ON $sg\alpha$ -US SPACES

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Abstract: In this paper, we introduce and study $sg\alpha$ -US topological spaces.

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1. Introduction

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. One of the most well known notions and also an inspiration source is the notion of α -open [2] sets introduced by Njastad in 1965. Quite recently, as a generalization of closed sets called $sg\alpha$ -closed sets were introduced in [3]. The aim of this paper is to introduce and study $sg\alpha$ -US spaces; $sg\alpha$ -convergence, sequentially $sg\alpha$ -compact sets, sequentially $sg\alpha$ -continuous functions, sequentially nearly $sg\alpha$ -continuous functions, sequentially $sg\alpha$ -compact preserving functions and

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sequentially sub- $sg\alpha$ -continuous functions.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and Y mean topological spaces. For a subset A of a topological space (X, τ) , $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of A and the interior of A , respectively.

We recall the following definitions, which are used in the sequel.

Definition 1. A subset A of a space (X, τ) is called semi-open [1] (resp. α -open [2]) if $A \subset \text{Cl}(\text{Int}(A))$ (resp. $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$). The complement of an α -open set is called an α -closed set.

The α -closure of a subset A of a space (X, τ) , denoted by $\alpha \text{Cl}(A)$ and is defined to be the intersection of all α -closed subsets of X containing A .

Definition 2. A subset A of a space (X, τ) is called $sg\alpha$ -closed [3] if $\alpha \text{Cl}(A) \subset U$ whenever $A \subset U$ and U is semi-open in X . The complement of a $sg\alpha$ -closed set is called a $sg\alpha$ -open set. The family of all $sg\alpha$ -open subsets of (X, τ) is denoted by $sg\alpha O(X)$.

The intersection of all $sg\alpha$ -closed sets of X containing A is called the $sg\alpha$ -closure [3] of A and is denoted by $sg\alpha \text{Cl}(A)$.

Definition 3. A space (X, τ) is said to be:

1. $sg\alpha$ - T_0 if to each pair of distinct points x, y of X there exists a $sg\alpha$ -open set A containing x but not y or a $sg\alpha$ -open set B containing y but not x .
2. $sg\alpha$ - T_1 [5] if for each pair of distinct points x and y of X , there exist $sg\alpha$ -open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$.
3. $sg\alpha$ - T_2 [5] if for each pair of distinct points x and y of X , there exist $sg\alpha$ -open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.
4. $sg\alpha$ - R_1 [4] if and only if for $x, y \in X$ with $sg\alpha \text{Cl}(\{x\}) \neq sg\alpha \text{Cl}(\{y\})$, there exist disjoint $sg\alpha$ -open sets U and V such that $sg\alpha \text{Cl}(\{x\}) \subset U$ and $sg\alpha \text{Cl}(\{y\}) \subset V$.

Theorem 4. [5] Let (X, τ) be a topological space. Then (X, τ) is $sg\alpha$ - T_2 if and only if it is $sg\alpha$ - R_1 and $sg\alpha$ - T_0 .

3. On $sg\alpha$ -US-Spaces

Definition 5. A sequence (x_n) is said to be $sg\alpha$ -converges to a point x of X , denoted by $(x_n) \xrightarrow{sg\alpha} x$ if (x_n) is eventually in every $sg\alpha$ -open set containing x .

Definition 6. A space (X, τ) is said to be $sg\alpha$ -US if every $sg\alpha$ -convergent sequence (x_n) in X $sg\alpha$ -converges to a unique point.

Definition 7. A subset S of (X, τ) is said to be sequentially $sg\alpha$ -closed if every sequence in S $sg\alpha$ -converges to a point in S .

Definition 8. A subset S of a space (X, τ) is said to be sequentially $sg\alpha$ -compact if every sequence in S has a subsequence which $sg\alpha$ -converges to a point in S .

Theorem 9. Every $sg\alpha$ - T_2 space is $sg\alpha$ -US.

Proof. Let (X, τ) be a $sg\alpha$ - T_2 space and (x_n) be a sequence in X . Suppose that (x_n) $sg\alpha$ -converges to two distinct points x and y . That is, (x_n) is eventually in every $sg\alpha$ -open set containing x and also in every $sg\alpha$ -open set containing y . This is a contradiction since X is $sg\alpha$ - T_2 . This shows that the space X is $sg\alpha$ -US. $\square$

Theorem 10. Every $sg\alpha$ -US space is $sg\alpha$ - T_1 .

Proof. Let (X, τ) be a $sg\alpha$ -US space and x and y be two distinct points of X . Consider the sequence (x_n) where $x_n = x$ for every n . Clearly (x_n) $sg\alpha$ -converges to x . Also, since $x \neq y$ and X is $sg\alpha$ -US, (x_n) cannot $sg\alpha$ -converges to y , that is, there exists a $sg\alpha$ -open set V containing y but not x . Similarly, if we consider the sequence (y_n) where $y_n = y$ for all n , and proceeding as above we get a $sg\alpha$ -open set U containing x but not y . This shows that the space X is $sg\alpha$ - T_1 . $\square$

Theorem 11. In a $sg\alpha$ -US space, every sequentially $sg\alpha$ -compact set is sequentially $sg\alpha$ -closed.

Proof. Let (X, τ) be a $sg\alpha$ -US space and Y a sequentially $sg\alpha$ -compact subset of X . Let (x_n) be a sequence in Y . Suppose that (x_n) $sg\alpha$ -converges to a point in $X - Y$. Let (x_{nk}) be a subsequence of (x_n) which $sg\alpha$ -converges to a point $y \in Y$ since Y is sequentially $sg\alpha$ -compact. Also, a subsequence (x_{nk}) of (x_n) $sg\alpha$ -converges to $x \in X - Y$. Since (x_{nk}) is a sequence in the $sg\alpha$ -US space X , $x = y$. This shows that Y is sequentially $sg\alpha$ -closed set. $\square$

Lemma 12. [3] Let A and Y be subsets of a space (X, τ) . If $A \in sg\alpha O(X)$ and Y is open in X , then $A \cap Y \in sg\alpha O(Y)$.

Theorem 13. Every open subset of a $sg\alpha$ -US space is a $sg\alpha$ -US.

Proof. Let (X, τ) be a $sg\alpha$ -US space and Y be an open subset of X . Let (x_n) be a sequence in Y . Suppose that (x_n) $sg\alpha$ -converge to x and y in Y . We shall prove that (x_n) $sg\alpha$ -converges to x and y in X . Let U be any $sg\alpha$ -open subset of X containing x and V be any $sg\alpha$ -open set of X containing y . Then by Lemma 12, $U \cap Y$ and $V \cap Y$ are $sg\alpha$ -open sets in Y . Therefore, (x_n) is eventually in $U \cap Y$ and $V \cap Y$ and so in U and V . Since X is $sg\alpha$ -US, this implies that $x = y$ and hence the subspace Y is $sg\alpha$ -US. $\square$

Theorem 14. A space (X, τ) is $sg\alpha$ - T_2 if and only if it is both $sg\alpha$ - R_1 and $sg\alpha$ -US.

Proof. Let (X, τ) be a $sg\alpha$ - T_2 space. Then (X, τ) is $sg\alpha$ - R_1 by Theorem 4 and $sg\alpha$ -US by Theorem 9. Conversely, let X be both $sg\alpha$ - R_1 and $sg\alpha$ -US space. By Theorem 10, we obtain that every $sg\alpha$ -US space is $sg\alpha$ - T_1 and X is both $sg\alpha$ - T_1 and $sg\alpha$ - R_1 , it follows from Theorem 4 that X is $sg\alpha$ - T_2 . $\square$

Now, we introduce the notion of sequentially $sg\alpha$ -continuous functions.

Definition 15. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

1. sequentially $sg\alpha$ -continuous at $x \in X$ if $f(x_n)$ $sg\alpha$ -converges to $f(x)$ whenever (x_n) is a sequence $sg\alpha$ -converging to x ;
2. sequentially $sg\alpha$ -continuous if f is sequentially $sg\alpha$ -continuous at all $x \in X$.

Definition 16. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be sequentially nearly $sg\alpha$ -continuous if for each point $x \in X$ and each sequence (x_n) in X $sg\alpha$ -converging to x , there exists a subsequence (x_{n_k}) of (x_n) such that $f(x_{n_k}) \xrightarrow{sg\alpha} f(x)$.

Definition 17. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be sequentially sub- $sg\alpha$ -continuous if for each point $x \in X$ and each sequence (x_n) in X $sg\alpha$ -converging to x , there exists a subsequence (x_{n_k}) of (x_n) and a point $y \in Y$ such that $f(x_{n_k}) \xrightarrow{sg\alpha} y$.

Definition 18. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be sequentially $sg\alpha$ -compact preserving if the image $f(K)$ of every sequentially $sg\alpha$ -compact set K of X is sequentially $sg\alpha$ -compact in Y .

Theorem 19. *Every function $f : (X, \tau) \rightarrow (Y, \sigma)$ is sequentially sub- $sg\alpha$ -continuous if Y is sequentially $sg\alpha$ -compact.*

Proof. Let (x_n) be a sequence in X $sg\alpha$ -converging to a point x of X . Then $(f(x_n))$ is a sequence in Y and as Y is sequentially $sg\alpha$ -compact, there exists a subsequence $(f(x_{n_k}))$ of $(f(x_n))$ $sg\alpha$ -converging to a point $y \in Y$. This shows that $f : (X, \tau) \rightarrow (Y, \sigma)$ is sequentially sub- $sg\alpha$ -continuous. $\square$

Theorem 20. *Every sequentially nearly $sg\alpha$ -continuous function is sequentially $sg\alpha$ -compact preserving.*

Proof. Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is sequentially nearly $sg\alpha$ -continuous function and let A be any sequentially $sg\alpha$ -compact set of Y . Let (y_n) be any sequence in $f(A)$. Then for each positive integer n , there exists a point $x_n \in A$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in the sequentially $sg\alpha$ -compact set A , there exists a subsequence (x_{n_k}) of (x_n) $sg\alpha$ -converging to a point $x \in A$. Since f is sequentially nearly $sg\alpha$ -continuous, then there exists a subsequence (x_j) of (x_{n_k}) such that $f(x_j) \xrightarrow{sg\alpha} f(x)$. Therefore, there exists a subsequence (y_j) of (y_n) $sg\alpha$ -converging to $f(x) \in f(A)$. This shows that $f(A)$ is sequentially $sg\alpha$ -compact set in Y . $\square$

Theorem 21. *Every sequentially $sg\alpha$ -compact preserving function is sequentially sub- $sg\alpha$ -continuous.*

Proof. Suppose $f : (X, \tau) \rightarrow (Y, \sigma)$ is a sequentially $sg\alpha$ -compact preserving function. Let x be any point of X and (x_n) any sequence in X $sg\alpha$ -converging to x . We shall denote the set $\{x_n : n = 1, 2, \dots\}$ by A and $B = A \cup \{x\}$. Then B is sequentially $sg\alpha$ -compact since $x_n \xrightarrow{sg\alpha} x$. Since f is sequentially $sg\alpha$ -compact set preserving, it follows that $f(B)$ is a sequentially $sg\alpha$ compact set of Y . Since $(f(x_n))$ is a sequence in $f(B)$, there exists a subsequence $(f(x_{n_k}))$ of $(f(x_n))$ $sg\alpha$ -converging to a point $y \in f(B)$. This implies that f is sequentially sub- $sg\alpha$ -continuous. $\square$

Theorem 22. *A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is sequentially $sg\alpha$ -compact preserving if and only if $f|_M : M \rightarrow f(M)$ is sequentially sub- $sg\alpha$ -continuous for each sequentially $sg\alpha$ -compact subset M of X .*

Proof. Suppose that $f : (X, \tau) \rightarrow (Y, \sigma)$ is a sequentially $sg\alpha$ -compact preserving function. Then $f(M)$ is sequentially $sg\alpha$ -compact set M of X . Therefore, by Theorem 19, $f|_M : M \rightarrow f(M)$ is sequentially sub- $sg\alpha$ -continuous function. Conversely, let M be any sequentially $sg\alpha$ -compact set in Y . We shall show

that $f(M)$ is sequentially $sg\alpha$ -compact set in Y . Let (y_n) be any sequence in $f(M)$. Then for each positive integer n , there exists a point $x_n \in M$ such that $f(x_n) = y_n$. Since (x_n) is a sequence in a sequentially $sg\alpha$ -compact set M , there exists a subsequence (x_{nk}) of (x_n) $sg\alpha$ -converging to a point $x \in M$. Since $f|_M: M \rightarrow f(M)$ is sequentially sub- $sg\alpha$ -continuous, there exists a subsequence (y_{nk}) of (y_n) $sg\alpha$ -converging to a point $y \in f(M)$. This implies that $f(M)$ is sequentially $sg\alpha$ -compact set in Y . Thus, $f: (X, \tau) \rightarrow (Y, \sigma)$ is sequentially $sg\alpha$ -compact preserving function. $\square$

The following theorem gives a sufficient condition for a sequentially sub- $sg\alpha$ -continuous function to be a sequentially $sg\alpha$ -compact preserving.

Theorem 23. *If a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is sequentially sub- $sg\alpha$ -continuous and $f(M)$ is sequentially $sg\alpha$ -closed set in Y for each sequentially $sg\alpha$ -compact set M of X , then f is sequentially $sg\alpha$ -compact preserving function.*

Proof. We use the previous Theorem. It suffices to prove that $f|_M: M \rightarrow f(M)$ is sequentially sub- $sg\alpha$ -continuous for each sequentially $sg\alpha$ -compact subset M of X . Let (x_n) be any sequence in M $sg\alpha$ -converging to a point $x \in M$. Then since f is sequentially sub- $sg\alpha$ -continuous, there exists a subsequence (x_{nk}) of (x_n) and a point $y \in Y$ such that $f(x_{nk})$ $sg\alpha$ -converges to y . Since $f(x_{nk})$ is a sequence in the sequentially $sg\alpha$ -closed set $f(M)$ of Y , we obtain $y \in f(M)$. This implies that $f|_M: M \rightarrow f(M)$ is sequentially sub- $sg\alpha$ -continuous. $\square$

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