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P-REGULAR NEARRINGS CHARACTERIZED BY THEIR BI-IDEALS

Aphisit Muangma[1], Aiyared Iampan[2 §]

[1,2]Department of Mathematics
School of Science
University of Phayao
Phayao, 56000, THAILAND

Abstract: Using the idea of quasi-ideals of P-regular nearrings, the concept of bi-ideals of P-regular nearrings is generalized, which is an extension of the concept of quasi-ideals of P-regular nearrings and some interesting characterizations of bi-ideals are obtained. As a result, we prove that every element of a bi-ideal B of a P-regular nearring can be represented as the sum of two elements of P and Q. Moreover, every element of the finite intersection $\bigcap_{i=1}^{n} B_i$ of bi-ideals of a P-regular distributive nearring N can be represented as the sum of two elements of P and $B_1NB_2N \dots NB_{n-1}NB_n$.

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1. Introduction and Preliminaries

The notion of nearrings is first defined by Pilz [8] in 1977 and that of bi-ideals by Chelvam and Ganesan [3] in 1987. As we know, nearrings are a generalization of rings, and bi-ideals are a generalization of quasi-ideals and ideals in nearrings. Many types of ideals on the algebraic structures were characterized by several

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§Correspondence author

authors such as: In 1983, Yakabe [10] introduced and characterized the notion of quasi-ideals of nearrings. In 1987, Chelvam and Ganesan [3] introduced and generalized the notion of quasi-ideals of nearrings which was introduced by [10] to bi-ideals. In 1989, Yakabe [11] characterized regular zero-symmetric nearrings without nonzero nilpotent elements in terms of quasi-ideals. In 1990, Andrunakievich [2] introduced P-regular rings. In 1991, Choi [4] extended the P-regularity of rings which was introduced by [2] to the P-regularity of nearrings. In 2005, Kim, Jun and Yon [7] introduced the notion of anti fuzzy ideals of near-rings and investigated some related properties. In 2008, Abbasi and Rizvi [1] studied prime ideals in near-rings. In 2009, Zhan and B. Davvaz [12] introduced the concept of $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy subnear-rings (ideals) of near-rings and obtain some of its related properties. In 2010, Choi [5] gave some characterizations of quasi-ideals of P-regular nearrings and proved that every element of a quasi-ideal Q of a P-regular nearring can be represented as the sum of two elements of P and Q. In 2011, Dheena and Manivasan [6] gave some characterizations of quasi-ideals of P-regular nearrings in the same way as of Choi [5]. In 2012, Sharma [9] studied the properties of intuitionistic fuzzy ideals of near ring with the help of their (α, β)-cut sets. The concept of quasi-ideals play an important role in studying the structure of nearrings. Now, the notion of bi-ideals is an important and useful generalization of quasi-ideals of nearrings. Therefore, we will study bi-ideals of nearrings in the same way as of quasi-ideals of nearrings which was studied by Choi [5].

To present the main results we discuss some elementary definitions that we use later.

Definition 1.1. [8] A *nearring* is a system consisting of a nonempty set N together with two binary operations on N called addition and multiplication such that

(1) N together with addition is a group,

(2) N together with multiplication is a semigroup, and

(3) $(a + b)c = ac + bc$ for all $a, b, c \in N$.

For two nonempty subsets A and B of a nearring N, let

$$A + B := \{a + b \mid a \in A \text{ and } b \in B\}$$

and

$$AB := \{ab \mid a \in A \text{ and } b \in B\}.$$

If $A = \{a\}$, then we also write $\{a\}+B$ as $a+B$, and $\{a\}B$ as aB, and similarly if $B = \{b\}$.

Definition 1.2. A nonempty subset S of a nearring N is called a *left (right) N-subgroup* of N if

(1) S together with addition is a subgroup of N, and

(2) $NS \subseteq S$ ($SN \subseteq S$).

Definition 1.3. A nonempty subset S of a nearring N is called an *ideal* of N if

(1) S together with addition is a normal subgroup of N,

(2) $SN \subseteq S$,

(3) $NS \subseteq S$, and

(4) $n_1(n_2 + s) - n_1n_2 \in S$ for all $s \in S$ and $n_1, n_2 \in N$.

Note that S is a *left ideal* of N if S satisfies (1), (3) and (4), and S is a *right ideal* of N if S satisfies (1) and (2).

Remark 1.4. By Definition 1.3, we have that

(1) S is a left ideal of N if and only if S is a normal left N-subgroup of N and $n_1(n_2 + s) - n_1n_2 \in S$ for all $s \in S$ and $n_1, n_2 \in N$.

(2) S is a right ideal of N if and only if S is a normal right N-subgroup of N.

Definition 1.5. A nearring N is called a *distributive nearring* if $a(b+c) = ab + ac$ for all $a, b, c \in N$.

Definition 1.6. A nonempty subset Q of a nearring N is called a *quasi-ideal* of N if

(1) Q together with addition is a subgroup of N, and

(2) $QN \cap NQ \subseteq Q$.

Definition 1.7. A nonempty subset B of a nearring N is called a *bi-ideal* of N if

(1) B together with addition is a subgroup of N, and

(2) $BNB \subseteq B$.

Definition 1.8. A nearring N is called *regular nearring* if for each $x \in N$ there exists $y \in N$ such that $xyx = x$.

Definition 1.9. Let N be a nearring with unity and P an ideal of N. Then N is said to be *P-regular nearring* if for each $x \in N$ there exists $y \in N$ such that $xyx - x \in P$.

2. Lemmas

Before the characterizations of bi-ideals of nearrings for the main results, we give some auxiliary results which are necessary in what follows.

Lemma 2.1. *[5] Let N be a nearring and $P = \{0\}$. If N is a P-regular nearring, then N is a regular nearring.*

Lemma 2.2. *Let $\mathcal{B}$ be a nonempty family of bi-ideals of a nearring N. Then $\bigcap \mathcal{B}$ is a bi-ideal of N.*

Proof. Clearly, $\bigcap \mathcal{B}$ together with addition is a subgroup of N. Now, for all $B \in \mathcal{B}$, we have

$$\bigcap \mathcal{B} N \bigcap \mathcal{B} \subseteq BNB \subseteq B.$$

Thus $\bigcap \mathcal{B} N \bigcap \mathcal{B} \subseteq \bigcap \mathcal{B}$. Hence $\bigcap \mathcal{B}$ is a bi-ideal of N. □

Corollary 2.3. *Any finite intersection of bi-ideals of a nearring is a bi-ideal.*

Lemma 2.4. *Every quasi-ideal of a nearring is a bi-ideal.*

Proof. Let Q be a quasi-ideal of a nearring N. Then Q together with addition is a subgroup of N. Thus $QNQ \subseteq QN$ and $QNQ \subseteq NQ$, so $QNQ \subseteq QN \cap NQ \subseteq Q$. Hence Q is a bi-ideal of N. □

3. Main Results

In this section, give some characterizations of bi-ideals of nearrings. Finally, we prove that every element of a bi-ideal B of a P-regular nearring can be represented as the sum of two elements of P and Q. Moreover, every element of the finite intersection $\bigcap_{i=1}^{n} B_i$ of bi-ideals of a P-regular distributive nearring N can be represented as the sum of two elements of P and $B_1NB_2N \dots NB_{n-1}NB_n$.

Theorem 3.1. *Let N be a P-regular nearring. Then for each $n \in N$ there exists $n' \in N$ such that $n'n \in P$.*

Theorem 3.2. *Let N be a P-regular distributive nearring. Then for every right ideal R and every left ideal L of N,*

$$(P + R) \cap (P + L) = P + RL.$$

Theorem 3.3. *Let N be a P-regular nearring and B a bi-ideal of N. Then every $x \in B$ there exist $p' \in P$ and $b' \in B$ such that $x = p' + b'$.*

Proof. Let $x \in B$. Since N is a P-regular nearring and $x \in B \subseteq N$, there exists $y \in N$ such that $xyx - x = p$ for some $p \in P$. Thus $x = -p + xyx$. Since B is a bi-ideal of N, we have $xyx \in BNB \subseteq B$. Since $p \in P$ and P together with addition is a subgroup of N, we have $-p \in P$. Put $p' = -p$ and $b' = xyx$. Thus

$$x = -p + xyx = p' + b' \in P + B. \qquad \square$$

Theorem 3.4. *Let N be a P-regular distributive nearring and B_1 and B_2 bi-ideals of N. If $b \in B_1 \cap B_2$ and $x \in N$, then the element b can be represented as*

$$b = p + b_1x_1b_2 \text{ and } b_1x_1b_2xP \subseteq P$$

for some $p \in P, x_1 \in N, b_1 \in B_1$ and $b_2 \in B_2$.

Proof. Let $b \in B_1 \cap B_2$. Since N is a P-regular nearring, there exists $x_1 \in N$ such that $bx_1b - b \in P$. By Lemma 2.2, we have $B_1 \cap B_2$ is a bi-ideal of N. Since $b \in B_1 \cap B_2$, we have $b \in B_1$ and $b \in B_2$. By Theorem 3.3, we have $b = p_1 + b_1$ for some $p_1 \in P$ and $b_1 \in B_1$, and $b = p_2 + b_2$ for some $p_2 \in P$ and $b_2 \in B_2$. Since $bx_1b - b \in P$, we have $bx_1b - b = p_3$ for some $p_3 \in P$. Thus $b = -p_3 + bx_1b$. Hence

$$b = -p_3 + bx_1b$$

$$\begin{aligned} &= -p_3 + (p_1 + b_1)x_1(p_2 + b_2) \\ &= -p_3 + p_1x_1p_2 + p_1x_1b_2 + b_1x_1p_2 + b_1x_1b_2. \end{aligned}$$

Since P is an ideal of N, we have $-p_3, p_1x_1p_2, p_1x_1b_2, b_1x_1p_2 \in P$. Then $-p_3 + p_1x_1p_2 + p_1x_1b_2 + b_1x_1p_2 = p_4$ for some $p_4 \in P$. Thus $b = p_4 + b_1x_1b_2$, so $b_1x_1b_2 = -p_4 + b$. Hence

$$b_1x_1b_2xP = (-p_4 + b)xP \subseteq -p_4xP + bxP \subseteq P + P \subseteq P. \qquad \square$$

Theorem 3.5. *Let N be a P-regular distributive nearring and $\{B_i \mid i \in \mathbb{Z}$ and $1 \leq i \leq n\}$ a nonempty family of bi-ideals of N. If $b \in \bigcap_{i=1}^{n} B_i$ and $x \in N$, then the element b can be represented as*

$$b = p + b_1x_1b_2x_2 \dots b_{n-1}x_{n-1}b_n \text{ and } b_1x_1b_2x_2 \dots b_{n-1}x_{n-1}b_nxP \subseteq P$$

for some $p \in P, x_1, x_2, \dots, x_{n-1} \in N$ and $b_i \in B_i$ for all $1 \leq i \leq n$.

Proof. If $b \in B_1$, then by Theorem 3.3, we have $b = p + b_1$ for some $p \in P$ and $b_1 \in B_1$. Thus

$$b_1xP = (-p + b)xP \subseteq -pxP + bxP \subseteq P + P \subseteq P.$$

Assume that the theorem is true for integer $n-1$. Let $b \in \bigcap_{i=1}^{n} B_i$. Since $\bigcap_{i=1}^{n} B_i \subseteq \bigcap_{i=1}^{n-1} B_i$ and $\bigcap_{i=1}^{n} B_i \subseteq B_n$, we have $b \in \bigcap_{i=1}^{n-1} B_i$ and $b \in B_n$. By assumption, we have

$$b = p_1 + b_1x_1b_2x_2 \dots b_{n-2}x_{n-2}b_{n-1} \tag{3.1}$$

and $b_1x_1b_2x_2 \dots b_{n-2}x_{n-2}b_{n-1}xP \subseteq P$ for some $p_1 \in P, x_1, x_2, \dots, x_{n-2} \in N$ and $b_i \in B_i$ for all $1 \leq i \leq n-1$. By Theorem 3.3, we have

$$b = p_2 + b_n \tag{3.2}$$

for some $p_2 \in P$ and $b_n \in B_n$. Since N is a P-regular nearring, there exists $x_{n-1} \in N$ such that $bx_{n-1}b - b \in P$. Thus $bx_{n-1}b - b = p_3$ for some $p_3 \in P$, so $b = -p_3 + bx_{n-1}b$. By (3.1) and (3.2), we have

$$bx_{n-1}b = (p_1 + b_1x_1b_2x_2 \dots b_{n-2}x_{n-2}b_{n-1})x_{n-1}(p_2 + b_n). \tag{3.3}$$

By (3.3), we have

$$b = -p_3 + bx_{n-1}b$$

$$\begin{aligned} &= -p_3 + (p_1 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1})x_{n-1}(p_2 + b_n) \\ &= -p_3 + p_1x_{n-1}p_2 + p_1x_{n-1}b_n + \\ &\quad b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}p_2 + \\ &\quad b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_n. \end{aligned}$$

Put $-p_3 + p_1x_{n-1}p_2 + p_1x_{n-1}b_n + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}p_2 = p_4$ for some $p_4 \in P$. Thus

$$b = p_4 + b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_n.$$

That is $b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_n = -p_4 + b$. Hence

$$\begin{aligned} b_1x_1b_2x_2 \ldots b_{n-2}x_{n-2}b_{n-1}x_{n-1}b_nxP &= (-p_4 + b)xP \\ &\subseteq -p_4xP + bxP \\ &\subseteq P + P \subseteq P. \end{aligned}$$

□

Theorem 3.6. *Let N be a P-regular nearring and B a bi-ideal of N. Then*

$$P + B = P + BNB.$$

Proof. Since B is a bi-ideal of N, we have $BNB \subseteq B$. Thus

$$P + BNB \subseteq P + B. \tag{3.4}$$

On the other hand, let $n \in P + B$. Then $n = p' + b'$ for some $p' \in P$ and $b' \in B$. Since N is a P-regular nearring, there exists $x \in N$ such that $b'xb' - b' \in P$. Thus $b'xb' - b' = p''$ for some $p'' \in P$, so $b' = -p'' + b'xb'$. Therefore

$$n = p' + b' = p' + (-p'' + b'xb') = (p' - p'') + b'xb' \in P + BNB.$$

Hence

$$P + B \subseteq P + BNB. \tag{3.5}$$

By (3.4) and (3.5), we have $P + B = P + BNB$. □

Theorem 3.7. *Let N be a P-regular nearring, and B_1 and B_2 bi-ideals of N. Then*

$$P + (B_1 \cap B_2) \subseteq P + (B_1NB_2 \cap B_2NB_1).$$

Proof. Let $b \in P + (B_1 \cap B_2)$. Then $b = p + b'$ for some $p \in P$ and $b' \in B_1 \cap B_2$. Thus $b' \in B_1$ and $b' \in B_2$. Since N is a P-regular nearring, there exists $x \in N$ such that $b'xb' - b' \in P$. Thus $b'xb' - b' = p'$ for some $p' \in P$, so $b' = -p' + b'xb'$. Hence

$$b = p + b' = p - p' + b'xb' = p'' + b'xb' \in P + (B_1NB_2 \cap B_2NB_1)$$

where $p'' = p - p'$. Therefore

$$P + (B_1 \cap B_2) \subseteq P + (B_1NB_2 \cap B_2NB_1). \tag{3.6}$$

□

Theorem 3.8. *Let N be a P-regular nearring, and $\{B_i \mid i \in \mathbb{Z}$ and $1 \leq i \leq n\}$ a nonempty family of bi-ideals of N. Then*

$$P + (\bigcap_{i=1}^{n} B_i) \subseteq$$
$$P + (B_1NB_n \cap B_2NB_n \cap \ldots \cap B_{n-1}NB_n \cap B_nNB_1 \cap B_nNB_2 \cap \ldots \cap B_nNB_{n-1}).$$

Proof. By Theorem 3.6, we have $P + B_1 = P + B_1NB_1$. That is $P + B_1 \subseteq P + B_1NB_1$. Assume that the theorem is true for integer $n-1$. By Theorem 3.7, we have

$$\begin{aligned} P + (\bigcap_{i=1}^{n} B_i) &= P + (\bigcap_{i=1}^{n-1} B_i \cap B_n) \\ &\subseteq P + ((\bigcap_{i=1}^{n-1} B_i)NB_n \cap B_nN(\bigcap_{i=1}^{n-1} B_i)) \\ &\subseteq P + ((B_1 \cap B_2 \cap \ldots \cap B_{n-1})NB_n \\ &\quad \cap B_nN(B_1 \cap B_2 \cap \ldots \cap B_{n-1})) \\ &\subseteq P + (B_1NB_n \cap B_2NB_n \cap \ldots \cap B_{n-1}NB_n \\ &\quad \cap B_nNB_1 \cap B_nNB_2 \cap \ldots \cap B_nNB_{n-1}). \end{aligned}$$

□

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