CONVERGENCE OF IMPLICIT ITERATION PROCESS FOR A COUNTABLE FAMILY OF CONTINUOUS PSEUDOCONTRACTIVE MAPPINGS

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Abstract: We study weak convergence of implicit iterations for a countable family of continuous pseudocontractive mappings and a nonexpansive mapping in Banach spaces. Moreover, necessary and sufficient conditions for strong convergence to a common fixed point of continuous hemicontactive mappings and a continuous quasi-nonexpansive mapping are given in real Banach spaces. The obtained results extend those announced by many authors.

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1. Introduction

Let $E$ be a real Banach space and $K$ a nonempty subset of $E$. Let $J$ denote the normalized duality mapping from $E$ into $2^{E^*}$ given by $J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \}, \forall x \in E$, where $E^*$ denotes the dual space of $E$ and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. If $E$ is smooth or $E^*$ is strictly convex, then $J$ is single-valued.

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Throughout this paper, we denote the single-valued duality mapping by $j$ and denote the set of fixed points of a nonlinear mapping $T : K \to E$ by

$$F(T) = \{ x \in K : Tx = x \}.$$ 

**Definition 1.1.** A mapping $T$ with domain $D(T)$ and range $R(T)$ in $E$ is called

(i) *pseudocontractive* [2] if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2;$$

equivalently [2, 8], for all $x, y \in D(T)$ and for all $s > 0,$

$$\|x - y\| \leq \|x - y + s(I - T)x - (I - T)y\|; \quad (1.1)$$

(ii) *hemicontactive* if for all $x \in D(T), \ x^* \in F(T)$, there exists $j(x - x^*) \in J(x - x^*)$ such that

$$\langle Tx - x^*, j(x - x^*) \rangle \leq \|x - x^*\|^2;$$

(iii) *λ-strictly pseudocontractive in the terminology of Browder-Petryshyn* [1] if for all $x, y \in D(T)$, there exists $\lambda > 0$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|x - y - (Tx - Ty)\|^2;$$

(iv) *strongly pseudocontractive* if for all $x, y \in D(T)$, there exists $\lambda \in (0, 1)$ and $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq \lambda \|x - y\|^2;$$

(v) *L-Lipschitzian* if for all $x, y \in D(T)$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|;$$

(vi) *nonexpansive* if for all $x, y \in D(T),$

$$\|Tx - Ty\| \leq \|x - y\|;$$

(vii) *quasi-nonexpansive* if for all $x \in D(T), \ x^* \in F(T),$

$$\|Tx - x^*\| \leq \|x - x^*\|.$$

**Remark 1.2.** It is obvious by definitions that:

1. Every strictly pseudocontractive mapping is pseudocontractive.
2. Every pseudocontractive mapping is hemicontactive.
3. Every $\lambda$-strictly pseudocontractive mapping is $(\frac{1 + \lambda}{\lambda})$-Lipschitzian; see [5].
Let \( \{T_i\}_{i=1}^N \) be a finite family of nonlinear self-mappings on a subset \( K \). Let \( \{x_n\} \) be defined by \( x_0 \in K \) and
\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n x_n, \quad n \geq 1
\] (1.2)
where \( \alpha_n \in (0, 1) \) and \( T_n = T_n \mod N \). The implicit iteration (1.2) was introduced by Xu and Ori [18] for a finite family of nonexpansive mappings in a Hilbert space. To be more precise, they proved the following theorem:

**Theorem 1.3.** [18] Let \( H \) be a real Hilbert space, \( K \) a nonempty, closed and convex subset of \( H \), and \( \{T_i\}_{i=1}^N \) a finite family of nonexpansive self-mappings on \( K \) such that \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be defined by (1.2). If \( \lim_{n \to \infty} \alpha_n = 0 \), then \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i\}_{i=1}^N \).

Motivated by Xu and Ori [18]'s idea, Osilike [12] extended the above theorem from the class of nonexpansive mappings to the more general class of strictly pseudocontractive mappings. He proved the following theorem:

**Theorem 1.4.** [12] Let \( H \) be a real Hilbert space, \( K \) a nonempty closed convex subset of \( H \), and \( \{T_i\}_{i=1}^N \) a finite family of strictly pseudocontractive self-mappings on \( K \) such that \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be defined by (1.2). If \( \lim_{n \to \infty} \alpha_n = 0 \), then \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i\}_{i=1}^N \).

In 2006, Chen et al. [5] extended Osilike [12]'s result from Hilbert spaces to \( q \)-uniformly smooth and uniformly convex Banach spaces. They proved the following theorem:

**Theorem 1.5.** [5] Let \( E \) be a real \( q \)-uniformly smooth Banach space which is also uniformly convex and satisfies Opial’s condition. Let \( K \) be a nonempty, closed and convex subset of \( E \), and \( T_i : K \to K, \ i = 1, 2, ..., N \) be a finite family of strictly pseudocontractive self-mappings on \( K \) such that \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be defined by (1.2). If \( 0 < a \leq \alpha_n \leq b < 1 \), then \( \{x_n\} \) converges weakly to a common fixed point of \( \{T_i\}_{i=1}^N \).

Recently, Zhou [19] extended the results of Xu and Ori [18], Osilike [12] and Chen et al. [5] to the more general uniformly convex Banach spaces and the more general class of Lipschitzian pseudocontractive mappings; in particular, he proved the following theorem:

**Theorem 1.6.** [19] Let \( E \) be a real uniformly convex Banach space with a Fréchet differentiable norm. Let \( K \) be a closed and convex subset of \( E \), and \( \{T_i\}_{i=1}^N \) be a finite family of Lipschitzian pseudocontractive self-mappings of \( K \) such that \( F = \bigcap_{i=1}^N F(T_i) \neq \emptyset \). Let \( \{x_n\} \) be defined by (1.2). If \( \{\alpha_n\} \) is...
chosen so that $\alpha_n \in (0, 1)$ with $\limsup_{n \to \infty} \alpha_n < 1$, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^N$.

Motivated and inspired by Xu and Ori [18], Osilike [12], Chen et al. [5] and Zhou [19], we consider the following implicit iteration:

$$x_0 \in K$$
$$x_n = \alpha_n y_{n-1} + (1 - \alpha_n) T_n x_n,$$
$$y_{n-1} = \beta_n x_{n-1} + (1 - \beta_n) S x_{n-1}, \quad n \geq 1$$

(1.3)

where $\alpha_n, \beta_n \in (0, 1)$ and $S, \{T_n\}_{n=1}^\infty$ are nonlinear mappings on a closed and convex subset $K$ of a real Banach space $E$.

In this paper, we prove strong convergence for the implicit iteration (1.3) in the frameworks of an arbitrary real Banach space. Then we prove weak convergence results in a uniformly convex Banach space which satisfies Opial’s condition or has a Fréchet differentiable norm. The results obtained in this paper extend the results of Zhou [19] from a finite family Lipschitzian pseudocontractions to a countable family of continuous pseudocontractions. Consequently, our results also extend and improve the results of Xu and Ori [18], Osilike [12], Rafiq [14], Song [16], Chen et al. [5] and some others.

The implicit iteration process for nonlinear mappings in the framework of Hilbert spaces and Banach spaces has been studied by several authors; see also [3, 4, 7, 9].

We will use the notation:

• $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

• $\omega(x_n) = \{x : x_n \rightharpoonup x\}$ denotes the weak $\omega$-limit set of $\{x_n\}$.

• $d(x, C) = \inf_{z \in C} \|x - z\|$.

2. Preliminaries

Let $E$ be a real Banach space and $S(E) = \{x \in E : \|x\| = 1\}$. Then $E$ is said to be smooth if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. The norm of $E$ is said to be Fréchet differentiable if for each $x \in S(E)$, the limit is attained uniformly for $y \in S(E)$.

A Banach space $E$ is called uniformly convex if for each $\epsilon > 0$ there is a $\delta > 0$ such that for $x, y \in E$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, $\|x + y\| \leq 2(1 - \delta)$ holds. A Banach space $E$ is called strictly convex if $\|x + y\|/2 < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. In a strictly convex Banach $E$ we have
that if \( \|x\| = \|y\| = \|\lambda x + (1 - \lambda)y\| \) for \( x, y \in E \) and \( \lambda \in (0, 1) \), then \( x = y \). It is well known that a uniformly convex Banach space is strictly convex.

In the sequel, we shall need the following definitions and lemmas.

**Definition 2.1.** A Banach space \( E \) is said to satisfy Opial’s condition \([13]\), if whenever \( \{x_n\} \) is a sequence in \( E \) which converge weakly to \( x \) as \( n \to \infty \), then

\[
\limsup_{n \to \infty} \|x_n - x\| < \limsup_{n \to \infty} \|x_n - y\|, \quad \forall y \in E, \ x \neq y.
\]

**Lemma 2.2.** \([19]\) Let \( E \) be a real uniformly convex Banach space, \( K \) a nonempty, closed and convex subset of \( E \), and \( T : K \to K \) be a continuous pseudocontractive mapping. Then, \( I - T \) is demiclosed at zero, that is, for all sequence \( \{x_n\} \subset K \) with \( x_n \rightharpoonup p \) and \( \|x_n - Tx_n\| \to 0 \) it follows that \( p = Tp \).

**Lemma 2.3.** \([19]\) Let \( E \) be a smooth Banach space and \( K \) be a nonempty and convex subset of \( E \). Given an integer \( N \geq 1 \), assume that for each \( i \in \Lambda \), \( S_i : K \to K \) is a \( \lambda_i \)-strictly pseudocontraction for some \( 0 \leq \lambda_i < 1 \). Assume that \( \{\mu_i\}_{i=1}^N \) is a positive sequence such that \( \sum_{i=1}^N \mu_i = 1 \). Then \( \sum_{i=1}^N \mu_i S_i : K \to K \) is a \( \lambda \)-strictly pseudocontraction with \( \lambda = \min \{\lambda_i : 1 \leq i \leq N\} \).

**Lemma 2.4.** \([19]\) Let \( E \) be a smooth Banach space and \( K \) be a nonempty and convex subset of \( E \). Given an integer \( N \geq 1 \), assume that \( \{S_i\}_{i=1}^N : K \to K \) is a finite family of \( \lambda_i \)-strictly pseudocontraction for some \( 0 \leq \lambda_i < 1 \) such that \( F = \bigcap_{i=1}^N F(S_i) \neq \emptyset \). Assume that \( \{\mu_i\}_{i=1}^N \) is a positive sequence such that \( \sum_{i=1}^N \mu_i = 1 \). Then \( F(\sum_{i=1}^N \mu_i S_i) = F \).

**Lemma 2.5.** \([15]\) Suppose that \( E \) is a uniformly convex Banach space and \( 0 < s \leq t_n \leq t < 1 \) for all positive integers \( n \). Also suppose that \( \{x_n\} \) and \( \{y_n\} \) are two sequences of \( E \) such that \( \limsup_{n \to \infty} \|x_n\| \leq r \), \( \limsup_{n \to \infty} \|y_n\| \leq r \) and

\[
\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r \quad \text{for some } r \geq 0.
\]

Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 2.6.** \([6]\) Let \( E \) be a real Banach space and \( K \) a nonempty, closed and convex subset of \( E \), and \( T : K \to K \) a continuous strongly pseudocontractive mapping. Then \( T \) has a unique fixed point in \( K \).

Let \( K \) be a nonempty, closed and convex subset of a real Banach space \( E \) and \( T \) a continuous strongly pseudocontractive mapping of \( K \). For every \( u \in K \) and \( t \in (0, 1) \), the mapping \( S_t : K \to K \) defined by

\[ S_t x = tu + (1 - t)Tx, \quad x \in K, \]

is a continuous and strongly pseudocontractive mapping; by utilizing Lemma
there exists a unique fixed point $x_t \in K$ of $S_t$ such that

$$x_t = tu + (1 - t)Tx_t, \quad t \in (0, 1).$$

**Lemma 2.7.** [17] Let $E$ be a real uniformly convex Banach space with a Fréchet differentiable norm. Let $K$ be a closed and convex subset of $E$, and \{\{T_n\}_{n=1}^\infty\} be a family of Lipschitzian self-mappings on $K$ such that $\sum_{n=1}^\infty (L_n - 1) < \infty$ and $F = \bigcap_{n=1}^\infty F(T_n) \neq \emptyset$. For arbitrary $x_1 \in K$, define $x_{n+1} = T_n x_n$ for all $n \geq 1$. Then for every $p, q \in F$, $\lim_{n \to \infty} \langle x_n, j(p - q) \rangle$ exists, in particular, for all $u, v \in \omega(x_n)$, and $p, q \in F$, $\langle u - v, j(p - q) \rangle = 0$.

### 3. Convergence in Banach Spaces

In this section, we prove a strong convergence of an implicit iteration for continuous hemicontractive mappings and a continuous quasi-nonexpansive mapping in a real arbitrary Banach space.

To prove our main results, we need the following lemma:

**Lemma 3.1.** Let $E$ be a real Banach space and $K$ a nonempty, closed and convex subset of $E$. Let $S$ be a continuous quasi-nonexpansive self-mapping on $K$ and \{\{T_n\}_{n=1}^\infty\} a countable family of continuous hemicontractive self-mappings on $K$ such that $F = \bigcap_{n=1}^\infty F(T_n) \cap F(S) \neq \emptyset$. Let \{\{x_n\}\} be defined by (1.3) and let \{\{a_n\}\}, \{\{\beta_n\}\} be real sequences in $(0, 1)$. Then:

(i) $\lim_{n \to \infty} \|x_n - p\|$, $\lim_{n \to \infty} \|y_n - p\|$ exist and $\lim_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|y_n - p\|$ for all $p \in F$,

(ii) $\lim_{n \to \infty} d(x_n, F)$, $\lim_{n \to \infty} d(y_n, F)$ exist and $\lim_{n \to \infty} d(x_n, F) = \lim_{n \to \infty} d(y_n, F)$.

**Proof.** Let $p \in F$ and $n \geq 1$. Then there exists $j(x_n - p) \in J(x_n - p)$ such that

$$\|x_n - p\|^2 = \langle x_n - p, j(x_n - p) \rangle$$

$$= \alpha_n \langle y_{n-1} - p, j(x_n - p) \rangle + (1 - \alpha_n) \langle T_n x_n - p, j(x_n - p) \rangle$$

$$\leq \alpha_n \|y_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2$$

$$\leq \alpha_n (\beta_n \|x_{n-1} - p\| + (1 - \beta_n) \|S x_{n-1} - p\|) \|x_n - p\|$$

$$+ (1 - \alpha_n) \|x_n - p\|^2$$

$$\leq \alpha_n \|x_{n-1} - p\| \|x_n - p\| + (1 - \alpha_n) \|x_n - p\|^2.$$ 

Hence

$$\|x_n - p\| \leq \|y_{n-1} - p\| \leq \|x_{n-1} - p\|.$$ 

(3.1)
This implies that (i) holds. By taking the infimum over all $p \in F$ in (3.1), we also obtain
\[ d(x_n, F) \leq d(y_{n-1}, F) \leq d(x_{n-1}, F). \] (3.2)
This shows that $\lim_{n \to \infty} d(x_n, F)$ exists. Moreover, by taking the limit as $n \to \infty$ to (3.2), $\lim_{n \to \infty} d(y_n, F) = \lim_{n \to \infty} d(x_n, F)$. Thus, (i) and (ii) are proved. \qed

Now, we prove our result.

**Theorem 3.2.** Let $E$ be a real Banach space and $K$ a nonempty, closed and convex subset of $E$. Let $S$ be a continuous quasi-nonexpansive self-mapping on $K$ and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous hemicontractive self-mappings on $K$ such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be defined by (1.3), and let $\{\alpha_n\}, \{\beta_n\}$ be real sequences in $(0, 1)$. Then, $\{x_n\}$ converges strongly to $x^* \in F$ if and only if $\liminf_{n \to \infty} d(x_n, F) = 0$.

**Proof.** Since the necessity is obvious, it suffices to show the sufficiency. Suppose $\liminf_{n \to \infty} d(x_n, F) = 0$. From Lemma 3.1 (ii) we have $\lim_{n \to \infty} d(x_n, F) = 0$. It follows from (3.1) that for $n, m \in \mathbb{N}$ and $p \in F$,
\[ \|x_{n+m} - x_n\| \leq \|x_{n+m} - p\| + \|x_n - p\| \leq 2\|x_n - p\|. \]
Consequently,
\[ \|x_{n+m} - x_n\| \leq 2d(x_n, F) \to 0, \]
as $n \to \infty$. Hence $\{x_n\}$ is a Cauchy sequence. By the completeness of $E$, we can assume that $\lim_{n \to \infty} x_n = x^*$ for some $x^* \in E$. Then
\[ d(x^*, F) = \lim_{n \to \infty} d(x_n, F) = 0. \]
Hence $x_n \to x^* \in F$ as $n \to \infty$. This completes the proof. \qed

**Remark 3.3.** Theorem 3.2 improves and extends Theorem 2.3 of Chen et al.[5], Theorem 2.2 of Boonchari and Saejung [3], and Theorem 2 of Osilike [12].
4. Convergence in Uniformly Convex Banach Spaces

In this section, we prove weak convergence theorems of implicit iteration process for continuous pseudocontractive mappings and a nonexpansive mapping in a uniformly convex Banach space.

Let \( K \) be a subset of a Banach space \( E \). Let \( \{T_n\} \) and \( \Gamma \) be families of mappings on \( K \) such that \( \emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n) \). Then, a countable family of mappings \( \{T_n\} \) is said to satisfy:

(i) The NST-condition [10] if for each bounded sequence \( \{z_n\} \) in \( K \),
\[
\lim_{n \to \infty} \|z_n - T_n z_n\| = 0 \quad \text{implies} \quad \lim_{n \to \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \Gamma.
\]

(ii) The NST*-condition [11] if for each bounded sequence \( \{z_n\} \) in \( K \),
\[
\lim_{n \to \infty} \|z_n - T_n z_n\| = \lim_{n \to \infty} \|z_n - z_{n+1}\| = 0 \quad \text{implies} \quad \lim_{n \to \infty} \|z_n - T z_n\| = 0 \quad \forall T \in \Gamma.
\]

**Remark 4.1.** It follows directly that if \( \{T_n\} \) satisfies the NST-condition, then \( \{T_n\} \) satisfies the NST*-condition.

Using the NST-condition and the NST*-condition, we obtain the following:

**Lemma 4.2.** Let \( E \) be a real uniformly convex Banach space and \( K \) a nonempty, closed and convex subset of \( E \). Let \( S \) be a nonexpansive self-mapping on \( K \) and \( \{T_n\}_{n=1}^{\infty} \) a countable family of continuous pseudocontractive self-mappings on \( K \) such that \( F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset \). Let \( \Gamma \) be any subclass of continuous pseudocontractive mappings such that \( \emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n) \). Let \( \{x_n\} \) be defined by (1.3) and let \( \{\alpha_n\}, \{\beta_n\} \) be real sequences with \( 0 < \alpha_n \leq a < 1 \) and \( 0 < b \leq \beta_n \leq c < 1 \) for some \( a, b, c \in (0, 1) \). If \( \{T_n\}_{n=1}^{\infty} \) satisfies the NST*-condition, then
\[
\lim_{n \to \infty} \|x_n - S x_n\| = \lim_{n \to \infty} \|x_n - T x_n\| = 0, \quad \forall T \in \Gamma.
\]

**Proof.** Let \( p \in F \). Then, by Lemma 3.1 (i), we have
\[
\lim_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|y_n - p\| = d,
\]
for some \( d \geq 0 \). By using (1.1) and (3.1), we also have
\[
\|x_n - p\| \leq \|x_n - p + \frac{1 - \alpha_n}{2\alpha_n} (x_n - T_n x_n)\| = \|x_n - p + \frac{1 - \alpha_n}{2} (y_{n-1} - T_n x_n)\|
\]
\[\begin{align*}
= & \left\| \alpha_n y_{n-1} + (1 - \alpha_n) T_n x_n - p + \frac{1 - \alpha_n}{2} (y_{n-1} - T_n x_n) \right\| \\
= & \left\| \frac{y_{n-1} + x_n}{2} - p \right\| \\
= & \left\| \frac{y_{n-1} - p}{2} + \frac{x_n - p}{2} \right\| \\
\leq & \frac{1}{2} \left\| y_{n-1} - p \right\| + \frac{1}{2} \left\| x_n - p \right\| \\
\leq & \frac{1}{2} \left\| x_{n-1} - p \right\| + \frac{1}{2} \left\| x_n - p \right\| \\
\leq & \left\| x_{n-1} - p \right\|,
\end{align*}\]

which implies that
\[
\lim_{n \to \infty} \left\| \frac{y_{n-1} - p}{2} + \frac{x_n - p}{2} \right\| = d.
\]

By Lemma 2.5, we obtain
\[
\lim_{n \to \infty} \left\| y_{n-1} - x_n \right\| = 0. \tag{4.1}
\]

On the other hand, we also have
\[
\left\| x_n - p \right\| \leq \left\| y_{n-1} - p \right\| \\
= \left\| \beta_n (x_{n-1} - p) + (1 - \beta_n)(S x_{n-1} - p) \right\| \\
\leq \left\| x_{n-1} - p \right\|.
\]

Hence
\[
\lim_{n \to \infty} \left\| \beta_n (x_{n-1} - p) + (1 - \beta_n)(S x_{n-1} - p) \right\| = d.
\]

It is easy to see that \( \lim_{n \to \infty} \left\| S x_{n-1} - p \right\| \leq d \). Since \( 0 < b \leq \beta_n \leq c < 1 \), it follows from Lemma 2.5 that
\[
\lim_{n \to \infty} \left\| x_n - S x_n \right\| = 0. \tag{4.2}
\]

Again by (1.3) we observe that
\[
x_n - T_n x_n = \frac{\alpha_n}{1 - \alpha_n} (y_{n-1} - x_n),
\]

which implies
\[
\left\| x_n - T_n x_n \right\| = \frac{\alpha_n}{1 - \alpha_n} \left\| y_{n-1} - x_n \right\|.
\]
From (4.1) and $0 < \alpha_n \leq a < 1$, we obtain
$$\lim_{n \to \infty} \|x_n - T_n x_n\| = 0.$$  
(4.3)

On the other hand, from (4.2), we also obtain
$$\|y_{n-1} - x_{n-1}\| = (1 - \beta_n) \|S x_{n-1} - x_{n-1}\| \to 0,$$  
(4.4)
as $n \to \infty$. So, by (4.1) and (4.4), we have
$$\lim_{n \to \infty} \|x_n - x_{n+1}\| = 0.$$  
(4.5)

Since $\{T_n\}$ satisfies the NST*-condition, it follows from (4.3) and (4.5) that
$$\lim_{n \to \infty} \|x_n - T x_n\| = 0,$$
for all $T \in \Gamma$. This completes the proof. \(\square\)

Now we prove our main results.

**Theorem 4.3.** Let $E$ be a real uniformly convex Banach space which satisfies Opial’s condition and $K$ a nonempty, closed and convex subset of $E$. Let $S$ be a nonexpansive self-mapping on $K$ and $\{T_n\}_{n=1}^\infty$ a countable family of continuous pseudocontractive self-mappings on $K$ such that $F = \bigcap_{n=1}^\infty F(T_n) \cap F(S) \neq \emptyset$. Let $\Gamma$ be any subclass of continuous pseudocontractive mappings such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^\infty F(T_n)$. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for some $a, b, c \in (0, 1)$. If $\{T_n\}_{n=1}^\infty$ satisfies the NST*-condition, then a sequence $\{x_n\}$ defined by (1.3) converges weakly to $x^* \in F$.

**Proof.** By Lemma 2.2, we know that $T$ is demiclosed at zero for all $T \in \Gamma$. It follows from Lemma 3.1 (i) and Lemma 4.2 that $\omega(x_n) \subset F(S) \cap F(\Gamma) \subset F$. Moreover, in a uniformly convex Banach space, Opial’s condition ensures that $\omega(x_n)$ is a singleton. We thus complete the proof. \(\square\)

**Remark 4.4.** If $S = I$, then Theorem 4.3 improves and extends Theorem 5 of Chen et al. [4] and Theorem 2.6 of Chen et al. [5] in several respects:
(i) From real $q$-uniformly smooth and uniformly convex Banach spaces to real uniformly convex Banach spaces.
(ii) From a finite family of strictly pseudocontractions to a countable family of continuous pseudocontractions.
(iii) Relax the restriction on $\{\alpha_n\}$ in Theorem 2.6 of [5].
Theorem 4.5. Let $E$ be a real uniformly convex Banach space with a Fréchet differentiable norm and $K$ a nonempty, closed and convex subset of $E$. Let $S$ be a nonexpansive self-mapping on $K$ and $\{T_n\}_{n=1}^{\infty}$ a countable family of continuous pseudocontractive self-mappings on $K$ such that $F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset$. Let $\Gamma$ be any subclass of continuous pseudocontractive mappings such that $\emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n)$. Let $\{\alpha_n\}, \{\beta_n\}$ be real sequences with $0 < \alpha_n \leq a < 1$ and $0 < b \leq \beta_n < c < 1$ for some $a, b, c \in (0, 1)$. If $\{T_n\}_{n=1}^{\infty}$ satisfies the NST*-condition, then a sequence $\{x_n\}$ defined by (1.3) converges weakly to $x^* \in F$.

Proof. As shown in Theorem 4.3, we get that $\omega_{\omega}(x_n) \subset F$. So, it suffices to prove that $\omega_{\omega}(x_n)$ is a singleton. First, we show that the mapping $D_n : K \to K$ defined by $D_n x = \frac{1}{\alpha_n+1} (I - (1-\alpha_n+1)T_{n+1})x$, $x \in K$ is one-to-one for all $n \geq 0$. From (1.1), for each $x, y \in K$, we have

$$
\|x - y\| \leq \left\| x - y + \frac{(1-\alpha_{n+1})}{\alpha_{n+1}} [(x - T_{n+1}x) - (y - T_{n+1}y)] \right\|
= \frac{1}{\alpha_{n+1}} \left\| x - y - (1 - \alpha_{n+1})T_{n+1}x + (1-\alpha_{n+1})T_{n+1}y \right\|
= \frac{1}{\alpha_{n+1}} \left\| (I - (1-\alpha_{n+1})T_{n+1})x - (I - (1-\alpha_{n+1})T_{n+1})y \right\|
= \|D_n x - D_n y\|. \tag{4.6}
$$

Thus $D_n$ is one-to-one for all $n \geq 0$. For each $n \geq 0$, let $C_n = D_n^{-1}$ and $Q_n = \beta_{n+1} I + (1-\beta_{n+1})S$. Then, we observe that (1.3) is equivalent to

$$
x_{n+1} = C_n Q_n x_n, \quad \forall n \geq 0.
$$

Next, we shall show that $C_n(K) \subset K$ for every $n \geq 0$. For $u \in K$, we know from Lemma 2.6 that the continuous and strongly pseudocontractive mapping $S_{n,u} : K \to K$ defined by

$$
S_{n,u} x = \alpha_n u + (1 - \alpha_n)T_{n}x \quad \forall n \geq 1, \ x \in K
$$

has a unique fixed point $p_n \in K$. So we have

$$
p_n = S_{n,u} p_n = \alpha_n u + (1 - \alpha_n)T_{n}p_n \quad \forall n \geq 1.
$$

This implies that

$$
u = \alpha_n^{-1}(I - (1 - \alpha_n)T_n)p_n.
$$
Hence, we obtain that $K \subset \alpha_n^{-1}(I - (1 - \alpha_n)T_n)(K)$ for all $n \geq 1$; consequently,
\[
\alpha_n(I - (1 - \alpha_n)T_n)^{-1}(K) \subset K \quad \forall n \geq 1.
\]

This shows that $C_n(K) \subset K$ for all $n \geq 0$. To apply Lemma 2.7, we will show that $C_nQ_n$ is nonexpansive. Since $S$ is nonexpansive, $Q_n$ is also nonexpansive. So it suffices to show that $C_n$ is nonexpansive. From (4.6), we see that $D_n = C_n^{-1}$ is one-to-one for all $n \geq 0$. For each $x', y' \in K$ and $n \geq 0$, we set $x' = C_n^{-1}x$ and $y' = C_n^{-1}y$. Then
\[
\|C_n x' - C_n y'\| \leq \|x' - y'\|, \quad \forall n \geq 0.
\]

Thus $C_n$ is nonexpansive for all $n \geq 0$. So is $C_nQ_n$. Moreover, $C_n^{-1}(K)$ is closed for all $n \geq 0$.

Next, we show that $\bigcap_{n=0}^\infty F(C_n) = \bigcap_{n=1}^\infty F(T_n)$. Let $z \in \bigcap_{n=0}^\infty F(C_n)$. Then
\[
z = C_nz = \alpha_{n+1}(I - (1 - \alpha_{n+1})T_{n+1})^{-1}z,
\]
this implies that
\[
\frac{1}{\alpha_{n+1}}(I - (1 - \alpha_{n+1})T_{n+1})z = z.
\]

Hence $z = T_nz$ for all $n \geq 1$. On the other hand, let $z \in \bigcap_{n=1}^\infty F(T_n)$. Then from (4.6) we have
\[
\|z - C_nz\| \leq \|D_nz - D_nC_nz\|
= \left\|\frac{1}{\alpha_{n+1}}(I - (1 - \alpha_{n+1})T_{n+1})z - z\right\|
= 0.
\]

Thus $z = C_nz$ for all $n \geq 0$. Hence, $\bigcap_{n=0}^\infty F(C_n) = \bigcap_{n=1}^\infty F(T_n)$.

Next, we will show that $\bigcap_{n=0}^\infty F(C_nQ_n) = F$. $F \subset \bigcap_{n=0}^\infty F(C_nQ_n)$ is obvious. Let $z \in \bigcap_{n=0}^\infty F(C_nQ_n)$ and $p \in F \subset \bigcap_{n=0}^\infty F(C_n)$. Then,
\[
\|z - p\| = \|C_nQ_nz - C_np\|
\leq \|Q_nz - p\|
= \|\beta_{n+1}z + (1 - \beta_{n+1})Sz - p\|
= \|\beta_{n+1}(z - p) + (1 - \beta_{n+1})(Sz - p)\|
\leq \beta_{n+1}\|z - p\| + (1 - \beta_{n+1})\|Sz - p\|
\leq \|z - p\|.
\]
It follows that \( \|z - p\| = \|Sz - p\| = \|\beta_{n+1}(z - p) + (1 - \beta_{n+1})(Sz - p)\| \). Since \( E \) is strictly convex, \( z = Sz \). We also have \( z = C_n Q_n z = C_n z \); consequently, \( \bigcap_{n=0}^{\infty} F(C_n Q_n) \subset F \). Hence, \( \bigcap_{n=0}^{\infty} F(C_n Q_n) = F \).

Finally, we will show that \( \omega_\omega(x_n) \) is a singleton. Suppose that \( x^*, y^* \in \omega_\omega(x_n) \). By Lemma 2.2, we know that \( x^*, y^* \in F = \bigcap_{n=0}^{\infty} F(C_n Q_n) \). By Lemma 2.7, we also get that \( \lim_{n \to \infty} (x_n, j(x^* - y^*)) \) exists. Suppose that \( \{x_{n_k}\} \) and \( \{x_{m_k}\} \) are subsequences of \( \{x_n\} \) such that \( x_{n_k} \to x^* \) and \( x_{m_k} \to y^* \). Then

\[
\|x^* - y^*\|^2 = \langle x^* - y^*, j(x^* - y^*) \rangle = \lim_{k \to \infty} \langle x_{n_k} - x_{m_k}, j(x^* - y^*) \rangle = 0.
\]

Hence \( x^* = y^* \); consequently, \( x_n \to x^* \in F \) as \( n \to \infty \). This completes the proof. \( \square \)

**Remark 4.6.** In Theorem 4.3 and Theorem 4.5, if \( T_n : K \to K \) is defined by \( T_n x = \gamma_n x + (1 - \gamma_n) T x, \ x \in K \) where \( T : K \to K \) is a continuous pseudocontraction, \( 0 < \gamma_n \leq d < 1 \) for some \( d \in (0,1) \). Then we see that \( \bigcap_{n=1}^{\infty} F(T_n) = F(T) \). \( \{T_n\}_{n=1}^{\infty} \) is a countable family of continuous pseudocontractions and satisfies the NST*-condition. Hence, a sequence \( \{x_n\} \) defined by \( x_0 \in K \) and

\[
x_n = \alpha_n y_{n-1} + (1 - \alpha_n) (\gamma_n x_n + (1 - \gamma_n) T x_n),
\]

\[
y_{n-1} = \beta_n x_{n-1} + (1 - \beta_n) S x_{n-1}, \ n \geq 1
\]

converges weakly to \( x^* \in \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) = F(T) \cap F(S) \).

**Remark 4.7.** If \( S = I \), then Theorem 4.5 extends Theorem 3.1 of Zhou [19] from a finite family of Lipschitzian pseudocontractions to a countable family of continuous pseudocontractions.

Since every strictly pseudocontractive mapping is continuous pseudocontractive, we immediately obtain the following results.

**Theorem 4.8.** Let \( E \) be a real uniformly convex Banach space and \( K \) a nonempty, closed and convex subset of \( E \). Let \( S \) be a nonexpansive self-mapping on \( K \) and \( \{T_n\}_{n=1}^{\infty} \) a countable family of strictly pseudocontractive self-mappings on \( K \) such that \( F = \bigcap_{n=1}^{\infty} F(T_n) \cap F(S) \neq \emptyset \). Let \( \Gamma \) be any subclass of strictly pseudocontractive mappings such that \( \emptyset \neq F(\Gamma) \subset \bigcap_{n=1}^{\infty} F(T_n) \). Let \( \{x_n\} \) be defined by (1.3), and let \( \{\alpha_n\}, \{\beta_n\} \) be real sequences with \( 0 < \alpha_n \leq a < 1 \) and \( 0 < b \leq \beta_n \leq c < 1 \) for some \( a, b, c \in (0,1) \). If \( \{T_n\}_{n=1}^{\infty} \) satisfies the NST*-condition, then the following statements hold:

(i) If \( E \) satisfies Opial’s condition, then \( \{x_n\} \) converges weakly to \( x^* \in F \).

(ii) If \( E \) has a Fréchet differentiable norm, then \( \{x_n\} \) converges weakly to \( x^* \in F \).
Theorem 4.9. Let $E$ be a real smooth and uniformly convex Banach space and $K$ a nonempty, closed and convex subset of $E$. Let $S$ be a nonexpansive self-mapping on $K$ and $\{S_i\}_{i=1}^N$ a finite family of $\lambda_i$-strictly pseudocontractive self-mappings on $K$ such that $F = \bigcap_{i=1}^N F(S_i) \cap F(S) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\mu_{n,i}\}_{i=1}^N$ be real sequences with $0 < \alpha_n \leq a < 1$, $0 < b \leq \beta_n \leq c < 1$, $0 < d \leq \mu_{n,i} < 1$ and $\sum_{i=1}^N \mu_{n,i} = 1$ for some $a, b, c, d \in (0, 1)$. Let $\{x_n\}$ be defined by the following manner: $x_0 \in K$ and
\[
x_n = \alpha_n y_{n-1} + (1 - \alpha_n) \sum_{i=1}^N \mu_{n,i} S_i x_n,
\]
\[
y_{n-1} = \beta_n x_{n-1} + (1 - \beta_n) S x_{n-1}, \quad n \geq 1.
\]
Then the following statements hold:
(i) If $E$ satisfies Opial’s condition, then $\{x_n\}$ converges weakly to $x^* \in F$.
(ii) If $E$ has a Fréchet differentiable norm, then $\{x_n\}$ converges weakly to $x^* \in F$.

Proof. For each $n \geq 1$, define $T_n x = \sum_{i=1}^N \mu_{n,i} S_i x$, $x \in K$. By Lemma 2.3 and Lemma 2.4, we see that $T_n : K \to K$ is a $\lambda$-strictly pseudocontractive mapping with $\lambda = \min \{\lambda_i : 1 \leq i \leq N\}$ and $\bigcap_{i=1}^N F(S_i) = \bigcap_{n=1}^\infty F(T_n)$.

Next, we will show that $\{T_n\}$ satisfies the NST*-condition. By Remark 4.1, it suffices to show that $\{T_n\}$ satisfies the NST-condition. Let $\{z_n\}$ be a bounded sequence in $K$ such that $\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$ and let $z \in \bigcap_{n=1}^\infty F(T_n)$. Then,
\[
\|z_n - z\|^2 = \langle z_n - z, j(z_n - z) \rangle
\]
\[
= \langle z_n - T_n z_n, j(z_n - z) \rangle + \langle T_n z_n - z, j(z_n - z) \rangle
\]
\[
\leq \|z_n - T_n z_n\| \|z_n - z\| + \sum_{i=1}^N \mu_{n,i} \langle S_i z_n - z, j(z_n - z) \rangle
\]
\[
\leq \|z_n - T_n z_n\| \|z_n - z\| + \|z_n - z\|^2 - \lambda \sum_{i=1}^N \mu_{n,i} \|z_n - S_i z_n\|^2,
\]
which implies that
\[
\lambda d \sum_{i=1}^N \|z_n - S_i z_n\|^2 \leq \lambda \sum_{i=1}^N \mu_{n,i} \|z_n - S_i z_n\|^2 \leq \|z_n - T_n z_n\| \|z_n - z\|.
\]
Since $\{z_n\}$ is bounded and $\lim_{n \to \infty} \|z_n - T_n z_n\| = 0$,
\[
\lim_{n \to \infty} \|z_n - S_i z_n\| = 0, \quad 1 \leq i \leq N.
\]
Hence, \( \{T_n\} \) satisfies the NST-condition. By Theorem 4.8, the statements (i) and (ii) hold.

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References


