JULIA SETS OF
FINITELY GENERATED RATIONAL SEMIGROUPS

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Abstract: Let \( G \) be a semigroup of rational functions with degree at least 2 with the semigroup operation being functional composition. We prove that if \( G \) is finitely generated and satisfies \( OSC(U) \), then \( J(G) \) has no interior points.

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1. Introduction

In the series of their papers, Hinkkanen and Martin tried to extend the classical theory of the dynamics associated to the iteration of a rational function of a complex variable to the more general setting of semigroups of rational functions, see [6, 7, 8] etc.

Let

\[ f_j : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \quad (j = 1, 2, \ldots) \]

be non-constant rational functions. We denote by

\[ G = \langle f_1, f_2, \ldots, f_n, \ldots \rangle \]

the semigroup generated by the family \( \{ f_j : j = 1, 2, \ldots \} \) with the semigroup
operation being functional composition, which is called a rational semigroup. If there exists a finite family \( \{ f_j : j = 1, 2, \ldots, m \} \) such that \( G = \langle f_1, \ldots, f_m \rangle \), then we call that \( G \) is a finitely generated rational semigroup.

The Fatou set of a rational semigroup \( G \) is defined by

\[
F(G) = \{ z \in \hat{\mathbb{C}} : G \text{ is normal in some neighbourhood of } z \}
\]

and the Julia set of \( G \) by \( J(G) = \hat{\mathbb{C}} \setminus F(G) \). We write \( F(f) \) and \( J(f) \) for \( F(\langle f \rangle) \) and \( J(\langle f \rangle) \). Then \( F(f) \) and \( J(f) \) are the Fatou set and Julia set, respectively, in the classical iteration theory of Fatou-Julia’s. It is obvious that the dynamics of a semigroup is more complicated than that of a single function. Many properties in the classical case cannot be preserved for the case of semigroups. For example, \( F(G) \) and \( J(G) \) may not be complete invariant. In general, we can only know that \( F(G) \) is forward invariant and \( J(G) \) is backward invariant under \( G \). And \( J(G) \) may not be \( \hat{\mathbb{C}} \) when \( J(G) \) has an interior point. See the examples in [6, 22].

We refer to [1, 9, 15] etc for more properties of the iteration of a single rational function and [2 ∼ 8, 10 ∼ 14, 16 ∼ 23] etc for more properties of the dynamics of rational semigroups.

In the sequel of this note, we always assume that \( G \) consists of rational functions with degree at least 2. We are mainly interested in finitely generated rational semigroups in this paper.

As in the iteration of a single function, the following problem is fundamental and interesting in the study of the dynamics of rational semigroups.

**Problem 1.1.** When does \( J(G) \) contain no interior points if \( G \) is finitely generated?

Example 3 in [6] shows that without extra condition(s), \( J(G) \) may contain interior points. In order to discuss Problem 1.1, we introduce the following condition.

**Definition 1.1.** Let \( G = \langle f_1, f_2, \ldots, f_m \rangle \) be a finitely generated rational semigroup. We say that \( G \) satisfies the open set condition if there exists an open set \( U \) satisfying that \( f_i^{-1}(U) \subseteq U \) for each \( i \) (1 ≤ \( i \) ≤ \( m \)) and \( f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset \) for each pair \((i, j)\), where \( 1 \leq i, j \leq m \) and \( i \neq j \).

For the sake of simplicity, we always denote the open set condition by \( \text{OSC}(U) \) in the following.

In Section 2, we will prove the following result.

**Theorem 1.1.** Let \( G = \langle f_1, f_2, \ldots, f_m \rangle \) be a finitely generated rational semigroup. If \( G \) satisfies \( \text{OSC}(U) \), then the Julia set \( J(G) \) of \( G \) has no interior
points.

**Remark 1.1.** In 2001, Sumi discussed Problem 1.1. As one of the main results in [16], Sumi proved that if $G$ is finitely generated and satisfies $OSC(U)$, and if $U \setminus J(G) \neq \emptyset$, then $J(G)$ has no interior points, see Proposition 4.3 in [16]. Theorem 1.1 shows that the condition "$U \setminus J(G) \neq \emptyset$" in Sumi’s result is unnecessary.

2. The Proof of Theorem 1.1

Let $G$ be a rational semigroup. A point $z$ is called exceptional if its backward orbit $O^{-}(z) = \{ w : \text{there exists an element } g \in G \text{ such that } g(w) = z \}$ is finite. The set of exceptional points is denoted by $E(G)$. That is,

$$E(G) = \{ z \in \hat{\mathbb{C}} : O^{-}(z) \text{ contains at most two points} \}.$$

**Lemma 2.1.** (Lemma 3.3 in [6]) $Card(E(G)) \leq 2$, where "card" means "cardinality".

Let’s introduce two more lemmas which are useful for the following proofs.

**Lemma 2.2.** (Lemma 3.2 in [6]) Let $G$ be a rational semigroup and $z \in \hat{\mathbb{C}} \setminus E(G)$. Then $J(G)$ is contained in the set of accumulation points of $O^{-}(z)$. That is

$$J(G) \subset \{ \text{accumulation points of } O^{-}(z) \}.$$

**Lemma 2.3.** (Lemma 1.1.4 (2) in [17]) Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Then

$$J(G) = \bigcup_{i=1}^{m} f_i^{-1}(J(G)).$$

By using Lemmas 2.1 and 2.2, we prove a result which is crucial for the proof of Theorem 1.1.

**Lemma 2.4.** Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. If there exists an open set $\Omega$ such that $\Omega \cap F(G) \neq \emptyset$ and $f_i^{-1}(\Omega \cap F(G)) \subset \Omega \cap F(G)$ for each $i$ $(1 \leq i \leq m)$, then $Int(J(G)) = \emptyset$, where $Int(J(G))$ denotes the interior of $J(G)$. 
Proof. Without loss of generality, we may assume that $J(G) \neq \emptyset$ and $z \in J(G)$. It follows from $\Omega \cap F(G) \neq \emptyset$ and Lemma 2.1 that there exists $z_0 \in \Omega \cap F(G)$ such that $z_0 \notin E(G)$.

Let $W$ be any neighborhood of $z$. Then Lemma 2.2 implies that there is an element $g \in G$ such that $g^{-1}(z_0) \in W$. Since

$$f^{-1}_i(\Omega \cap F(G)) \subset \Omega \cap F(G) \quad (i = 1, 2, \ldots, m),$$

we see that $g^{-1}(z_0) \in \Omega \cap F(G) \subset F(G)$. Hence $W \cap F(G) \neq \emptyset$. This shows that $z$ is not an interior point of $J(G)$. The proof is completed.

The following result is obvious.

**Lemma 2.5.** Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. Suppose that $G$ satisfies $OSC(U)$. If we let $V_i = f^{-1}_i(U)$ and $V = \bigcup_{i=1}^m V_i$, then $V$ is a proper subset of $U$ and $\bigcup_{i=1}^m f^{-1}_i(U)$ is a proper subset of $V$.

**Lemma 2.6.** Let $G = \langle f_1, f_2, \ldots, f_m \rangle$ be a finitely generated rational semigroup. If $G$ satisfies $OSC(U)$, and if $V_i$ and $V$ are as in Lemma 2.5, then $V \cap F(G) \neq \emptyset$ and for each $j \ (1 \leq j \leq m),$

$$f^{-1}_j(V \cap F(G)) \subset V \cap F(G).$$

Proof. Since $G$ satisfies $OSC(U)$, we know that $V_i \subset U$. This yields

$$f^{-1}_j(V_i) \subset V_j \quad (2.1)$$

for each pair $(i, j)$, where $1 \leq i, j \leq m$.

Claim I. $J(G) \subset \overline{V}$.

In fact, for any point $w \in V \setminus E(G)$, (2.1) implies that for each $g \in G$,  

$$\{g^{-1}(w)\} \subset V.$$  

Claim I follows from Lemma 2.2.

Claim II. $V \cap F(G) \neq \emptyset$.

Suppose, on the contrary, that $V \cap F(G) = \emptyset$. Then $V \subset J(G)$. Claim I implies

$$J(G) = \overline{V} = \bigcup_{i=1}^m \overline{V_i}.$$  

We deduce from Lemma 2.3 that
\[
\bigcup_{i=1}^{m} \overline{V}_i = \bigcup_{j=1}^{m} f_j^{-1}(\bigcup_{i=1}^{m} \overline{V}_i).
\] (2.2)

Lemma 2.5 shows that \( \bigcup_{j=1}^{m} f_j^{-1}(\bigcup_{i=1}^{m} \overline{V}_i) \) is a proper subset of \( \bigcup_{i=1}^{m} \overline{V}_i \). From (2.2), this is a contradiction. Our second Claim is proved.

The inclusion \( f_j^{-1}(V \cap F(G)) \subset V \) for each \( j \) (1 ≤ \( j \) ≤ \( m \)) is obvious. Now we come to show that

\[
f_j^{-1}(V \cap F(G)) \subset F(G).
\]

Suppose that there exist a point \( z \in V \cap F(G) \) and some \( k \) (1 ≤ \( k \) ≤ \( m \)) such that \( f_k^{-1}(z) \in J(G) \). Then

\[
f_k^{-1}(z) \in f_k^{-1}(V) \cap J(G).
\]

On the other hand, Lemma 2.3 shows that

\[
f_k^{-1}(z) \in \bigcup_{i=1}^{m} (f_i^{-1}(J(G)) \cap f_k^{-1}(V)).
\]

Since \( G \) satisfies OSC\( (U) \), we have

\[
f_i^{-1}(V) \cap f_j^{-1}(V) = \emptyset
\]

for each pair \((i, j)\), where \( i \neq j \) and 1 ≤ \( i, j \) ≤ \( m \).

Hence

\[
f_i^{-1}(V) \cap f_j^{-1}(V) = \emptyset
\]

Then Claim I implies that

\[
f_k^{-1}(z) \in f_k^{-1}(J(G)) \cap f_k^{-1}(V),
\]

i.e.

\[
f_k^{-1}(z) \in f_k^{-1}(J(G) \cap V).
\]

We see that \( z \in J(G) \cap V \). This contradiction completes the proof.
3. The proof of Theorem 1.1

Let $V$ be as in Lemma 2.5, and let $\Omega = V$. The proof follows from Lemmas 2.4 and 2.6.

References


