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HARDY INEQUALITY IN BANACH FUNCTION SPACE

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Abstract: Hardy-inequality has been characterized for sum of two integral operators in weighted Banach function space.

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1. Introduction

The concept of Banach Function Space (BFS) was introduced in [7]. A good treatment on the theory of BFS is available in [1]. Hardy inequality for various Hardy-type integral operators in BFS has been studied in [2–4, 6].

In this paper, we give necessary and sufficient conditions for the boundedness of operator T defined as:

$$(Tf)(x) = \phi_1(x) \int_a^x \psi_1(t)f(t)dt + \phi_2(x) \int_x^b \psi_2(t)f(t)dt \qquad (1)$$

between Weighted BFS (X, v) and (Y, u) for the interval $(0, \infty)$ in Section 2 [where $a = 0, b = \infty$ in (1.1)] and for the general interval (a, b) in Section 3. In Section 3 we also studied the boundedness of conjugate to T, which is denoted by T^*. Here ϕ_i, $\psi_i; i = 1, 2$ are measurable and finite functions defined on (a, b) (not necessarily non-negative), X and Y be BFS satisfying the Berezhnoi

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ℓ-condition [3] (also see [6]) and u, v are weight functions, that is, measurable functions, positive and finite almost everywhere in the interval (a, b) on real line such that - $\infty \leq a < b \leq \infty$. Throughout the paper, f is Lebesgue measurable functions defined on (a, b).

Boundedness of the operator T between weighted Lebesgue spaces $L^p(v)$ and $L^q(u)$ was proved in [8] (also see [5, Theorem 2.3]) for the case $p, q > 1$ and in [5, Remark 2.4] for the case $p > 1$, $0 < q \leq 1$.

X' denotes the associate space of BFS X. Norm of function $f \in X$ and $g \in X'$ are denoted, respectively, as $\|f\|_X$ and $\|g\|_{X'}$. For the definition of BFS, associate space of BFS and their norm, one can refer [1] or [6, Section 2]. We define weighted BFS (X, u) to be the space of all measurable functions f for which

$$\|f\|_{X,u} = \|fu\|_X < \infty.$$

$\|f\|_{X,u}$ denotes norm of a function $f \in (X, u)$. $\chi_{(a,b)}$ denotes characteristic function defined on (a, b).

2. Main Result

Theorem 2.1. *Let (X, v) and (Y, u) be weighted BFS satisfying ℓ-condition. Suppose the operator K be defined as*

$$(Kf)(x) = \phi_1(x)\int_0^x \psi_1(t)f(t)dt + \phi_2(x)\int_x^\infty \psi_2(t)f(t)dt.$$

Then the inequality

$$\|Kf\|_{Y,u} \leq C\|f\|_{X,v} \tag{2}$$

holds for a constant C if and only if $\max(A, B) < \infty$*; where*

$$A = \sup_{t>0} \|\chi_{[t,\infty]}u|\phi_1|\|_Y \|\chi_{[0,t]}v^{-1}|\psi_1|\|_{X'}$$
$$B = \sup_{t>0} \|\chi_{[0,t]}u|\phi_2|\|_Y \|\chi_{[t,\infty]}v^{-1}|\psi_2|\|_{X'}.$$

Proof. Necessity. Suppose $f = g\chi_{(\alpha,\beta)} \in (X, u)$ such that $f\psi_1 \geq 0$ and $0 < \alpha < \beta < \infty$. Then (2.1) becomes

$$C||f||_{X,v} = C||\chi_{(\alpha,\beta)}gv||_X$$
$$\geq \|\phi_1(x)\int_0^x \psi_1(t)f(t)dt + \phi_2(x)\int_x^\infty \psi_2(t)f(t)dt||_{Y,u}$$

$$\geq \|\chi_{[\beta,\infty)}|\phi_1(x)\int_0^x \psi_1(t)f(t)dt + \phi_2(x)\int_x^\infty \psi_2(t)f(t)dt|u(x)||_Y$$
$$= \|\chi_{[\beta,\infty)}|\phi_1(x)\int_0^x \psi_1(t)g(t)\chi_{(\alpha,\beta)}(t)dt|u(x)||_Y$$
$$\geq \|\chi_{[\beta,\infty)}u|\phi_1|||_Y \int_0^\infty \chi_{[\alpha,\beta]}(t)|\psi_1(t)|(v(t))^{-1}g(t)v(t)dt.$$

Consequently, applying associate norm as defined in [6, (2.1)], we have

$$\|\chi_{[\beta,\infty)}u|\phi_1|\,||_Y||\chi_{(\alpha,\beta]}v^{-1}|\psi_1|\,||_{X'} \leq C < \infty.$$

Since C is independent of α and β, we have, when $\alpha \to 0$ and then taking supremum over $\beta > 0, A < \infty$.

Necessity of $B < \infty$ can also be proved analogously by substituting the function $h(x)$ defined as

$$h(x) = g_1(x)\chi_{(\alpha,\beta)}(x) \in (X, u)$$

such that $h\psi_2 \geq 0$ and $0 < \alpha < \beta < \infty$ in inequality (2.1).

Sufficiency. The following Lemma extends a result of [6, Theorem 4] (also see [3]), which is easy to prove:

Lemma 1. *Let (X, v), (Y, u) be weighted BFS satisfying ℓ-condition and H be defined as*

$$(Hf)(x) = \phi_1(x)\int_0^x \psi_1(t)f(t)dt.$$

Then the inequality

$$||Hf||_{Y,u} \leq C||f||_{X,v}$$

holds for a constant C if and only if $A < \infty$.

Analogously the following can also be easily proved:

Lemma 2. *Suppose (X, v) and (Y, u) be weighted BFS satisfying ℓ-condition and H_1 be defined as*

$$(H_1f)(x) = \phi_2(x)\int_x^\infty \psi_2(t)f(t)dt.$$

Then the inequality

$$||H_1f||_{Y,u} \leq C||f||_{X,v}$$

holds for a constant C if and only if $B < \infty$.

Sufficiency now follows from Lemma 1, Lemma 2 and the inequality

$$||Kf||_{Y,u} \leq ||Hf||_{Y,u} + ||H_1f||_{Y,u}.$$

3. Operator T and T*

In the following two theorems, we state two natural analogues of Theorem 2.1 without proof (which can be obtained by suitable modifications in Theorem 2.1) which describes, respectively, boundedness of T and T*:

Theorem 3.1. *Suppose (X, v) and (Y, u) be weighted BFS satisfying ℓ-condition and T be defined as (1.1), then the inequality*

$$||\chi_{(a,b)}Tf||_{Y,u} \leq C||\chi_{(a,b)}f||_{X,v} \tag{3}$$

holds for a constant C if and only if $\max(A_1, B_1) < \infty$*, where*

$$A_1 = \sup_{a<t<b} ||\chi_{[t,b]}u|\phi_1|\,||_Y||\chi_{[a,t]}v^{-1}|\psi_1|\,||_{X'}$$
$$B_1 = \sup_{a<t<b} ||\chi_{[a,t]}u|\phi_2|\,||_Y||\chi_{[t,b]}v^{-1}|\psi_2|\,||_{X'}.$$

We define conjugate operator of T denoted by T* as follows (see [6]):

$$(T^*f)(x) = \psi_1(x)\int_x^b \phi_1(t)f(t)dt + \psi_2(x)\int_a^x \phi_2(t)f(t)dt.$$

Theorem 3.2. *Suppose that conditions given in Theorem 3.1 holds, then the inequality (3.1) holds for T replaced by T^* if and only if max $(A_1^*, B_1^*) < \infty$, where*

$$A_1^* = \sup_{a<t<b} ||\chi_{[t,b]}u|\psi_2|\,||_Y||\chi_{[a,t]}v^{-1}|\phi_2|\,||_{X'}$$
$$B_1^* = \sup_{a<t<b} ||\chi_{[a,t]}u|\psi_1|\,||_Y||\chi_{[t,b]}v^{-1}|\phi_1|\,||_{X'}.$$

Remark 3.3. Examples of BFS are classical Lebesgue spaces $L^p, 1 \leq p \leq \infty$, Orlicz spaces, Lorentz, Marcinkiewicz and symmetric spaces, X^p spaces $(1 \leq p < \infty)$, etc. Consequently, results of this paper can be extended in above mentioned examples of BFS.

Remark 3.4. The Berezhnoi ℓ-condition corresponds to the case $\max(r, s) \leq \min(p, q)$ in the Lorentz $L^{rs} - L^{pq}$ setting and to the case $p \leq q$ in the $L^p - L^q$ setting. Therefore, for $X = L^p$ and $Y = L^q$, the Theorem 3.1 reduces to [5, Theorem 2.3] for $1 < p \leq q < \infty$.

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