MATCHING AND EDGE COVERING NUMBER ON LEXICOGRAPHICAL PRODUCT OF COMPLETE GRAPHS

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching and edge covering number, respectively. The lexicographical product $G_1 \bullet G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2) | [u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. In this paper, we determined generalization of matching and edge covering number on lexicographical product of complete graphs and any simple graph.

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1. Introduction

In this paper, graphs must be simple graphs which can be the trivial graph. Let G_1 and G_2 be graphs. The lexicographical product of graph G_1 and G_2 , denote by $G_1 \bullet G_2$, is the graph with $V(G_1 \bullet G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \bullet G_2) = \{(u_1v_1)(u_2v_2) | [u_1u_2 \in E(G_1)] \cup [u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)]\}$. There are some properties about lexicographical product of graph. We recall these here.

Proposition 1. Let $H = G_1 \bullet G_2 = (V(H), E(H))$ then:

- (i) $|V(H)| = |V(G_1)||V(G_2)|$;
- (ii) $|E(H)| = |V(G_1)||V(G_2)|^2 + |V(G_1)||E(G_2)|$;
- (iii) for every $(u, v) \in V(H)$, $d_H((u, v)) = 2|V(G_2)| + d_{G_2}(v)$.

Theorem 2. Let G_1 and G_2 be connected graphs, The graph $H = G_1 \bullet G_2$ is connected if and only if G_1 is connected .

Next we get that general form of graph of lexicographical product of K_n and a simple graph.

Proposition 3. Let G be connected graph order m , the graph of $K_n \bullet G$ is

$$[(\bigcup_{i=1}^{n-1} H_i)] \cup \bigcup_{i=1}^n R_i; \quad H_i = \bigcup_{j=i+1}^n H_{ij}$$

where $V(H_i) = W_i \cup W_j$; $W_i = \{(i, 1), \dots, (i, m)\}$; $E(H_{ij}) = \{(i, v)(j, v) / v \in V(G)\}$ and $V(R_i) = W_i$; $E(R_i) = \{(i, u)(i, v) / uv \in E(G)\}$ Moreover, H_{ij} isomorphic to complete bipartite graph $K_{|V(G)|, |V(G)|}$ and R_i isomorphic to G .

Example

Figure 1: The graph of $K_4 \bullet G$

Next, we give the definitions about some graph parameters. A subset of the edge set E of G is said to be matching or an independent edge set of G , if no two distinct edges in M have a common vertex. A matching M is maximum matching in G if there is no matching M' of G with $|M'| > |M|$. The cardinality of maximum matching of G is called the matching number of G , denoted by $\alpha'(G)$.

An edge of graph G is said to cover the two vertices incident with it, and an edge cover of a graph G is a set of edges covering all the vertices of G . The minimum cardinality of an edge cover of a graph G is called the edge covering number of G , denoted by $\beta'(G)$.

By definitions of edge covering number and matching number, clearly that $\alpha'(K_n) = \lfloor \frac{n}{2} \rfloor$ and $\beta'(K_n) = \lceil \frac{n}{2} \rceil$.

2. Matching Number of the Graph of $K_n \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, the lemma 6 that shows character of matching for each H_{ij} and R_i .

Definition 4. Given a matching M , an M -alternating path is a path that alternates between edges in M and edges not in M . An M -alternating odd path whose endpoints are unsaturated by M is an M -augmenting path.

Theorem 5. A matching M in a graph G is a maximum matching in G if and only if G has no M -augmenting path.

Next, we giving the lemma 6 which show character of matching for each H_{ij} and R_i .

Lemma 6. Let $K_n \otimes G = [(\bigcup_{i=1}^{n-1} H_i)] \cup \bigcup_{i=1}^n R_i$; $H_i = \bigcup_{j=i+1}^n H_{ij}$. Then $\alpha'(H_{ij}) = |V(G)|$ and $\alpha'(R_i) = \alpha'(G)$.

Proof. By proposition 3, we get $H_{ij} \cong K_{|V(G)|, |V(G)|}$, $R_i \cong G$.

Hence $\alpha'(H_{ij}) = |V(G)|$ and $\alpha'(R_i) = \alpha'(G)$. □

Next, we establish theorem 7 for a matching number of $K_n \otimes G$

Theorem 7. Let G be connected graph order m , then

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\} & , n \text{ is even} \\ \{\frac{mn}{2} + \alpha'(G)\} & , n \text{ is odd} \end{cases}$$

Proof. Let $V(K_n) = \{u_i, i = 1, 2, \dots, n\}$, $V(G) = \{v_j, j = 1, 2, \dots, m\}$, $W_i = \{(u_i, v_j) \in V(K_n \bullet G) / j = 1, 2, \dots, m\}$, $i = 1, 2, \dots, n$. By lemma 6, we have

$\alpha'(H_{ij}) = |V(G)|$. Since $K_n \bullet G$ is $[\bigcup_{i=1}^{n-1} H_i] \cup \bigcup_{i=1}^n R_i$ which have matching in H_k

be $\{(u_k, v_j), (u_{k+1}, v_j) / j = 1, 2, \dots, m\}; k = 1, 3, \dots, 2\lfloor \frac{n}{2} \rfloor - 1$ where n is even. In the case n is odd, we get matching in H_k and R_n . Hence

$$\alpha'(K_n \bullet G) \geq \begin{cases} \{\frac{mn}{2}\} & , n \text{ is even} \\ \{\frac{mn}{2} + \alpha'(G)\} & , n \text{ is odd} \end{cases}$$

Figure 2: The matching when n is odd, m is even

In the case n is even, if $\alpha'(K_n \bullet G) > \frac{mn}{2}$. It is impossible because every vertices of $K_n \bullet G$ are matching out already.

In the case n is odd, we let M is maximum matching of G . If $\alpha'(K_n \bullet G) > \frac{mn}{2} + \alpha'(G)$, then there exist a matching M is augmenting path. That is not true because each vertices in $K_n \bullet G$ always incident with edges in M and another edges which are not in M .

Hence

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\} & , n \text{ is even} \\ \{\frac{mn}{2} + \alpha'(G)\} & , n \text{ is odd} \end{cases}$$

□

3. Edge Covering Number of the Graph of $K_n \otimes G$

We begin this section by giving the lemma 8 that shows a relation of matching number and edge covering number and the lemma 9 that show character of edge cover number for each H_i .

Lemma 8. *Let G be a simple graph with order n . Then $\alpha'(G) + \beta'(G) = n$*
 Next we establish theorem 9 for a edge covering number of $K_n \bullet G$.

Theorem 9. *Let G be connected graph of order m , then*

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\} & , n \text{ is even} \\ \{\beta'(G) - \frac{mn}{2}\} & , n \text{ is odd} \end{cases}$$

Proof. By theorem 7 and lemma 8, we can also show that

$$\begin{aligned} \alpha'(K_n \bullet G) + \beta'(K_n \bullet G) &= mn \\ \frac{mn}{2} + \beta'(K_n \bullet G) &= mn \\ \beta'(K_n \bullet G) &= \frac{mn}{2}, n \text{ is even.} \end{aligned}$$

$$\begin{aligned} \text{and,} \quad \alpha'(K_n \bullet G) + \beta'(K_n \bullet G) &= mn \\ [\frac{mn}{2} + \alpha'(G)] + \beta'(K_n \bullet G) &= mn \\ \beta'(K_n \bullet G) &= mn - \frac{mn}{2} - \alpha'(G) \\ &= \beta'(G) - \frac{mn}{2}, n \text{ is odd.} \end{aligned}$$

Hence

$$\alpha'(K_n \bullet G) = \begin{cases} \{\frac{mn}{2}\} & , n \text{ is even} \\ \{\beta'(G) - \frac{mn}{2}\} & , n \text{ is odd} \end{cases}$$

□

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