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THE LAPLACIAN ON AN AFFINE HOMOGENEOUS SPACE

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Abstract: The solution of the eigenvalue problem of the Laplacian on the affine homogeneous space $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ is obtained. Here $\hat{\mathbf{g}}$ and $\hat{\boldsymbol{\eta}}$ are affine Lie algebras, $\hat{\mathbf{g}} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\hat{k} \oplus \mathbf{C}d$ and $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\hat{k} \oplus \mathbf{C}d$. $\boldsymbol{\eta} \subset \mathbf{g}$ are two Lie algebras with $\mathbf{g}$ semi-simple and $\boldsymbol{\eta}$ reductive and having the same rank.

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1. Introduction

Homogeneous spaces play an important role in mathematics and physics, (see, for example [1], [2], [3], [4] and references therein).

In [5], we gave the solution of the eigenvalue problem of the Laplacian on a homogeneous space G/H, where G is a compact, semi-simple Lie group, H is a closed subgroup of G, and the rank of H is equal to the rank of G.

In this paper we generalize the result and give the solution of the eigenvalue problem of the Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$. Here $\hat{\mathbf{g}}$ and $\hat{\boldsymbol{\eta}}$ are affine Lie algebras, $\hat{\mathbf{g}} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\hat{k} \oplus \mathbf{C}d$ and $\hat{\boldsymbol{\eta}} = \boldsymbol{\eta} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\hat{k} \oplus \mathbf{C}d$. $\boldsymbol{\eta} \subset \mathbf{g}$ are two Lie algebras with $\mathbf{g}$ semi-simple and $\boldsymbol{\eta}$ reductive and having the same rank.

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The layout of the paper is as follows. In Section 2, we review the result in [5]. In Secction 3, we first give the definition of the Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$. Then we give the eigenvalues of the Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$. Finally, The lowest eigenvalue and its eigenspace of the Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ is obtained.

2. The Laplacian on G/H

We briefly review the solution of the eigenvalue problem of the Laplacian on homogeneous space G/H. Here G is a compact, semi-simple Lie group, H is a closed subgroup of G, and the rank of H is equal to the rank of G. More detailed account can be found in [5].

Let $\mathbf{g}$ and $\boldsymbol{\eta}$ be the Lie algebras of G and H, respectively.

We suppose that G/H is reductive, i.e. $\mathbf{g}$ has an orthognal decomposition $\mathbf{g} = \boldsymbol{\eta} \oplus \mathbf{m}$ with $[\boldsymbol{\eta}, \mathbf{m}] \subset \mathbf{m}$ and $[\mathbf{m}, \mathbf{m}] \subset \mathbf{g}$.

We can choose a common Cartan subalgebra

$$\mathbf{h} \subset \boldsymbol{\eta} \subset \mathbf{g}.$$

Let $\Phi_{\mathbf{g}}$ be the set of roots of $\mathbf{g}$. The roots $\Phi_{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ form a subset of the roots of $\mathbf{g}$, i.e.,

$$\Phi_{\boldsymbol{\eta}} \subset \Phi_{\mathbf{g}}.$$

Choosing a positive root system $\Phi_{\mathbf{g}}^+$ for $\mathbf{g}$ also determines a positive root system $\Phi_{\boldsymbol{\eta}}^+$ for $\boldsymbol{\eta}$, where

$$\Phi_{\boldsymbol{\eta}}^+ \subset \Phi_{\mathbf{g}}^+.$$

Let $\rho_{\mathbf{g}} = \frac{1}{2}\sum_{\alpha \in \Phi_{\mathbf{g}}^+} \alpha$ and $\rho_{\boldsymbol{\eta}} = \frac{1}{2}\sum_{\alpha \in \Phi_{\boldsymbol{\eta}}^+} \alpha$ denote the Weyl vector of $\mathbf{g}$ and $\boldsymbol{\eta}$ respectively.

Let U_μ be a given irreducible representation of $\boldsymbol{\eta}$ with highest weight μ. Let $G \times_H U$ be the associated vector bundle of the principal bundle $P(G/H, H)$. The Hilbert space of square integrable sections of $G \times_H U_\mu$ decomposes into the direct sum of the eigenspaces of the Laplacian on G/H, which are irreducible representations V_λ of $\mathbf{g}$ with highest weights λ's. and this induces the following expression for the Laplacian on G/H which was discussed in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and appears explicitly in [19].

Definition 1. The Laplacian on G/H is

$$\Delta = C_2(\mathbf{g}, \cdot) - C_2(\boldsymbol{\eta}, U). \tag{1}$$

Here $C_2(\mathbf{g}, \cdot)$ is the quadratic Casimir element of $\mathbf{g}$ calculated in an irreducible representation of $\mathbf{g}$. $C_2(\boldsymbol{\eta}, U)$ is the quadratic Casimir element of $\boldsymbol{\eta}$ calculated in a given irreducible representation U.

Thus we have the following result.

Theorem 2. *Given an irreducible representation U_μ of $\boldsymbol{\eta}$ with highest weight μ. The eigenvalue of Δ labelled by a highest weight λ reads*

$$E_\lambda = (\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\mu + \rho_{\boldsymbol{\eta}}, \mu + \rho_{\boldsymbol{\eta}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) + (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}) \tag{2}$$

with

$$(\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) \geq (\mu + \rho_{\boldsymbol{\eta}}, \mu + \rho_{\boldsymbol{\eta}}).$$

The multiplicity of the eigenvalue E_λ is given by the Weyl dimension formula:

$$dim V_\lambda = \frac{\prod_{\alpha \in \Phi_{\mathrm{g}}^+} (\lambda + \rho_{\mathbf{g}}, \alpha)}{\prod_{\alpha \in \Phi_{\mathrm{g}}^+} (\rho_{\mathbf{g}}, \alpha)}. \tag{3}$$

Moreover, If there exists an element $w \in W_{\mathbf{g}}$ in the Weyl group of $\mathbf{g}$ such that the weight $w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}$ is dominant for $\mathbf{g}$. Then the lowest eigenvalue of Δ is

$$E_{w(\mu+\rho_{\boldsymbol{\eta}})-\rho_{\mathrm{g}}} = (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}), \tag{4}$$

and the multiplicity of the lowest eigenvalue of Δ is

$$dim V_{w(\mu+\rho_{\boldsymbol{\eta}})-\rho_{\mathrm{g}}} = \frac{\prod_{\alpha \in \Phi_{\mathrm{g}}^+} (w(\mu + \rho_{\boldsymbol{\eta}}), \alpha)}{\prod_{\alpha \in \Phi_{\mathrm{g}}^+} (\rho_{\mathbf{g}}, \alpha)}. \tag{5}$$

Remark. 1. If $\lambda = w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}$ is not dominant for $\mathbf{g}$, the lowest eigenvalue of Δ does not exist. Thus we can always choose μ such that λ is dominant.

2. $V_{w(\mu+\rho_{\boldsymbol{\eta}})-\rho_{\mathrm{g}}}$ is, up to a sign, equal to the G-equivariant index of the Kostant's Dirac operator on G/H [20], [21], [22], [23].

3. The Solution of Eigenvalue Problem of the Laplacian on $\hat{\mathrm{g}}/\hat{\eta}$

3.1. The Laplacian and its Eigenvalues on $\hat{\mathrm{g}}/\hat{\eta}$

An introduction to the affine Lie algebra can be found in [24], [4]. Let $\hat{\mathbf{g}} = \mathbf{g} \otimes \mathbf{C}[t, t^{-1}] \oplus \mathbf{C}\hat{k} \oplus \mathbf{C}d$ be an affine Lie algebra. Here $\mathbf{C}[t, t^{-1}]$ is the set of Laurent polinomials in the variable t, $\hat{k}$ is the central extension operator and d is the energy operator. Let $\hat{\lambda} = (\lambda; k_\lambda; n_\lambda)$ and $\hat{\mu} = (\mu; k_\mu; n_\mu)$ be two affine weights of $\hat{\mathbf{g}}$. Here λ and μ are weights of $\mathbf{g}$, k_λ and k_μ are levels, n_λ and n_μ

are energies. The scalar product of $\hat{\lambda}$ and $\hat{\mu}$ induced by the extended Killing form is

$$(\hat{\lambda}, \hat{\mu}) = (\lambda, \mu) + k_\lambda n_\mu + k_\mu n_\lambda. \tag{6}$$

The Casimir operator $C_2(\hat{\mathbf{g}})$ on $\hat{\mathbf{g}}$ is the generalization of the Casimir operator on $\mathbf{g}$. The detailed account for the construction can be found, for example, in [25], [26].

$$\begin{aligned} C_2(\hat{\mathbf{g}}) &= \sum_{a=1}^{l} \sum_{m\in\mathbf{Z}} : X^a_m X^a_{-m} : +2(k+g)d \\ &= \sum_{a=1}^{l} [\sum_{m\leq -1} X^a_m X^a_{-m} + \sum_{m\geq 0} X^a_{-m} X^a_m] \\ &\quad +2(k+g)d. \end{aligned} \tag{7}$$

Here $:\ :$ is the usual normal ordering, g is the dual Coxeter number of $\mathbf{g}$, $X^a_m = X^a \otimes t^m$ with X^a generators of $\mathbf{g}$, l is the dimension of $\mathbf{g}$ and $\mathbf{Z}$ is the set of integers. The Casimir operator commutes with $\hat{\mathbf{g}}$. If $V_{\hat{\lambda}}$ is a highest weight representation generated by a highest weight vector $\mid \hat{\lambda} >$ of $\hat{\lambda} = (\lambda; k; n)$, then

$$\begin{aligned} C_2(\hat{\mathbf{g}}) \mid \hat{\lambda} > &= [\sum_{a=1}^{l} X^a_0 X^a_0 + 2(k+g)d] \mid \hat{\lambda} > \\ &= [\sum_{a=1}^{l} X^a X^a + 2(k+g)d] \mid \hat{\lambda} > \\ &= [(\hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}, \hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}) - (\hat{\rho}_{\hat{\mathbf{g}}}, \hat{\rho}_{\hat{\mathbf{g}}})] \mid \hat{\lambda} > \\ &= [(\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) \\ &\quad +2(k+g)n] \mid \hat{\lambda} >, \end{aligned} \tag{8}$$

where $\hat{\rho}_{\hat{\mathbf{g}}} = (\rho_{\mathbf{g}}; g; 0)$ is the affine Weyl vector of $\hat{\mathbf{g}}$ with g the dual Coxeter number of $\mathbf{g}$. The following convention is often adopted (see [4]).

Convention 3. For a highest weight, $\hat{\lambda}(d) = n = 0$.

One has

$$C_2(\hat{\mathbf{g}}) \mid \hat{\lambda} >= [(\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}})] \mid \hat{\lambda} >= C_2(\mathbf{g}) \mid \hat{\lambda} > . \tag{9}$$

The Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ is the generalization of the Laplacian on $\mathbf{g}/\boldsymbol{\eta}$. We have

Definition 4. The Laplacian on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ is

$$\hat{\Delta} = C_2(\hat{\mathbf{g}}, \cdot) - C_2(\hat{\boldsymbol{\eta}}, U). \tag{10}$$

Here $C_2(\hat{\mathbf{g}}, \cdot)$ is the Casimir operator of $\hat{\mathbf{g}}$ calculated in an irreducible representation of $\hat{\mathbf{g}}$. $C_2(\hat{\boldsymbol{\eta}}, U)$ is the Casimir operator of $\hat{\boldsymbol{\eta}}$ calculated in a given irreducible representation U.

Proposition 5. *Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\boldsymbol{\eta}}$ with highest weight $\hat{\mu} = (\mu; k_\mu; 0)$. The eigenvalue of $\hat{\Delta}$ on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ labelled by highest weight $\hat{\lambda} = (\lambda; k_\lambda; 0)$ reads*

$$\begin{aligned} E_{\hat{\lambda}} &= (\hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}, \hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}) - (\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}, \hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}) - (\hat{\rho}_{\hat{\mathbf{g}}}, \hat{\rho}_{\hat{\mathbf{g}}}) + (\hat{\rho}_{\hat{\boldsymbol{\eta}}}, \hat{\rho}_{\hat{\boldsymbol{\eta}}}) \\ &= (\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) - (\mu + \rho_{\boldsymbol{\eta}}, \mu + \rho_{\boldsymbol{\eta}}) \\ &\quad -(\rho_{\mathbf{g}}, \rho_{\mathbf{g}}) + (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}) \end{aligned} \tag{11}$$

with

$$(\lambda + \rho_{\mathbf{g}}, \lambda + \rho_{\mathbf{g}}) \geq (\mu + \rho_{\boldsymbol{\eta}}, \mu + \rho_{\boldsymbol{\eta}}).$$

The corresponding eigenspace is the irreducible representation of $\hat{\mathbf{g}}$, $V_{(\lambda; k_\lambda; 0)}$ with highest weight $(\lambda; k_\lambda; 0)$.

3.2. The Lowest Eigenvalue and its Eigenspace of $\hat{\Delta}$ on $\hat{\mathbf{g}}/\hat{\eta}$

In order to determine the lowest eigenvalue and its eigenspace of $\hat{\Delta}$ on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$, we first review the relation between the Weyl groups $W_{\mathbf{g}}$ of $\mathbf{g}$ and $W_{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$ [21]. $W_{\boldsymbol{\eta}}$ is a subgroup of $W_{\mathbf{g}}$. Choose the positive roots consistently. Then the positive Weyl chamber $\mathcal{W}_{\mathbf{g}}$ of $\mathbf{g}$ is contained in the positive Weyl chamber $\mathcal{W}_{\boldsymbol{\eta}}$ of $\boldsymbol{\eta}$. Let

$$C \subset W_{\mathbf{g}} \tag{12}$$

denote the set of elements that map $\mathcal{W}_{\mathbf{g}}$ into $\mathcal{W}_{\boldsymbol{\eta}}$. So the cardinality of C is the index of $W_{\boldsymbol{\eta}}$ in $W_{\mathbf{g}}$, and

$$\mathcal{W}_{\boldsymbol{\eta}} = \bigcup_{c \in C} c(\mathcal{W}_{\mathbf{g}}), \tag{13}$$

while

$$W_{\mathbf{g}} = W_{\boldsymbol{\eta}} \cdot C. \tag{14}$$

Let λ be a dominant weight of $\mathbf{g}$ and V_λ be the corresponding irreducible representation of $\mathbf{g}$. For each $c \in C$, let

$$c \bullet \lambda = c(\lambda + \rho_{\mathbf{g}}) - \rho_{\boldsymbol{\eta}}. \tag{15}$$

Then $c \bullet \lambda$ is a dominant weight for $\boldsymbol{\eta}$.

It can be seen that the above relations are also satisfied in the corresponding affine Weyl groups. Now we generalize the result in Section 2 to the case of the affine Lie algebras. Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\boldsymbol{\eta}}$ with highest weight $\hat{\mu}$, $\hat{\mu}$ corresponds to a unique highest weight $\hat{\lambda}$ of an irreducible representation $V_{\hat{\lambda}}$ of $\hat{\mathbf{g}}$. More precisely, there exists $c \in \hat{C} \subset \hat{W}_{\hat{\mathbf{g}}}$, where $\hat{W}_{\hat{\mathbf{g}}}$ is the affine Weyl group of $\hat{\mathbf{g}}$, such that $\hat{\mu} = c \bullet \hat{\lambda} = c(\hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}) - \hat{\rho}_{\hat{\boldsymbol{\eta}}}$, where $\hat{\rho}_{\hat{\mathbf{g}}} = (\rho_{\mathbf{g}}; g; 0)$ and $\hat{\rho}_{\hat{\boldsymbol{\eta}}} = (\rho_{\boldsymbol{\eta}}; \eta; 0)$ are affine Weyl vectors of $\hat{\mathbf{g}}$ and $\hat{\boldsymbol{\eta}}$, respectively; g and η are dual Coxeter numbers of $\mathbf{g}$ and $\boldsymbol{\eta}$, respectively. This means that

$$\hat{\lambda} = w(\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}) - \hat{\rho}_{\hat{\mathbf{g}}}, \tag{16}$$

where $w = c^{-1} \in \hat{W}_{\hat{\mathbf{g}}}$. If $\hat{\lambda}$ is dominant for $\hat{\mathbf{g}}$, the eigenspace of the lowest eigenvalue is $V_{w(\hat{\mu}+\hat{\rho}_{\hat{\boldsymbol{\eta}}})-\hat{\rho}_{\hat{\mathfrak{g}}}}$ with highest weight $w(\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}) - \hat{\rho}_{\hat{\mathbf{g}}}$.

Let $\hat{\lambda} = (\lambda; k; n)$ be a weight of $\hat{\mathbf{g}}$ ($\hat{\boldsymbol{\eta}}$). Let $\hat{\alpha} = (\alpha; 0; m)$ be an affine root of $\hat{\mathbf{g}}$ ($\hat{\boldsymbol{\eta}}$). The Weyl reflection with respect to $\hat{\alpha}$ reads

$$\begin{aligned} s_{\hat{\alpha}}\hat{\lambda} &= \hat{\lambda} - (\hat{\lambda}, \hat{\alpha}^\vee)\hat{\alpha} \\ &= (\lambda - [(\lambda, \alpha) + km]\alpha^\vee; k; n - [(\lambda, \alpha) + km]\frac{2m}{|\alpha|^2}) \\ &= (s_\alpha(\lambda + km\alpha^\vee); k; n - [(\lambda, \alpha) + km]\frac{2m}{|\alpha|^2}). \end{aligned} \tag{17}$$

Here $\hat{\alpha}^\vee$ is affine coroot of $\hat{\alpha}$ and $\alpha^\vee$ is coroot of α. Now we impose the condition that the value of $\hat{\lambda}(d)$ is unchanged under the Weyl reflection, This forces $m = 0$, i. e.,

$$\hat{\alpha} = (\alpha; 0; 0) = \alpha, \tag{18}$$

and (17) reduces to

$$w\hat{\lambda} = (w\lambda; k; n), \tag{19}$$

where $w \in W_{\mathbf{g}}$, the Weyl group of $\mathbf{g}$. It means that only the Weyl group does not change $\hat{\lambda}(d)$ and one can obtain the related affine weights just by making use of the Weyl group. For a highest affine weight $\hat{\lambda}$, We have the convention that $\hat{\lambda}(d) = 0$. This greatly simplifies the calculation.

Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\boldsymbol{\eta}}$ with highest weight $\hat{\mu} = (\mu; k_\mu; 0)$, from (16) and (17),

$$\hat{\lambda} = w(\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}) - \hat{\rho}_{\hat{\mathbf{g}}} = (w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}; k + \eta - g; 0), \tag{20}$$

where $w \in W_{\mathbf{g}}$, g and η are dual Coxeter numbers of $\mathbf{g}$ and $\boldsymbol{\eta}$, respectively. If $\hat{\lambda}$ is dominant for $\hat{\mathbf{g}}$, the eigenspace of the lowest eigenvalue is the irreducible representation of $\hat{\mathbf{g}}$, $V_{(w(\mu+\rho_{\boldsymbol{\eta}})-\rho_{\mathfrak{g}};k+\eta-g;0)}$ with highest weight $(w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}; k + \eta - g; 0)$. It follows that $(\hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}, \hat{\lambda} + \hat{\rho}_{\hat{\mathbf{g}}}) = (w(\mu + \rho_{\boldsymbol{\eta}}), w(\mu + \rho_{\boldsymbol{\eta}})) = (\mu + \rho_{\boldsymbol{\eta}}, \mu + \rho_{\boldsymbol{\eta}}) = (\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}, \hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}})$. By Proposition 4, the lowest eigenvalue is $E_{\hat{w}(\hat{\mu}+\hat{\rho}_{\hat{\boldsymbol{\eta}}})-\hat{\rho}_{\hat{\mathfrak{g}}}} = (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}})$. Thus we have the following result:

Theorem 6. *Given an irreducible representation $U_{\hat{\mu}}$ of $\hat{\boldsymbol{\eta}}$ with highest weight $\hat{\mu} = (\mu; k_\mu; 0)$. If there exists an element $w \in W_{\mathbf{g}}$ in the Weyl group of* $\mathbf{g}$ *such that the weight $w(\hat{\mu} + \hat{\rho}_{\hat{\boldsymbol{\eta}}}) - \hat{\rho}_{\hat{\mathbf{g}}} = (w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}; k + \eta - g; 0)$ is dominant for $\hat{\mathbf{g}}$. Then the eigenspace of the lowest eigenvalue of $\hat{\Delta}$ is the irreducible representation of $\hat{\mathbf{g}}$, $V_{(w(\mu+\rho_{\boldsymbol{\eta}})-\rho_{\mathbf{g}};k+\eta-g;0)}$ with highest weight $(w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}; k + \eta - g; 0)$. The lowest eigenvalue of $\hat{\Delta}$ is*

$$E_{w(\hat{\mu}+\hat{\rho}_{\hat{\boldsymbol{\eta}}})-\hat{\rho}_{\hat{\mathbf{g}}}} = (\rho_{\boldsymbol{\eta}}, \rho_{\boldsymbol{\eta}}) - (\rho_{\mathbf{g}}, \rho_{\mathbf{g}}). \tag{21}$$

Remark. Due to the convention $\hat{\lambda}(d) = 0$ for highest weights, The eigenvalues of $\hat{\Delta}$ on $\hat{\mathbf{g}}/\hat{\boldsymbol{\eta}}$ are the same as the eigenvalues of Δ on $\mathbf{g}/\boldsymbol{\eta}$. However, the corresponding eigenspaces are different.

If $\hat{\lambda} = (w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}; k + \eta - g; 0)$ is not dominant for $\hat{\mathbf{g}}$, the lowest eigenvalue of $\hat{\Delta}$ does not exist. Thus we can always choose $\hat{\mu}$ such that $\hat{\lambda}$ is dominant. It can be shown that $\hat{\lambda}$ is dominant for $\hat{\mathbf{g}}$, if $w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}$ is dominant for $\mathbf{g}$, and the level $k + \eta - g \geq (w(\mu + \rho_{\boldsymbol{\eta}}) - \rho_{\mathbf{g}}, \theta)$, where θ is the highest root of $\mathbf{g}$.

References

[1] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience, (1969).

[2] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, (1978).

[3] F. W. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, Graduate Texts in Mathematics, Vol. 94, Springer-Verlag, New York, (1983).

[4] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*, Springer-Verlag New York, Inc. (1997).

[5] L. Hu, The Laplacian on homogeneous spaces, *J. Math. Phys.*, **49** (2008), 053513.

[6] C. N. Yang, Generalization of Diracs monopole to SU_2 gauge fields, *J. Math. Phys.*, **19** (1978), 320-328.

[7] H. Muto and H. Urakawa, On the least positive eigenvalue of Laplacian for compact homogeneous spaces, *Osaka J. Math.*, **21** (1980), 471-484 .

[8] A. Salam and Strathdee, On Kaluza-Klein theory, *Ann. Phys.*, **141** (1982), 316-352.

[9] M. Stone, Supersymmetry and the quantum mechanics of spin, *Nucl. Phys. B*, **314** (1989), 557-586.

[10] V. D. Lyakhovsky, N. N. Shtykov and D. V. Vassilevich, DeWitt-Schwinger expansion for projective and Grassmann spaces, *Lett. Math. Phys.*, **21** (1991), 89-95.

[11] T. Jaroszewicz and P. S. Kurzepa, Polyakov spin and Laplacians on homogeneous spaces, *Annals of Physics*, **213** (1992), 135-165.

[12] N. P. Landsman, Induced representations, gauge fields, and quantization on homogeneous spaces, *Rev. Math. Phys.*, **4** (1992), 503-528.

[13] P. Lévay, D. McMullan and I.Tsutsui, The canonical connection in quantum mechanics, *J. Math. Phys.*, **37** (1996), 625-636.

[14] S.-C. Zhang and J. Hu, A four dimensional generalization of the quantum Hall effect, *Science*, **294** (2001), 823-828.

[15] D. Karabali and V. P. Nair, Quantum Hall effect in higher dimensions, *Nucl. Phys. B*, **641** (2002), 533-546.

[16] B. Dolan, The spectrum of the Dirac operator on coset spaces with homogeneous gauge fields, *JHEP*, **0305** (2003), 018.

[17] B. A. Bernevig, J. Hu, N. Toumbas and S.-C. Zhang, The eight dimensional quantum Hall effect and the octonions, *Phys. Rev. Lett.* **91** (2003), 236803-236807.

[18] S. Bellucci, P. Y. Casteill and A. Nersessian, Four-dimensional Hall mechanics as a particle on CP(3), *Phys. Lett. B*, **574** (2003), 121-128.

[19] G. Meng, Geometric construction of the Quantum Hall Effect in all even dimensions, *J. Phys. A*, **36** 9415 (2003), 9415-9424.

[20] R. Bott, The index theorem for homogeneous differential operators, In: *Differential and Combinatorial Topology*, edited by S. S. Cairns, Princeton University Press, (1965), 167-186.

[21] B. Gross, B. Kostant, P. Ramond and S. Sternberg, The Weyl character formula, the half-spin representations, and equal rank subgroups, *Proc. Natl. Acad. Sci. USA*, **95** (1998), 8441-8442.

[22] B. Kostant, A cubic Dirac operator and the emergence of the Euler number multiplets of representations for equal rank subgroups, *Duke Math. J.*, **100** (1999), 447-501.

[23] G. D. Landweber, Harmonic spinors on homogeneous spaces, *Representation Theory*, **4** (2000), 466-473.

[24] V. G. Kac, *Infinite Dimensional Lie Algebras*, Third edition, Cambridge University Press, Cambridge (1990).

[25] A. Pressley and G. Segal, *Loop Groups*, Oxford University Press, (1988).

[26] A. Wassermann, Kac-Moody and Virasoro Algebras, *arXiv:*1004.1287v2.