{\sqrt{\beta} \pi m} \psi_{n,m} e_n \right) \right\|^2 d\mu(\phi) d\mu(\psi) \\
\leq & \frac{C}{m^2} \left(\int_E \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| d\mu(\phi) \right) \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \int_E \psi_{n,m}^2 d\mu(\psi) \right) \\
= & \frac{C}{m^2} \left(\int_E \left\| V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right\| d\mu(\phi) \right) \left(\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-1}} \right),
\end{aligned}$$

where we have used (2.2). By the fact that

$$\sum_{m=1}^{\infty} \frac{1}{m^2} < \infty,$$

we obtain (4.8). □

Thus, we have an asymptotic formula for Z.

Theorem 4.5. *For all $\beta > 0$,*

$$\begin{aligned}
\frac{\mathrm{Tr} e^{-\beta \hbar H_\hbar}}{\mathrm{Tr} e^{-\beta \hbar H_0}} = & \int_E \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) d\mu(\phi) \\
& - \frac{\beta^3 \hbar^2}{2} \sum_{m=1}^{\infty} \int_{E^2} d\mu(\phi) d\mu(\psi) \exp\left(-\beta V \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \right) \\
& \times V'' \left(\sqrt{\frac{2}{\beta}} A^{-1/2} \phi \right) \left(A^{1/2} \left(\frac{1}{\sqrt{\beta} \pi m} \sum_{n=1}^{\infty} \psi_{n,m} e_n \right), \right. \\
& \left. A^{1/2} \left(\frac{1}{\sqrt{\beta} \pi m} \sum_{n=1}^{\infty} \psi_{n,m} e_n \right) \right) \\
& + o(\hbar^2)
\end{aligned}$$

as $\hbar \to 0$.

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