THE LAPLACIAN ON AN AFFINE HOMOGENEOUS SPACE

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Abstract: The solution of the eigenvalue problem of the Laplacian on the affine homogeneous space \(\hat{g}/\hat{\eta}\) is obtained. Here \(\hat{g}\) and \(\hat{\eta}\) are affine Lie algebras, \(\hat{g} = g \otimes C[t, t^{-1}] \oplus C\hat{k} \oplus Cd\) and \(\hat{\eta} = \eta \otimes C[t, t^{-1}] \oplus C\hat{k} \oplus Cd\). \(\eta \subset g\) are two Lie algebras with \(g\) semi-simple and \(\eta\) reductive and having the same rank.

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1. Introduction

Homogeneous spaces play an important role in mathematics and physics, (see, for example [1], [2], [3], [4] and references therein).

In [5], we gave the solution of the eigenvalue problem of the Laplacian on a homogeneous space \(G/H\), where \(G\) is a compact, semi-simple Lie group, \(H\) is a closed subgroup of \(G\), and the rank of \(H\) is equal to the rank of \(G\).

In this paper we generalize the result and give the solution of the eigenvalue problem of the Laplacian on \(\hat{g}/\hat{\eta}\). Here \(\hat{g}\) and \(\hat{\eta}\) are affine Lie algebras, \(\hat{g} = g \otimes C[t, t^{-1}] \oplus C\hat{k} \oplus Cd\) and \(\hat{\eta} = \eta \otimes C[t, t^{-1}] \oplus C\hat{k} \oplus Cd\). \(\eta \subset g\) are two Lie algebras with \(g\) semi-simple and \(\eta\) reductive and having the same rank.

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The layout of the paper is as follows. In Section 2, we review the result in [5]. In Section 3, we first give the definition of the Laplacian on $\hat{g}/\hat{\eta}$. Then we give the eigenvalues of the Laplacian on $\hat{g}/\hat{\eta}$. Finally, The lowest eigenvalue and its eigenspace of the Laplacian on $\hat{g}/\hat{\eta}$ is obtained.

2. The Laplacian on $G/H$

We briefly review the solution of the eigenvalue problem of the Laplacian on homogeneous space $G/H$. Here $G$ is a compact, semi-simple Lie group, $H$ is a closed subgroup of $G$, and the rank of $H$ is equal to the rank of $G$. More detailed account can be found in [5]. Let $g$ and $\eta$ be the Lie algebras of $G$ and $H$, respectively. We suppose that $G/H$ is reductive, i.e. $g$ has an orthogonal decomposition $g = \eta \oplus m$ with $[\eta, m] \subset m$ and $[m, m] \subset g$. We can choose a common Cartan subalgebra $h \subset \eta \subset g$.

Let $\Phi_g$ be the set of roots of $g$. The roots $\Phi_\eta$ of $\eta$ form a subset of the roots of $g$, i.e.,

$$\Phi_\eta \subset \Phi_g.$$ 

Choosing a positive root system $\Phi^+_g$ for $g$ also determines a positive root system $\Phi^+_\eta$ for $\eta$, where

$$\Phi^+_\eta \subset \Phi^+_g.$$ 

Let $\rho_g = \frac{1}{2} \sum_{\alpha \in \Phi^+_g} \alpha$ and $\rho_\eta = \frac{1}{2} \sum_{\alpha \in \Phi^+_\eta} \alpha$ denote the Weyl vector of $g$ and $\eta$ respectively.

Let $U_\mu$ be a given irreducible representation of $\eta$ with highest weight $\mu$. Let $G \times_H U$ be the associated vector bundle of the principal bundle $P(G/H, H)$. The Hilbert space of square integrable sections of $G \times_H U_\mu$ decomposes into the direct sum of the eigenspaces of the Laplacian on $G/H$, which are irreducible representations $V_\lambda$ of $g$ with highest weights $\lambda$’s. and this induces the following expression for the Laplacian on $G/H$ which was discussed in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and appears explicitly in [19].

**Definition 1.** The Laplacian on $G/H$ is

$$\Delta = C_2(g, \cdot) - C_2(\eta, U).$$ (1)

Here $C_2(g, \cdot)$ is the quadratic Casimir element of $g$ calculated in an irreducible representation of $g$. $C_2(\eta, U)$ is the quadratic Casimir element of $\eta$ calculated in a given irreducible representation $U$. 

Thus we have the following result.

**Theorem 2.** Given an irreducible representation \( U_\mu \) of \( \eta \) with highest weight \( \mu \). The eigenvalue of \( \Delta \) labelled by a highest weight \( \lambda \) reads

\[
E_\lambda = (\lambda + \rho_\mathfrak{g}, \lambda + \rho_\mathfrak{g}) - (\mu + \rho_\eta, \mu + \rho_\eta) - (\rho_\mathfrak{g}, \rho_\mathfrak{g}) + (\rho_\eta, \rho_\eta) 
\]

with

\[
(\lambda + \rho_\mathfrak{g}, \lambda + \rho_\mathfrak{g}) \geq (\mu + \rho_\eta, \mu + \rho_\eta).
\]

The multiplicity of the eigenvalue \( E_\lambda \) is given by the Weyl dimension formula:

\[
dim V_\lambda = \frac{\prod_{\alpha \in \Phi^+}(\lambda + \rho_\mathfrak{g}, \alpha)}{\prod_{\alpha \in \Phi^+}(\rho_\mathfrak{g}, \alpha)}. \tag{3}
\]

Moreover, if there exists an element \( w \in W_\mathfrak{g} \) in the Weyl group of \( \mathfrak{g} \) such that the weight \( w(\mu + \rho_\eta) - \rho_\mathfrak{g} \) is dominant for \( \mathfrak{g} \). Then the lowest eigenvalue of \( \Delta \) is

\[
E_{w(\mu + \rho_\eta) - \rho_\mathfrak{g}} = (\rho_\eta, \rho_\eta) - (\rho_\mathfrak{g}, \rho_\mathfrak{g}), \tag{4}
\]

and the multiplicity of the lowest eigenvalue of \( \Delta \) is

\[
dim V_{w(\mu + \rho_\eta) - \rho_\mathfrak{g}} = \frac{\prod_{\alpha \in \Phi^+}(w(\mu + \rho_\eta), \alpha)}{\prod_{\alpha \in \Phi^+}(\rho_\mathfrak{g}, \alpha)}. \tag{5}
\]

**Remark.** 1. If \( \lambda = w(\mu + \rho_\eta) - \rho_\mathfrak{g} \) is not dominant for \( \mathfrak{g} \), the lowest eigenvalue of \( \Delta \) does not exist. Thus we can always choose \( \mu \) such that \( \lambda \) is dominant.

2. \( V_{w(\mu + \rho_\eta) - \rho_\mathfrak{g}} \) is, up to a sign, equal to the \( \mathcal{G} \)-equivariant index of the Kostant’s Dirac operator on \( G/H \) \([20], [21], [22], [23]\).}

### 3. The Solution of Eigenvalue Problem of the Laplacian on \( \hat{\mathfrak{g}}/\hat{\eta} \)

#### 3.1. The Laplacian and its Eigenvalues on \( \hat{\mathfrak{g}}/\hat{\eta} \)

An introduction to the affine Lie algebra can be found in \([24], [4]\). Let \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \hat{\mathfrak{k}} \oplus \mathbb{C}d \) be an affine Lie algebra. Here \( \mathbb{C}[t, t^{-1}] \) is the set of Laurent polynomials in the variable \( t \), \( \hat{\mathfrak{k}} \) is the central extension operator and \( d \) is the energy operator. Let \( \hat{\lambda} = (\lambda; k_\lambda; n_\lambda) \) and \( \hat{\mu} = (\mu; k_\mu; n_\mu) \) be two affine weights of \( \hat{\mathfrak{g}} \). Here \( \lambda \) and \( \mu \) are weights of \( \mathfrak{g} \), \( k_\lambda \) and \( k_\mu \) are levels, \( n_\lambda \) and \( n_\mu \)
are energies. The scalar product of $\hat{\lambda}$ and $\hat{\mu}$ induced by the extended Killing form is
\begin{equation}
(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda.
\end{equation}

The Casimir operator $C_2(\hat{\mathfrak{g}})$ on $\hat{\mathfrak{g}}$ is the generalization of the Casimir operator on $\mathfrak{g}$. The detailed account for the construction can be found, for example, in [25], [26].

\begin{equation}
C_2(\hat{\mathfrak{g}}) = \sum_{a=1}^l \sum_{m \in \mathbb{Z}} X_a^m X_{-m}^a : + 2(k + g)d
\end{equation}

Here $: :$ is the usual normal ordering, $g$ is the dual Coxeter number of $\mathfrak{g}$, $X_a^m = X_a \otimes t^m$ with $X_a$ generators of $\mathfrak{g}$, $l$ is the dimension of $\mathfrak{g}$ and $\mathbb{Z}$ is the set of integers. The Casimir operator commutes with $\hat{\mathfrak{g}}$. If $V_\lambda$ is a highest weight representation generated by a highest weight vector $| \lambda \rangle$ of $\hat{\mathfrak{g}}$, then

\begin{equation}
C_2(\hat{\mathfrak{g}}) | \lambda \rangle = [\sum_{a=1}^l X_a^0 X_0^a + 2(k + g)d] | \lambda \rangle = [\sum_{a=1}^l X_a X^a + 2(k + g)d] | \lambda \rangle = [(\lambda + \hat{\rho}_g, \lambda + \hat{\rho}_g) - (\hat{\rho}_g, \hat{\rho}_g)] | \lambda \rangle = [(\lambda + \rho_g, \lambda + \rho_g) - (\rho_g, \rho_g) + 2(k + g)n] | \lambda \rangle,
\end{equation}

where $\hat{\rho}_g = (\rho_g; g; 0)$ is the affine Weyl vector of $\hat{\mathfrak{g}}$ with $g$ the dual Coxeter number of $\mathfrak{g}$. The following convention is often adopted (see [4]).

**Convention 3.** For a highest weight, $\hat{\lambda}(d) = n = 0$.

One has
\begin{equation}
C_2(\hat{\mathfrak{g}}) | \lambda \rangle = [(\lambda + \rho_g, \lambda + \rho_g) - (\rho_g, \rho_g)] | \lambda \rangle = C_2(\mathfrak{g}) | \lambda \rangle.
\end{equation}

The Laplacian on $\hat{\mathfrak{g}}/\hat{\eta}$ is the generalization of the Laplacian on $\mathfrak{g}/\eta$. We have

**Definition 4.** The Laplacian on $\hat{\mathfrak{g}}/\hat{\eta}$ is
\begin{equation}
\hat{\Delta} = C_2(\hat{\mathfrak{g}}, \cdot) - C_2(\hat{\eta}, U).
\end{equation}

Here $C_2(\hat{\mathfrak{g}}, \cdot)$ is the Casimir operator of $\hat{\mathfrak{g}}$ calculated in an irreducible representation of $\hat{\mathfrak{g}}$. $C_2(\hat{\eta}, U)$ is the Casimir operator of $\hat{\eta}$ calculated in a given irreducible representation $U$. 

Proposition 5. Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\eta}$ with highest weight $\hat{\mu} = (\mu; k_\mu; 0)$. The eigenvalue of $\hat{\Delta}$ on $\hat{g}/\hat{\eta}$ labelled by highest weight $\hat{\lambda} = (\lambda; k_\lambda; 0)$ reads

$$E_{\hat{\lambda}} = (\hat{\lambda} + \hat{\rho}_g, \hat{\lambda} + \hat{\rho}_g) - (\hat{\mu} + \hat{\rho}_\eta, \hat{\mu} + \hat{\rho}_\eta) - (\hat{\rho}_g, \hat{\rho}_g) + (\hat{\rho}_\eta, \hat{\rho}_\eta)$$

with

$$(\lambda + \rho_g, \lambda + \rho_g) \geq (\mu + \rho_\eta, \mu + \rho_\eta).$$

The corresponding eigenspace is the irreducible representation of $\hat{g}$, $V_{(\lambda; k_\lambda; 0)}$ with highest weight $V_{(\lambda; k_\lambda; 0)}$.

3.2. The Lowest Eigenvalue and its Eigenspace of $\hat{\Delta}$ on $\hat{g}/\hat{\eta}$

In order to determine the lowest eigenvalue and its eigenspace of $\hat{\Delta}$ on $\hat{g}/\hat{\eta}$, we first review the relation between the Weyl groups $W_g$ of $g$ and $W_\eta$ of $\eta$ [21]. $W_\eta$ is a subgroup of $W_g$. Choose the positive roots consistently. Then the positive Weyl chamber $W_g$ of $g$ is contained in the positive Weyl chamber $W_\eta$ of $\eta$. Let

$$C \subset W_g$$

denote the set of elements that map $W_g$ into $W_\eta$. So the cardinality of $C$ is the index of $W_\eta$ in $W_g$, and

$$W_\eta = \bigcup_{c \in C} c(W_g),$$

while

$$W_g = W_\eta \cdot C.$$  

Let $\lambda$ be a dominant weight of $g$ and $V_\lambda$ be the corresponding irreducible representation of $g$. For each $c \in C$, let

$$c \cdot \lambda = c(\lambda + \rho_g) - \rho_\eta.$$  

Then $c \cdot \lambda$ is a dominant weight for $\eta$.

It can be seen that the above relations are also satisfied in the corresponding affine Weyl groups. Now we generalize the result in Section 2 to the case of the affine Lie algebras. Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\eta}$ with highest weight $\hat{\mu}$, $\hat{\mu}$ corresponds to a unique highest weight $\hat{\lambda}$ of an irreducible
representation $V_{\hat{\lambda}}$ of $\hat{\mathfrak{g}}$. More precisely, there exists $c \in \hat{C} \subset \hat{W}_{\hat{\mathfrak{g}}}$, where $\hat{W}_{\hat{\mathfrak{g}}}$ is the affine Weyl group of $\hat{\mathfrak{g}}$, such that $\hat{\mu} = c \bullet \hat{\lambda} = c(\hat{\lambda} + \hat{\rho}_g) - \hat{\rho}_\eta$, where $\hat{\rho}_g = (\rho_g; g; 0)$ and $\hat{\rho}_\eta = (\rho_\eta; \eta; 0)$ are affine Weyl vectors of $\hat{\mathfrak{g}}$ and $\hat{\eta}$, respectively; $g$ and $\eta$ are dual Coxeter numbers of $\mathfrak{g}$ and $\eta$, respectively. This means that

$$\hat{\lambda} = w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g,$$

where $w = c^{-1} \in \hat{W}_{\hat{\mathfrak{g}}}$. If $\hat{\lambda}$ is dominant for $\hat{\mathfrak{g}}$, the eigenspace of the lowest eigenvalue is $V_{w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g}$ with highest weight $w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g$.

Let $\hat{\lambda} = (\lambda; k; n)$ be a weight of $\hat{\mathfrak{g}}$ ($\hat{\eta}$). Let $\hat{\alpha} = (\alpha; 0; m)$ be an affine root of $\hat{\mathfrak{g}}$ ($\hat{\eta}$). The Weyl reflection with respect to $\hat{\alpha}$ reads

$$s_\hat{\alpha} \hat{\lambda} = (\hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee) \hat{\alpha} \equiv (\lambda - ([\lambda, \alpha] + km)\alpha^\vee; k; n - ([\lambda, \alpha] + km)\frac{2m}{|\alpha|^2})$$

$$= (s_\alpha(\lambda + km\alpha^\vee); k; n - ([\lambda, \alpha] + km)\frac{2m}{|\alpha|^2}).$$

(17)

Here $\hat{\alpha}^\vee$ is affine coroot of $\hat{\alpha}$ and $\alpha^\vee$ is coroot of $\alpha$. Now we impose the condition that the value of $\hat{\lambda}(d)$ is unchanged under the Weyl reflection, This forces $m = 0$, i.e.,

$$\hat{\alpha} = (\alpha; 0; 0) = \alpha,$$

and (17) reduces to

$$w\hat{\lambda} = (w\lambda; k; n),$$

(19)

where $w \in W_{\mathfrak{g}}$, the Weyl group of $\mathfrak{g}$. It means that only the Weyl group does not change $\hat{\lambda}(d)$ and one can obtain the related affine weights just by making use of the Weyl group. For a highest affine weight $\hat{\lambda}$, We have the convention that $\hat{\lambda}(d) = 0$. This greatly simplifies the calculation.

Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\eta}$ with highest weight $\hat{\mu} = (\mu; k\mu; 0)$, from (16) and (17),

$$\hat{\lambda} = w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g = (w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0),$$

(20)

where $w \in W_{\mathfrak{g}}$, $g$ and $\eta$ are dual Coxeter numbers of $\mathfrak{g}$ and $\eta$, respectively. If $\hat{\lambda}$ is dominant for $\hat{\mathfrak{g}}$, the eigenspace of the lowest eigenvalue is the irreducible representation of $\hat{\mathfrak{g}}$, $V_{w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0}$ with highest weight $(w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0)$. It follows that $(\hat{\lambda} + \hat{\rho}_\mathfrak{g}, \hat{\lambda} + \hat{\rho}_\mathfrak{g}) = (w(\mu + \rho_\eta), w(\mu + \rho_\eta)) = (\mu + \rho_\eta, \mu + \rho_\eta) = (\hat{\mu} + \hat{\rho}_\eta, \hat{\mu} + \hat{\rho}_\eta)$. By Proposition 4, the lowest eigenvalue is $E_{w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g} = (\rho_\eta, \rho_\eta) = (\rho_g, \rho_g)$. Thus we have the following result:
**Theorem 6.** Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\eta}$ with highest weight $\hat{\mu} = (\mu; k_\mu; 0)$. If there exists an element $w \in W_g$ in the Weyl group of $g$ such that the weight $w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g = (w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0)$ is dominant for $\hat{g}$. Then the eigenspace of the lowest eigenvalue of $\hat{\Delta}$ is the irreducible representation $\hat{\eta}$, $V_{(w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0)}$ with highest weight $(w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0)$. The lowest eigenvalue of $\hat{\Delta}$ is

$$E_{w(\hat{\mu} + \hat{\rho}_\eta) - \hat{\rho}_g} = (\rho_\eta, \rho_\eta) - (\rho_g, \rho_g).$$

**Remark.** Due to the convention $\hat{\lambda}(d) = 0$ for highest weights, The eigenvalues of $\Delta$ on $\hat{g}/\hat{\eta}$ are the same as the eigenvalues of $\Delta$ on $g/\eta$. However, the corresponding eigenspaces are different.

If $\hat{\lambda} = (w(\mu + \rho_\eta) - \rho_g; k + \eta - g; 0)$ is not dominant for $\hat{g}$, the lowest eigenvalue of $\Delta$ does not exist. Thus we can always choose $\hat{\mu}$ such that $\hat{\lambda}$ is dominant. It can be shown that $\hat{\lambda}$ is dominant for $\hat{g}$, if $w(\mu + \rho_\eta) - \rho_g$ is dominant for $g$, and the level $k + \eta - g \geq (w(\mu + \rho_\eta) - \rho_g, \theta)$, where $\theta$ is the highest root of $g$.

**References**


