HARDY INEQUALITY IN BANACH FUNCTION SPACE

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Abstract: Hardy-inequality has been characterized for sum of two integral operators in weighted Banach function space.

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1. Introduction

The concept of Banach Function Space (BFS) was introduced in [7]. A good treatment on the theory of BFS is available in [1]. Hardy inequality for various Hardy-type integral operators in BFS has been studied in [2–4, 6].

In this paper, we give necessary and sufficient conditions for the boundedness of operator \( T \) defined as:

\[
(Tf)(x) = \phi_1(x) \int_a^x \psi_1(t)f(t)dt + \phi_2(x) \int_x^b \psi_2(t)f(t)dt
\]

between Weighted BFS \((X, v)\) and \((Y, u)\) for the interval \((0, \infty)\) in Section 2 [where \(a = 0, b = \infty\) in (1.1)] and for the general interval \((a, b)\) in Section 3. In Section 3 we also studied the boundedness of conjugate to \( T \), which is denoted by \( T^* \). Here \( \phi_i, \psi_i; i = 1, 2 \) are measurable and finite functions defined on \((a, b)\) (not necessarily non-negative), \( X \) and \( Y \) be BFS satisfying the Berezhnoi
ℓ-condition [3] (also see [6]) and $u, v$ are weight functions, that is, measurable functions, positive and finite almost everywhere in the interval $(a, b)$ on real line such that $-\infty \leq a < b \leq \infty$. Throughout the paper, $f$ is Lebesgue measurable functions defined on $(a, b)$.

Boundedness of the operator $T$ between weighted Lebesgue spaces $L^p(v)$ and $L^q(u)$ was proved in [8] (also see [5, Theorem 2.3]) for the case $p, q > 1$ and in [5, Remark 2.4] for the case $p > 1$, $0 < q \leq 1$.

$X'$ denotes the associate space of BFS $X$. Norm of function $f \in X$ and $g \in X'$ are denoted, respectively, as $\|f\|_X$ and $\|g\|_{X'}$. For the definition of BFS, associate space of BFS and their norm, one can refer [1] or [6, Section 2]. We define weighted BFS $(X, u)$ to be the space of all measurable functions $f$ for which

$$\|f\|_{X, u} = \|fu\|_X < \infty.$$ 

$\|f\|_{X, u}$ denotes norm of a function $f \in (X, u)$. $\chi(a, b)$ denotes characteristic function defined on $(a, b)$.

2. Main Result

**Theorem 2.1.** Let $(X, v)$ and $(Y, u)$ be weighted BFS satisfying ℓ-condition. Suppose the operator $K$ be defined as

$$(Kf)(x) = \phi_1(x) \int_0^x \psi_1(t) f(t) dt + \phi_2(x) \int_x^\infty \psi_2(t) f(t) dt.$$ 

Then the inequality

$$\|Kf\|_{Y, u} \leq C \|f\|_{X, v}$$ 

holds for a constant $C$ if and only if $\max(A, B) < \infty$; where

$$A = \sup_{t > 0} \|\chi_{[t, \infty]} u|\phi_1|\|_Y \|\chi_{[0, t]} v^{-1}|\psi_1|\|_{X'}$$

$$B = \sup_{t > 0} \|\chi_{[0, t]} u|\phi_2|\|_Y \|\chi_{[t, \infty]} v^{-1}|\psi_2|\|_{X'}.$$ 

**Proof.** Necessity. Suppose $f = g\chi(\alpha, \beta) \in (X, u)$ such that $f\psi_1 \geq 0$ and $0 < \alpha < \beta < \infty$. Then (2.1) becomes

$$C \|f\|_{X, v} = C \|\chi(\alpha, \beta)gv\|_X$$

$$\geq \|\phi_1(x) \int_0^x \psi_1(t) f(t) dt + \phi_2(x) \int_x^\infty \psi_2(t) f(t) dt\|_{Y, u}.$$
\[ |\chi_{[\beta,\infty)}\phi_1(x)\int_0^x \psi_1(t) f(t) dt + \phi_2(x) \int_x^\infty \psi_2(t) f(t) dt|_Y \]

\[ = |\chi_{[\beta,\infty)}\phi_1(x)\int_0^x \psi_1(t) g(t) \chi_{(\alpha,\beta]}(t) dt|_Y \]

\[ \geq |\chi_{[\beta,\infty)}\phi_1(x)\int_0^\infty \chi_{[\alpha,\beta]}(t) |\psi_1(t)| (v(t))^{-1} g(t) v(t) dt|_Y \]

Consequently, applying associate norm as defined in [6, (2.1)], we have

\[ \|\chi_{[\beta,\infty)}u|\phi_1||_Y \geq \|\chi_{[\beta,\infty)}u||_Y \int_0^\infty \chi_{(\alpha,\beta]}(t) |\psi_1(t)| (v(t))^{-1} g(t) v(t) dt \]

Since C is independent of \( \alpha \) and \( \beta \), we have, when \( \alpha \to 0 \) and then taking supremum over \( \beta > 0, A < \infty \).

Necessity of \( B < \infty \) can also be proved analogously by substituting the function \( h(x) \) defined as

\[ h(x) = g_1(x) \chi_{(\alpha,\beta]}(x) \in (X, u) \]

such that \( h\psi_2 \geq 0 \) and \( 0 < \alpha < \beta < \infty \) in inequality (2.1).

**Sufficiency.** The following Lemma extends a result of [6, Theorem 4] (also see [3]), which is easy to prove:

**Lemma 1.** Let \((X, v), (Y, u)\) be weighted BFS satisfying \( \ell \)-condition and \( H \) be defined as

\[ (Hf)(x) = \phi_1(x) \int_0^x \psi_1(t) f(t) dt. \]

Then the inequality

\[ \|Hf\|_{Y,u} \leq C\|f\|_{X,v} \]

holds for a constant \( C \) if and only if \( A < \infty \).

Analogously the following can also be easily proved:

**Lemma 2.** Suppose \((X, v)\) and \((Y, u)\) be weighted BFS satisfying \( \ell \)-condition and \( H_1 \) be defined as

\[ (H_1f)(x) = \phi_2(x) \int_x^\infty \psi_2(t) f(t) dt. \]

Then the inequality

\[ \|H_1f\|_{Y,u} \leq C\|f\|_{X,v} \]

holds for a constant \( C \) if and only if \( B < \infty \).

Sufficiency now follows from Lemma 1, Lemma 2 and the inequality

\[ \|Kf\|_{Y,u} \leq \|Hf\|_{Y,u} + \|H_1f\|_{Y,u}. \]
3. Operator T and T*

In the following two theorems, we state two natural analogues of Theorem 2.1 without proof (which can be obtained by suitable modifications in Theorem 2.1) which describes, respectively, boundedness of T and T*:

**Theorem 3.1.** Suppose $(X,v)$ and $(Y,u)$ be weighted BFS satisfying $\ell$-condition and $T$ be defined as (1.1), then the inequality

$$
\|\chi_{(a,b)}Tf\|_{Y,u} \leq C\|\chi_{(a,b)}f\|_{X,v}
$$

(3)

holds for a constant $C$ if and only if $\max(A_1,B_1) < \infty$, where

$$
A_1 = \sup_{a < t < b} \|\chi_{[t,b]}u|\phi_1|\|\chi_{[a,t]}v^{-1}|\psi_1|\|_{X'}
$$

$$
B_1 = \sup_{a < t < b} \|\chi_{[a,t]}u|\phi_2|\|\chi_{[t,b]}v^{-1}|\psi_2|\|_{X'}.
$$

We define conjugate operator of $T$ denoted by $T^*$ as follows (see [6]):

$$(T^*f)(x) = \psi_1(x) \int_x^b \phi_1(t)f(t)dt + \psi_2(x) \int_a^x \phi_2(t)f(t)dt.$$

**Theorem 3.2.** Suppose that conditions given in Theorem 3.1 holds, then the inequality (3.1) holds for $T$ replaced by $T^*$ if and only if $\max(A_1^*,B_1^*) < \infty$, where

$$
A_1^* = \sup_{a < t < b} \|\chi_{[t,b]}u|\psi_2|\|\chi_{[a,t]}v^{-1}|\phi_2|\|_{X'}
$$

$$
B_1^* = \sup_{a < t < b} \|\chi_{[a,t]}u|\psi_1|\|\chi_{[t,b]}v^{-1}|\phi_1|\|_{X'}.
$$

**Remark 3.3.** Examples of BFS are classical Lebesgue spaces $L^p, 1 \leq p \leq \infty$, Orlicz spaces, Lorentz, Marcinkiewicz and symmetric spaces, $X^p$ spaces ($1 \leq p < \infty$), etc. Consequently, results of this paper can be extended in above mentioned examples of BFS.

**Remark 3.4.** The Berezhnoi $\ell$-condition corresponds to the case $\max(r, s) \leq \min(p, q)$ in the Lorentz $L^{rs} - L^{pq}$ setting and to the case $p \leq q$ in the $L^p - L^q$ setting. Therefore, for $X = L^p$ and $Y = L^q$, the Theorem 3.1 reduces to [5, Theorem 2.3] for $1 < p \leq q < \infty$. 

References


