MATCHING AND EDGE COVERING NUMBER ON LEXICOGRAPHICAL PRODUCT OF COMPLETE GRAPHS

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching and edge covering number, respectively. The lexicographical product $G_1 \bullet G_2$ of graph of $G_1$ and $G_2$ has vertex set $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2)|[u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. In this paper, we determined generalization of matching and edge covering number on lexicographical product of complete graphs and any simple graph.

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1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let $G_1$ and $G_2$ be graphs. The lexicographical product of graph $G_1$ and $G_2$, denote by $G_1 \bullet G_2$, is the graph with $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2)|[u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. There are some properties about lexicographical product of graph. We recall these here.

Proposition 1. Let $H = G_1 \bullet G_2 = (V(H), E(H))$ then:

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(i) $|V(H)| = |V(G_1)||V(G_2)|;$
(ii) $|E(H)| = |V(G_1)||V(G_2)|^2 + |V(G_1)||E(G_2)|;$
(iii) for every $(u, v) \in V(H), d_H((u, v)) = 2|V(G_2)| + d_{G_2}(v)$.

**Theorem 2.** Let $G_1$ and $G_2$ be connected graphs, The graph $H = G_1 \cdot G_2$ is connected if and only if $G_1$ is connected.

Next we get that general form of graph of lexicographical product of $K_n$ and a simple graph.

**Proposition 3.** Let $G$ be connected graph order $m$, the graph of $K_n \cdot G$ is

$$[(\bigcup_{i=1}^{n-1} H_i) \cup \bigcup_{i=1}^n R_i; \quad H_i = \bigcup_{j=i+1}^n H_{ij}$$

where $V(H_i) = W_i \cup W_j; \quad W_i = \{(i, 1), ..., (i, m)\}; \quad E(H_{ij}) = \{(i, v)(j, v)/v \in V(G)\}$ and $V(R_i) = W_i; E(R_i) = \{(i, u)(i, v)/uv \in E(G)\}$ Moreover, $H_{ij}$ isomorphic to complete bipartite graph $K_{|V(G)|, |V(G)|}$ and $R_i$ isomorphic to $G$.

Example

![Figure 1: The graph of $K_4 \cdot G$](image)

Next, we give the definitions about some graph parameters. A subset of the edge set $E$ of $G$ is said to be matching or an independent edge set of $G$, if no two distinct edges in $M$ have a common vertex. A matching $M$ is maximum matching in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality of maximum matching of $G$ is called the matching number of $G$, denoted by $\alpha'(G)$.
An edge of graph \( G \) is said to cover the two vertices incident with it, and an edge cover of a graph \( G \) is a set of edges covering all the vertices of \( G \). The minimum cardinality of an edge cover of a graph \( G \) is called the edge covering number of \( G \), denoted by \( \beta'(G) \).

By definitions of edge covering number and matching number, clearly that \( \alpha'(K_n) = \left\lfloor \frac{n}{2} \right\rfloor \) and \( \beta'(K_n) = \left\lceil \frac{n}{2} \right\rceil \).

2. Matching Number of the Graph of \( K_n \otimes G \)

We begin this section by giving the definition and theorem for alternating path and augmenting path, the lemma 6 that shows character of matching for each \( H_{ij} \) and \( R_i \).

**Definition 4.** Given a matching \( M \), an \( M \)-alternating path is a path that alternates between edges in \( M \) and edges not in \( M \). An \( M \)-alternating odd path whose endpoints are unsaturated by \( M \) is an \( M \)-augmenting path.

**Theorem 5.** A matching \( M \) in a graph \( G \) is a maximum matching in \( G \) if and only if \( G \) has no \( M \)-augmenting path.

Next, we give the lemma 6 which shows character of matching for each \( H_{ij} \) and \( R_i \).

**Lemma 6.** Let \( K_n \otimes G = [(\bigcup_{i=1}^{n-1} H_i)] \cup \bigcup_{i=1}^{n} R_i; \quad H_i = \bigcup_{j=i+1}^{n} H_{ij}. \) Then \( \alpha'(H_{ij}) = |V(G)| \) and \( \alpha'(R_i) = \alpha'(G). \)

**Proof.** By proposition 3, we get \( H_{ij} \cong K_{|V(G)|,|V(G)|}, \; R_i \cong G. \) Hence \( \alpha'(H_{ij}) = |V(G)| \) and \( \alpha'(R_i) = \alpha'(G). \) \( \square \)

Next, we establish theorem 7 for a matching number of \( K_n \otimes G \)

**Theorem 7.** Let \( G \) be connected graph order \( m \), then

\[
\alpha'(K_n \bullet G) = \begin{cases} 
\left\{ \frac{mn}{2} \right\}, & n \text{ is even} \\
\left\{ \frac{mn}{2} + \alpha'(G) \right\}, & n \text{ is odd}
\end{cases}
\]

**Proof.** Let \( V(K_n) = \{u_i, i = 1, 2, ..., n\}, \; V(G) = \{v_j, j = 1, 2, ..., m\}, \; W_i = \{(u_i, v_j) \in V(K_n \bullet G)/j = 1, 2, ..., m\}, \; i = 1, 2, ..., n. \) By lemma 6, we have \( \alpha'(H_{ij}) = |V(G)|. \) Since \( K_n \bullet G \) is \( \bigcup_{i=1}^{n-1} H_i \cup \bigcup_{i=1}^{n} R_i \) which have matching in \( H_k \)
be \{ (u_k, v_j), (u_{k+1}, v_j) / j = 1, 2, ..., m \}; k = 1, 3, ..., 2 \left\lfloor \frac{n}{2} \right\rfloor - 1 where n is even. In the case n is odd, we get matching in \( H_k \) and \( R_n \). Hence

\[
\alpha'(K_n \bullet G) \geq \begin{cases} 
\left\lfloor \frac{mn}{2} \right\rfloor & \text{n is even} \\
\left\lfloor \frac{mn}{2} \right\rfloor + \alpha'(G) & \text{n is odd}
\end{cases}
\]

Figure 2: The matching when \( n \) is odd, \( m \) is even

In the case \( n \) is even, if \( \alpha'(K_n \bullet G) > \frac{mn}{2} \). It is impossible because every vertices of \( K_n \bullet G \) are matching out already.

In the case \( n \) is odd, we let \( M \) is maximum matching of \( G \). If \( \alpha'(K_n \bullet G) > \frac{mn}{2} + \alpha'(G) \), then there exist a matching \( M \) is augmenting path. That is not true because each vertices in \( K_n \bullet G \) always incident with edges in \( M \) and another edges which are not in \( M \).

Hence

\[
\alpha'(K_n \bullet G) = \begin{cases} 
\left\lfloor \frac{mn}{2} \right\rfloor & \text{n is even} \\
\left\lfloor \frac{mn}{2} \right\rfloor + \alpha'(G) & \text{n is odd}
\end{cases}
\]

3. Edge Covering Number of the Graph of \( K_n \otimes G \)

We begin this section by giving the lemma 8 that shows a relation of matching number and edge covering number and the lemma 9 that show character of edge cover number for each \( H_i \).
Lemma 8. Let $G$ be a simple graph with order $n$. Then $\alpha'(G) + \beta'(G) = n$

Next we establish theorem 9 for an edge covering number of $K_n \bullet G$.

Theorem 9. Let $G$ be connected graph of order $m$, then

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\}, & n \text{ is even} \\ \{\beta'(G) - \frac{mn}{2}\}, & n \text{ is odd} \end{cases}$$

Proof. By theorem 7 and lemma 8, we can also show that

$$\alpha'(K_n \bullet G) + \beta'(K_n \bullet G) = mn$$

$$\frac{mn}{2} + \beta'(K_n \bullet G) = mn$$

$$\beta'(K_n \bullet G) = \frac{mn}{2}, \ n \text{ is even}.$$ 

and,

$$\alpha'(K_n \bullet G) + \beta'(K_n \bullet G) = mn$$

$$[\frac{mn}{2} + \alpha'(G)] + \beta'(K_n \bullet G) = mn$$

$$\beta'(K_n \bullet G) = mn - \frac{mn}{2} - \alpha'(G)$$

$$\quad = \beta'(G) - \frac{mn}{2}, \ n \text{ is odd}.$$ 

Hence

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\}, & n \text{ is even} \\ \{\beta'(G) - \frac{mn}{2}\}, & n \text{ is odd} \end{cases}$$

\[\square\]

References


