JULIA SETS OF FINITELY GENERATED RATIONAL SEMIGROUPS

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Abstract: Let G be a semigroup of rational functions with degree at least 2 with the semigroup operation being functional composition. We prove that if G is finitely generated and satisfies $OSC(U)$, then $J(G)$ has no interior points.

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1. Introduction

In the series of their papers, Hinkkanen and Martin tried to extend the classical theory of the dynamics associated to the iteration of a rational function of a complex variable to the more general setting of semigroups of rational functions, see [6, 7, 8] etc.

Let

$$f_j : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \quad (j = 1, 2, \dots)$$

be non-constant rational functions. We denote by

$$G = \langle f_1, f_2, \dots, f_n, \dots \rangle$$

the semigroup generated by the family $\{f_j : j = 1, 2, \dots\}$ with the semigroup

operation being functional composition, which is called a *rational semigroup*. If there exists a finite family $\{f_j : j = 1, 2, \dots, m\}$ such that $G = \langle f_1, \dots, f_m \rangle$, then we call that G is a *finitely generated rational semigroup*.

The Fatou set of a rational semigroup G is defined by

$$F(G) = \{z \in \hat{\mathbb{C}} : G \text{ is normal in some neighbourhood of } z\}$$

and the Julia set of G by $J(G) = \hat{\mathbb{C}} \setminus F(G)$. We write $F(f)$ and $J(f)$ for $F(\langle f \rangle)$ and $J(\langle f \rangle)$. Then $F(f)$ and $J(f)$ are the Fatou set and Julia set, respectively, in the classical iteration theory of Fatou-Julia's. It is obvious that the dynamics of a semigroup is more complicated than that of a single function. Many properties in the classical case cannot be preserved for the case of semigroups. For example, $F(G)$ and $J(G)$ may not be complete invariant. In general, we can only know that $F(G)$ is forward invariant and $J(G)$ is backward invariant under G . And $J(G)$ may not be $\hat{\mathbb{C}}$ when $J(G)$ has an interior point. See the examples in [6, 22].

We refer to [1, 9, 15] etc for more properties of the iteration of a single rational function and [2 ~ 8, 10 ~ 14, 16 ~ 23] etc for more properties of the dynamics of rational semigroups.

In the sequel of this note, we always assume that G consists of rational functions with degree at least 2. We are mainly interested in finitely generated rational semigroups in this paper.

As in the iteration of a single function, the following problem is fundamental and interesting in the study of the dynamics of rational semigroups.

Problem 1.1. When does $J(G)$ contain no interior points if G is finitely generated?

Example 3 in [6] shows that without extra condition(s), $J(G)$ may contain interior points. In order to discuss Problem 1.1, we introduce the following condition.

Definition 1.1. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. We say that G satisfies *the open set condition* if there exists an open set U satisfying that $f_i^{-1}(U) \subset U$ for each i ($1 \leq i \leq m$) and $f_i^{-1}(U) \cap f_j^{-1}(U) = \emptyset$ for each pair (i, j) , where $1 \leq i, j \leq m$ and $i \neq j$.

For the sake of simplicity, we always denote *the open set condition* by $OSC(U)$ in the following.

In Section 2, we will prove the following result.

Theorem 1.1. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. If G satisfies $OSC(U)$, then the Julia set $J(G)$ of G has no interior

points.

Remark 1.1. In 2001, Sumi discussed Problem 1.1. As one of the main results in [16], Sumi proved that if G is finitely generated and satisfies $OSC(U)$, and if $U \setminus J(G) \neq \emptyset$, then $J(G)$ has no interior points, see Proposition 4.3 in [16]. Theorem 1.1 shows that the condition " $U \setminus J(G) \neq \emptyset$ " in Sumi's result is unnecessary.

2. The Proof of Theorem 1.1

Let G be a rational semigroup. A point z is called exceptional if its backward orbit $O^-(z) = \{w : \text{there exists an element } g \in G \text{ such that } g(w) = z\}$ is finite. The set of exceptional points is denoted by $E(G)$. That is,

$$E(G) = \{z \in \hat{\mathbb{C}} : O^-(z) \text{ contains at most two points}\}.$$

Lemma 2.1. (Lemma 3.3 in [6]) $Card(E(G)) \leq 2$, where "card" means "cardinality".

Let's introduce two more lemmas which are useful for the following proofs.

Lemma 2.2. (Lemma 3.2 in [6]) Let G be a rational semigroup and $z \in \hat{\mathbb{C}} \setminus E(G)$. Then $J(G)$ is contained in the set of accumulation points of $O^-(z)$. That is

$$J(G) \subset \{\text{accumulation points of } O^-(z)\}.$$

Lemma 2.3. (Lemma 1.1.4 (2) in [17]) Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Then

$$J(G) = \bigcup_{i=1}^m f_i^{-1}(J(G)).$$

By using Lemmas 2.1 and 2.2, we prove a result which is crucial for the proof of Theorem 1.1.

Lemma 2.4. Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. If there exists an open set Ω such that $\Omega \cap F(G) \neq \emptyset$ and $f_i^{-1}(\Omega \cap F(G)) \subset \Omega \cap F(G)$ for each i ($1 \leq i \leq m$), then $\text{Int}(J(G)) = \emptyset$, where $\text{Int}(J(G))$ denotes the interior of $J(G)$.

Proof. Without loss of generality, we may assume that $J(G) \neq \emptyset$ and $z \in J(G)$. It follows from $\Omega \cap F(G) \neq \emptyset$ and Lemma 2.1 that there exists $z_0 \in \Omega \cap F(G)$ such that $z_0 \notin E(G)$.

Let W be any neighborhood of z . Then Lemma 2.2 implies that there is an element $g \in G$ such that $g^{-1}(z_0) \in W$. Since

$$f_i^{-1}(\Omega \cap F(G)) \subset \Omega \cap F(G) \quad (i = 1, 2, \dots, m),$$

we see that $g^{-1}(z_0) \in \Omega \cap F(G) \subset F(G)$. Hence $W \cap F(G) \neq \emptyset$. This shows that z is not a interior point of $J(G)$. The proof is completed.

The following result is obvious.

Lemma 2.5. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. Suppose that G satisfies $OSC(U)$. If we let $V_i = f_i^{-1}(U)$ and $V = \bigcup_{i=1}^m V_i$, then V is a proper subset of U and $\bigcup_{i=1}^m f_i^{-1}(V)$ is a proper subset of V .*

Lemma 2.6. *Let $G = \langle f_1, f_2, \dots, f_m \rangle$ be a finitely generated rational semigroup. If G satisfies $OSC(U)$, and if V_i and V are as in Lemma 2.5, then $V \cap F(G) \neq \emptyset$ and for each j ($1 \leq j \leq m$),*

$$f_j^{-1}(V \cap F(G)) \subset V \cap F(G).$$

Proof. Since G satisfies $OSC(U)$, we know that $V_i \subset U$. This yields

$$f_j^{-1}(V_i) \subset V_j \tag{2.1}$$

for each pair (i, j) , where $1 \leq i, j \leq m$.

Claim I $J(G) \subset \overline{V}$.

In fact, for any point $w \in V \setminus E(G)$, (2.1) implies that for each $g \in G$,

$$\{g^{-1}(w)\} \subset V.$$

Claim I follows from Lemma 2.2.

Claim II $V \cap F(G) \neq \emptyset$.

Suppose, on the contrary, that $V \cap F(G) = \emptyset$. Then $V \subset J(G)$. Claim I implies

$$J(G) = \overline{V} = \bigcup_{i=1}^m \overline{V_i}.$$

We deduce from Lemma 2.3 that

$$\bigcup_{i=1}^m \overline{V}_i = \bigcup_{j=1}^m f_j^{-1}(\bigcup_{i=1}^m \overline{V}_i). \quad (2.2)$$

Lemma 2.5 shows that $\bigcup_{j=1}^m f_j^{-1}(\bigcup_{i=1}^m \overline{V}_i)$ is a proper subset of $\bigcup_{i=1}^m \overline{V}_i$. From (2.2), this is a contradiction. Our second Claim is proved.

The inclusion $f_j^{-1}(V \cap F(G)) \subset V$ for each j ($1 \leq j \leq m$) is obvious. Now we come to show that

$$f_j^{-1}(V \cap F(G)) \subset F(G).$$

Suppose that there exist a point $z \in V \cap F(G)$ and some k ($1 \leq k \leq m$) such that $f_k^{-1}(z) \in J(G)$. Then

$$f_k^{-1}(z) \in f_k^{-1}(V) \cap J(G).$$

On the other hand, Lemma 2.3 shows that

$$f_k^{-1}(z) \in \bigcup_{i=1}^m (f_i^{-1}(J(G)) \cap f_k^{-1}(V)).$$

Since G satisfies $OSC(U)$, we have

$$f_i^{-1}(V) \cap f_j^{-1}(V) = \emptyset$$

for each pair (i, j) , where $i \neq j$ and $1 \leq i, j \leq m$.

Hence

$$f_i^{-1}(\overline{V}) \cap f_j^{-1}(V) = \emptyset.$$

Then Claim I implies that

$$f_k^{-1}(z) \in f_k^{-1}(J(G)) \cap f_k^{-1}(V),$$

i.e.

$$f_k^{-1}(z) \in f_k^{-1}(J(G) \cap V).$$

We see that $z \in J(G) \cap V$. This contradiction completes the proof.

3. The proof of Theorem 1.1

Let V be as in Lemma 2.5, and let $\Omega = V$. The proof follows from Lemmas 2.4 and 2.6.

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