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MATCHING AND EDGE COVERING NUMBER ON TENSOR PRODUCT OF FAN GRAPH

Siriluk Intaja[1], Thanin Sitthiwirattham[2 §]
[1,2]Department of Mathematics
Faculty of Applied Science
King Mongkut's University of Technology
North Bangkok, Bangkok, 10800, THAILAND
[2]Centre of Excellence in Mathematics, CHE
Sri Ayutthaya Road, Bangkok, 10400, THAILAND

Abstract: Let $\alpha^{'}(G)$ and $\beta^{'}(G)$ be the matching number and edge covering number of G, respectively. The tensor product $G_1 \otimes G_2$ of graph of G_1 and G_2 has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$. In this paper, we determined generalization of matching number and edge covering on tensor product of fan graph and any simple graph.

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Key Words: tensor product, matching number, edge covering number

1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let G_1 and G_2 be graphs. The tensor product of graph G_1 and G_2, denote by $G_1 \otimes G_2$, is the graph with $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters.

A subset M of the edge set E of G is said to be a matching or an independent edge set of G if no two distinct edges in M have a common vertex. A matching

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§Correspondence author

M is called a maximum matching in G if there is no matching $M^{'}$ of G with $|M^{'}| > |M|$. The cardinality of a maximum matching of G is call the matching number of denoted by $\alpha^{'}(G)$.

An edge of graph G is said to cover the two vertices incident with it, and an edge cover of a graph G is a set of edges covering all the vertices of G. The minimum cardinality of an edge cover of a graph G is called the edge covering number of, G denote by $\beta^{'}(G)$.

By definitions of matching number and edge covering number, clearly that

$$\alpha^{'}(F_{m,n}) = \begin{cases} n, & m \geq n \\ m + \lfloor \frac{n-m}{2} \rfloor, & m < n \end{cases}$$

and

$$\beta^{'}(F_{m,n}) = \begin{cases} m, & m \geq n \\ m + \lceil \frac{n-m}{2} \rceil, & m < n. \end{cases}$$

Proposition 1.1 *Let* $H = G_1 \otimes G_2 = (V(H), E(H))$ *then:*
(i) $|V(H)| = |V(G_1)||V(G_2)|$
(ii) $|E(H)| = 2|E(G_1)||E(G_2)|$
(iii) *for every* $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

Theorem 1.2 *Let* G_1 *and* G_2 *be connected graphs, The graph* $H = G_1 \otimes G_2$ *is connected if and only if* G_1 *or* G_2 *contains an odd cycle.*

Theorem 1.3 *Let* G_1 *and* G_2 *be connected graphs with no odd cycle then* $G_1 \otimes G_2$ *has exactly two connected components.*

Next we get that general from of graph of tensor product of $F_{m,n}$ and a simple graph.

Proposition 1.4 *Let* G *be a connected graph of order* p, *the graph of* $F_{m,n} \otimes G$ *is*

$$\bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i \quad ; \quad H_i = \bigcup_{j=m+1}^{m+n} H_{ij} \quad \text{and} \quad M_i = H_{i(i+1)},$$

where

$$V(H_{ij}) = S_i \cup S_j, \; S_i = \{(i, 1), (i, 2), ..., (i, p)\},$$

$$S_j = \{(j, 1), (j, 2), ..., (j, p)\}; i < j \text{ and } E(H_{ij}) = \{(i, u)(j, v)/uv \in E(G)\}.$$

Moreover, if G has no odd cycle then each H_{ij} has exactly two connected components isomorphic to G.

Example

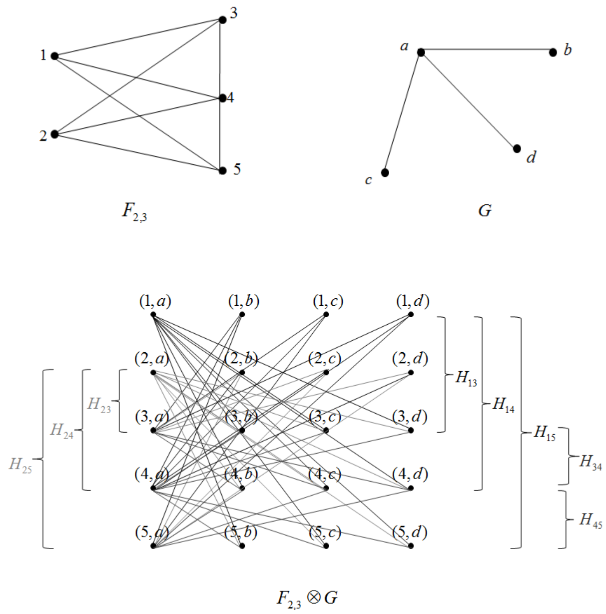

Figure 1: The graph of $F_{m,n} \otimes G$

2. Matching Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 2.3. that show character of matching for each H_{ij}.

Definition 2.1. *Let Given a matching M,* an *$M-$ alternating path is a path that alternates between edges in M and edges not in M. An $M-$ alternating odd path whose endpoints are unsaturated by M* is an *M -augmenting path.*

Theorem 2.2. *A matching M in a graph G is a maximum matching in G if and only if G* has no *$M-$augmenting path.*

Lemma 2.3. *Let* $F_{m,n} \otimes G = \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i; H_i = \bigcup_{j=m+1}^{m+n} H_{ij}$ *and* $M_i = H_{i(i+1)}$, *then* $\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G)$.

Proof. Suppose G has no odd cycle, by proposition 1.4, we have that H_{ij} and $H_{i(i+1)}$ each contains 2 components isomorphic to G. Therefore $H_{ij} = H_{i(i+1)} = 2G$. So $\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G)$.

If G has odd cycle, we have $d_{H_{ij}}(u_i, v) = d_{H_{ij}}(u_j, v) = d_G(v)$, for $(u_i, v) \in S_i$ and $(u_j, v) \in S_j, i < j$.

Let $E^* = \{e_i | e_i$ is any one edge in edge in each odd cycle C_i in $G; i = 1, 2, ..., l\}, | E^* | \leq l$ and let M be a maximum matching set of G. Now consider the tensor product $\bigcup_{i=1}^{m} H_i^* \cup \bigcup_{i=m+1}^{m+n-1} H_{i(i+1)}^* = F_{m,n} \otimes (G - E^*); i < j, i = 1, 2, ..., m$ where $V(H_{ij}^*) = V(H_{ij}), E(H_{ij}^*) = (i, u)(j, v) \mid uv \in (G - E^*)$. We get $H_{ij} = H_{i(i+1)} = 2(G - E^*)$ then

$$\alpha^{'}(H_{ij}^*) = \alpha^{'}(H_{i(i+1)}^*) = 2\alpha^{'}(G - E^*) = \begin{cases} 2[\alpha^{'}(G) - |E^{**}|], & \text{if } E^{**}e_i \in M \\ 2\alpha^{'}(G), & \text{otherwise.} \end{cases}$$

When we add E^* comeback, we get $\alpha^{'}(H_{ij}) = \alpha^{'}(H_{i(i+1)}) = 2|E^{**}|$. Hence $\alpha^{'}(H_{ij}) = \alpha^{'}(H_{i(i+1)}) = 2^{'}\alpha(G)$. □

Next, we establish theorem 2.3 for a maximum matching number of $F_{m,n} \otimes G$.

Lemma 2.3 Let G be connected graph order p, then

$$\alpha^{'}(F_{m,n} \otimes G) = \begin{cases} 2n\alpha^{'}(G), & m = n \text{ or } m < n \text{ and} \\ & M \text{ is perfectmatching} \\ 2n\alpha^{'}(G) + (m-n)|M^{**}|, & m > n \\ 2m\alpha^{'}(G) + |M^{**}|, & m < n \text{ and } n - m = 1 \\ (m+n)\alpha^{'}(G), & m < n \text{ and } n - m \text{ is even} \\ (m+n-1)\alpha^{'}(G) + |M^{**}|, & m < n \text{ and } n - m \text{ is odd,} \end{cases}$$

where $M^{**} = \{uv \in E(G) \mid u$ is not matched in maximum matching in G and $v \in N_G(u)\}$.

Proof. Let $V(\overline{K}_m) = \{u_i/i = 1, 2, ..., m\}, V(P_n) = \{v_j | j = 1, 2, ..., n\}$ and $V(G) = \{v_j/j = 1, 2, ..., p\}$. Since

$$\alpha^{'}(F_{m,n}) = \begin{cases} m, & m \geq n \\ m + \lfloor \frac{n}{2} \rfloor, & m < n. \end{cases}$$

Assume that the maximum matching set of $F_{m,n}$ and G be

$$M_1 = \begin{cases} \{u_iu_j \mid i = j = 1,2,3,...,n\}, & m \geq n \\ \{u_iu_j \mid i = j = 1,2,3,...,m\} \cup & \\ \{v_{m+1}v_{m+2}, v_{m+3}v_{m+4}, ..., v_{m+2\lfloor \frac{n-m}{2} \rfloor +1}v_{m+2\lfloor \frac{n-m}{2} \rfloor}\}, & m < n; \end{cases}$$

$M_2 = \{v_jv_{j+1} | j = 1,3,...,2k-1\}$,respectively.

<u>case1</u> $m = n$. By lemma 2.3 we have $\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G)$. Let $M_i^* = \{(u_i, w_l)(v_j, w_k) | i = j$ and $w_l, w_k \in M_2\}$. Therefore a matched in $F_{m,n} \otimes G$ is $\bigcup_{i=1}^{m} M_i^*$.

Hence $\alpha'(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| = 2n\alpha'(G)$.

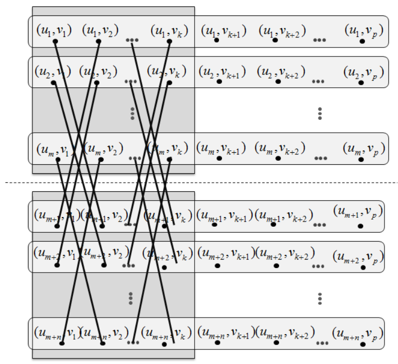

Figure 2: The case of $m = n$

Suppose that $\alpha'(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| = 2n\alpha'(G)$. Then there exist least two vertices $(u_i, w_j), (u_j, w_i) \in F_{m,n} \otimes G$ with edge $(u_i, w_j), (u_j, w_i) \notin M_i^*$. But is not possible.

Hence $\alpha'(F_{m,n} \otimes G) = 2n\alpha'(G)$ where $m = n$.

<u>case2</u> $m > n$. We have the matching $\bigcup_{i=1}^{m} M_i^*$. By definition of matching, we get another matching in G be set of edges $M^{**} = \{uv \in E(G) \mid u$ is not matched in maximum matching in G and $v \in N_G(u)\}$. So $\bigcup_{i=1}^{m} M_i^* \cup (m-n)M^{**}$.

Hence $\alpha'(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| + (m-n)|M^{**}| = 2n\alpha'(G) + (m-n)|M^{**}|$.

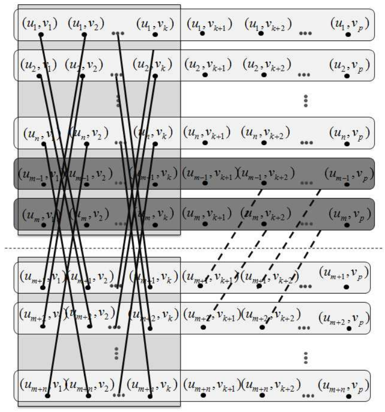

Figure 3: The case of $m > n$

Suppose that $\alpha'(F_{m,n} \otimes G) > |\bigcup_{i=1}^{m} M_i^*| + (m-n)|M^{**}| = 2n\alpha'(G) + (m-n)|M^{**}|$, then there exist a matching M is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in M.

Hence $\alpha'(F_{m,n} \otimes G) = 2n\alpha'(G) + (m-n)|M^{**}|$, where $m > n$.

case3 $m > n$ and $n - m = 1$. We have the matching $\bigcup_{i=1}^{m} M_i^*$.By definition of matching,we get another matching in G be set of edges $M^{**} = \{uv \in E(G) \mid u$ is not matched in maximum matching in G and $v \in N_G(u)\}$. So $\bigcup_{i=1}^{m} M_i^* \cup M^{**}$.

Hence $\alpha'(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| + |M^{**}| = 2m\alpha'(G) + |M^{**}|$.

Suppose that $\alpha'(F_{m,n} \otimes G) > |\bigcup_{i=1}^{m} M_i^*| + |M^{**}| = 2m\alpha'(G) + |M^{**}|$, then there exist a matching M is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in M.

Hence $\alpha'(F_{m,n} \otimes G) = 2m\alpha'(G) + |M^{**}|$, where $m > n$ and $n - m = 1$.

case4 $m > n$ and $n - m$ is even. We have the matching $\bigcup_{i=1}^{m} M_i^* \cup (n-m)\alpha'(G)$.

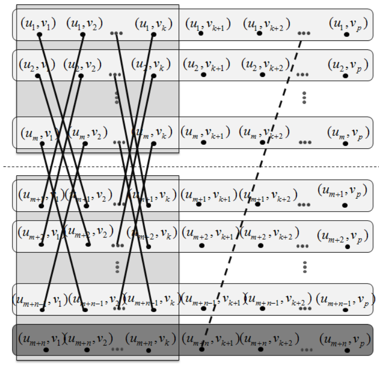

Figure 4: The case of $m > n$ and $n - m = 1$

Hence $\alpha^{'}(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| + (n-m)\alpha^{'}(G) = 2m\alpha^{'}(G) + (n-m)\alpha^{'}(G) = (m+n)\alpha^{'}(G)$

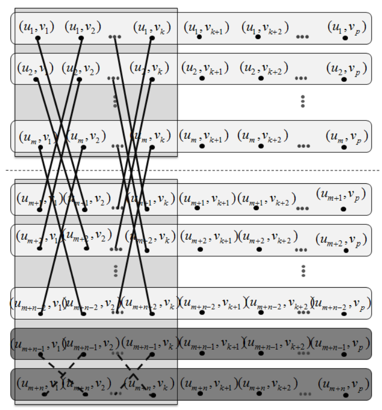

Figure 5: The case of $m > n$ and $n - m$ is even

Suppose that $\alpha^{'}(F_{m,n} \otimes G) > |\bigcup_{i=1}^{m} M_i^*| + (n-m)\alpha^{'}(G) = 2m\alpha^{'}(G) + (n-m)\alpha^{'}(G) = (m+n)\alpha^{'}(G)$, then there exist a matching M is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in M.

Hence $\alpha^{'}(F_{m,n} \otimes G) = 2m\alpha^{'}(G) + (n-m)\alpha^{'}(G) = (m+n)\alpha^{'}(G)$, where $m > n$ and $n - m$ is even.

<u>case5</u> $m > n$ and $n - m$ is odd. We have the matching $\bigcup_{i=1}^{m} M_i^* \cup (n-m-1)\alpha'(G) \cup M^{**}$, when $M^{**} = \{uv \in E(G) \mid u$ is not matched in maximum matching in G and $v \in N_G(u)\}$.

Hence $\alpha'(F_{m,n} \otimes G) \geq |\bigcup_{i=1}^{m} M_i^*| + (n-m-1)\alpha'(G) + |M^{**}| = 2m\alpha'(G) + (n-m-1)\alpha'(G) + |M^{**}| = (m+n-1)\alpha'(G) + |M^{**}|$

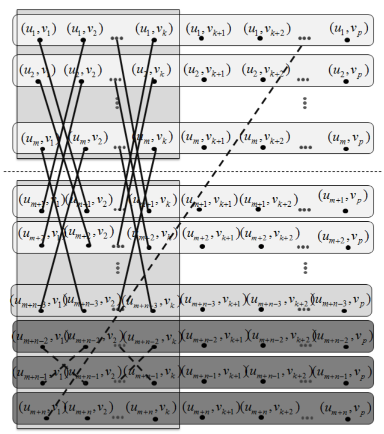

Figure 6: The case of $m > n$ and $n - m$ is odd

Suppose that $|\bigcup_{i=1}^{m} M_i^*| + (n-m-1)\alpha'(G) + |M^{**}| \geq 2m\alpha'(G) + (n-m-1)\alpha'(G) + |M^{**}| = (m+n-1)\alpha'(G) + |M^{**}|$,then there exist a matching M is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in M.

Hence $\alpha'(F_{m,n} \otimes G) = |\bigcup_{i=1}^{m} M_i^*| + (n-m-1)\alpha'(G) + |M^{**}| = (m+n-1)\alpha'(G) + |M^{**}|$, where $m > n$ and $n - m$ is odd. $\square$

3. Edge Covering Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving Lemma 3.1 that shows a relation of Matching number and Edge covering number.

Lemma 3.1 [2]. *Let G be a simple graph with order n. Then $\alpha(G)+\beta(G) = n$.*

Next, we establish theorem 3.2 for a minimum Edge covering number of $F_{m,n} \otimes G$.

Theorem 3.2 *Let G be connected graph order p, then*

$$\beta^{'}(F_{m,n} \otimes G) = \begin{cases} 2n\beta^{'}(G), & m = n \text{ or } m < n \text{ and} \\ & M \text{ is perfect matching} \\ 2n\beta^{'}(G) + (m-n)(p - |M^{**}|), & m > n \\ 2m\beta^{'}(G) + (n-m)p - |M^{**}|, & m < n \text{ and } n - m = 1 \\ (m+n)\beta^{'}(G), & m < n \text{ and } n - m \text{ is even} \\ (m+n-1)\beta^{'}(G) - |M^{**}|, & m < n \text{ and } n - m \text{ is odd,} \end{cases}$$

where $M^{**} = \{uv \in E(G) \mid u$ is not matched in maximum matching in G and $v \in N_G(u)\}$.

Proof. By theorem 2.3 and lemma 3.1, we can also show that $\alpha^{'}(F_{m,n} \otimes G) + \beta^{'}(F_{m,n} \otimes G) = (m+n)p$.

<u>case 1</u> $m = n$

$$\begin{aligned} \beta^{'}(F_{m,n} \otimes G) &= (m+n)p - 2n\alpha^{'}(G) \\ &= (n+n)p - 2n\alpha^{'}(G) \\ &= 2np - 2n\alpha^{'}(G) \\ &= 2n(p - \alpha^{'}(G)) \\ &= 2n\beta^{'}(G) \end{aligned}$$

Hence $\beta^{'}(F_{m,n} \otimes G) = 2n\beta^{'}(G)$, where $m = n$.

<u>case 2</u> $m > n$

$$\begin{aligned} \beta^{'}(F_{m,n} \otimes G) &= (m+n)p - 2n\alpha^{'}(G) - (m-n)|M^{**}| \\ &= ((m-n) + 2n)p - 2n\alpha^{'}(G) - (m-n)|M^{**}| \\ &= (m-n)p + 2np - 2n\alpha^{'}(G) - (m-n)|M^{**}| \\ &= 2n(p - \alpha^{'}(G)) + (m-n)(p - |M^{**}|) \\ &= 2n(p - \beta^{'}(G)) + (m-n)(p - |M^{**}|) \end{aligned}$$

Hence $\beta^{'}(F_{m,n} \otimes G) = 2n(p - \beta^{'}(G)) + (m-n)(p - |M^{**}|)$, where $m > n$.

case 3 $m > n$ and $n - m = 1$

$$\begin{aligned}\beta'(F_{m,n} \otimes G) &= (m+n)p - 2m\alpha'(G) - |M^{**}| \\ &= ((-m+n)+2m)p - 2m\alpha'(G) - |M^{**}| \\ &= (n-m)p + 2mp - 2m\alpha'(G) - |M^{**}| \\ &= 2m(p - \alpha'(G)) + (n-m)p - |M^{**}| \\ &= 2m\beta'(G)) + (n-m)p - |M^{**}|\end{aligned}$$

Hence $\beta'(F_{m,n} \otimes G) = 2m\beta'(G)) + (n-m)p - |M^{**}|$, where $m > n$ and $n - m = 1$.

case 4 $m > n$ and $n - m$ is even

$$\begin{aligned}\beta'(F_{m,n} \otimes G) &= (m+n)p - (m+n)\alpha'(G) \\ &= (m+n)(p - \alpha'(G)) \\ &= (m+n)\beta'(G)\end{aligned}$$

Hence $\beta'(F_{m,n} \otimes G) = (m+n)\beta'(G)$, where $m > n$ and $n - m$ is even.

case 5 $m > n$ and $n - m$ is odd

$$\begin{aligned}\beta'(F_{m,n} \otimes G) &= (m+n)p - (m+n-1)\alpha'(G) - |M^{**}| \\ &= (m+n)p - (m+n)\alpha'(G) + \alpha'(G) - |M^{**}| \\ &= (m+n)(p - \alpha'(G)) + p - \beta'(G) - |M^{**}| \\ &= (m+n)\beta'(G) - \beta'(G) + p - |M^{**}| \\ &= (m+n-1)\beta'(G) + p - |M^{**}|\end{aligned}$$

Hence $\beta'(F_{m,n} \otimes G) = (m+n-1)\beta'(G) + p - |M^{**}|$, where $m > n$ and $n - m$ is odd. □

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