MATCHING AND EDGE COVERING NUMBER
ON TENSOR PRODUCT OF FAN GRAPH

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Abstract: Let $\alpha'(G)$ and $\beta'(G)$ be the matching number and edge covering number of $G$, respectively. The tensor product $G_1 \otimes G_2$ of graph of $G_1$ and $G_2$ has vertex set $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. In this paper, we determined generalization of matching number and edge covering on tensor product of fan graph and any simple graph.

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1. Introduction

In this paper, graphs must be simple graphs which can be trivial graph. Let $G_1$ and $G_2$ be graphs. The tensor product of graph $G_1$ and $G_2$, denote by $G_1 \otimes G_2$, is the graph with $V(G_1 \otimes G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \otimes G_2) = \{(u_1v_1)(u_2v_2) \mid u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$.

Next, we give the definitions about some graph parameters.

A subset $M$ of the edge set $E$ of $G$ is said to be a matching or an independent edge set of $G$ if no two distinct edges in $M$ have a common vertex. A matching...
$M$ is called a maximum matching in $G$ if there is no matching $M'$ of $G$ with $|M'| > |M|$. The cardinality of a maximum matching of $G$ is call the matching number of denoted by $\alpha'(G)$.

An edge of graph $G$ is said to cover the two vertices incident with it, and an edge cover of a graph $G$ is a set of edges covering all the vertices of $G$. The minimum cardinality of an edge cover of a graph $G$ is called the edge covering number of $G$ denote by $\beta'(G)$.

By definitions of matching number and edge covering number, clearly that

$$\alpha'(F_{m,n}) = \begin{cases} n, & m \geq n \\ m + \lfloor \frac{n-m}{2} \rfloor, & m < n \end{cases}$$

and

$$\beta'(F_{m,n}) = \begin{cases} m, & m \geq n \\ m + \lceil \frac{n-m}{2} \rceil, & m < n \end{cases}$$

**Proposition 1.1** Let $H = G_1 \otimes G_2 = (V(H), E(H))$ then:

(i) $|V(H)| = |V(G_1)||V(G_2)|$

(ii) $|E(H)| = 2|E(G_1)||E(G_2)|$

(iii) for every $(u, v) \in V(H), d_H((u, v)) = d_{G_1}(u)d_{G_2}(v)$.

**Theorem 1.2** Let $G_1$ and $G_2$ be connected graphs, The graph $H = G_1 \otimes G_2$ is connected if and only if $G_1$ or $G_2$ contains an odd cycle.

**Theorem 1.3** Let $G_1$ and $G_2$ be connected graphs with no odd cycle then $G_1 \otimes G_2$ has exactly two connected components.

Next we get that general from of graph of tensor product of $F_{m,n}$ and a simple graph.

**Proposition 1.4** Let $G$ be a connected graph of order $p$, the graph of $F_{m,n} \otimes G$ is

$$\bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i ; \quad H_i = \bigcup_{j=m+1}^{m+n} H_{ij} \quad \text{and} \quad M_i = H_{i(i+1)},$$

where

$$V(H_{ij}) = S_i \cup S_j, \quad S_i = \{(i, 1), (i, 2), ..., (i, p)\},$$

$$S_j = \{(j, 1), (j, 2), ..., (j, p)\}; \quad i < j \quad \text{and} \quad E(H_{ij}) = \{(i, u)(j, v) / uv \in E(G)\}.$$  

Moreover, if $G$ has no odd cycle then each $H_{ij}$ has exactly two connected components isomorphic to $G$.

Example
2. Matching Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving the definition and theorem for alternating path and augmenting path, Lemma 2.3. that show character of matching for each $H_{ij}$.

**Definition 2.1.** Let $G$ be a matching $M$, an $M$-alternating path is a path that alternates between edges in $M$ and edges not in $M$. An $M$-alternating odd path whose endpoints are unsaturated by $M$ is an $M$-augmenting path.

**Theorem 2.2.** A matching $M$ in a graph $G$ is a maximum matching in $G$ if and only if $G$ has no $M$-augmenting path.

**Lemma 2.3.** Let $F_{m,n} \otimes G = \bigcup_{i=1}^{m} H_i \cup \bigcup_{i=m+1}^{m+n-1} M_i; H_i = \bigcup_{j=m+1}^{m+n} H_{ij}$ and $M_i = H_{i(i+1)}$, then $\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G)$.

**Proof.** Suppose $G$ has no odd cycle, by proposition 1.4, we have that $H_{ij}$ and $H_{i(i+1)}$ each contains 2 components isomorphic to $G$. Therefore $H_{ij} = H_{i(i+1)} = 2G$. So $\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G)$. 

Figure 1: The graph of $F_{m,n} \otimes G$
If \( G \) has odd cycle, we have \( d_{H_{ij}}(u_i, v) = d_{H_{ij}}(u_j, v) = d_G(v) \), for \((u_i, v) \in S_i \) and \((u_j, v) \in S_j \), \( i < j \).

Let \( E^* = \{ e_i | e_i \text{ is any one edge in each odd cycle } C_i \text{ in } G; i = 1, 2, ..., l \} \), \( |E^*| \leq l \) and let \( M \) be a maximum matching set of \( G \). Now consider the tensor product

\[
\bigcup_{i=1}^{m} H_{ij}^{*} \cup \bigcup_{i=m+1}^{m+n-1} H_{i(i+1)}^{*} = F_{m,n} \otimes (G - E^*); i < j, i = 1, 2, ..., m \text{ where } \]

\( V(H_{ij}^*) = V(H_{ij}), E(H_{ij}^*) = (i, u)(j, v) \mid u v \in (G - E^*) \). We get \( H_{ij} = H_{i(i+1)} = 2(G - E^*) \) then

\[
\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha'(G - E^*) = \begin{cases} 
2[\alpha'(G) - |E^*|], & \text{if } E^* e_i \in M \\
2\alpha'(G), & \text{otherwise.}
\end{cases}
\]

When we add \( E^* \) come back, we get \( \alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2|E^*| \). Hence

\[
\alpha'(H_{ij}) = \alpha'(H_{i(i+1)}) = 2\alpha(G).
\]

Next, we establish theorem 2.3 for a maximum matching number of \( F_{m,n} \otimes G \).

**Lemma 2.3** Let \( G \) be connected graph order \( p \), then

\[
\alpha'(F_{m,n} \otimes G) = \begin{cases} 
2\alpha'(G), & m = n \text{ or } m < n \text{ and } M \text{ is perfect matching} \\
2\alpha'(G) + (m - n)|M^*|, & m > n \\
2m\alpha'(G) + |M^*|, & m < n \text{ and } n - m = 1 \\
(m + n)\alpha'(G), & m < n \text{ and } n - m \text{ is even} \\
(m + n - 1)\alpha'(G) + |M^*|, & m < n \text{ and } n - m \text{ is odd},
\end{cases}
\]

where \( M^* = \{ uv \in E(G) \mid u \text{ is not matched in maximum matching in } G \text{ and } v \in N_G(u) \} \).

**Proof.** Let \( V(K_m) = \{ u_i/i = 1, 2, ..., m \}, V(P_n) = \{ v_j/j = 1, 2, ..., n \} \) and \( V(G) = \{ v_j/j = 1, 2, ..., p \} \). Since

\[
\alpha'(F_{m,n}) = \begin{cases} 
m, & m \geq n \\
m + \lfloor \frac{m}{2} \rfloor, & m < n.
\end{cases}
\]
Assume that the maximum matching set of $F_{m,n}$ and $G$ be

$$M_1 = \begin{cases} \{u_iu_j \mid i = j = 1, 2, 3, ..., n\}, & m \geq n \\ \{u_iu_j \mid i = j = 1, 2, 3, ..., m\} \cup \{v_{m+1}v_{m+2}, v_{m+3}v_{m+4}, ..., v_{m+2\left\lfloor \frac{n-m}{2} \right\rfloor+1}v_{m+2\left\lfloor \frac{n-m}{2} \right\rfloor} \}, & m < n; \end{cases}$$

$M_2 = \{v_jv_{j+1} \mid j = 1, 3, ..., 2k - 1\}$, respectively.

**case1** $m = n$. By lemma 2.3 we have $\alpha'(H_{ij}) = \alpha'(H_{(i+1)j}) = 2\alpha'(G)$.

Let $M_i^* = \{(u_i, w_l)(v_j, w_k) \mid i = j \text{ and } w_l, w_k \in M_2\}$. Therefore a matched in $F_{m,n} \otimes G$ is $\bigcup_{i=1}^{m} M_i^*$.

Hence $\alpha'(F_{m,n} \otimes G) \geq \left| \bigcup_{i=1}^{m} M_i^* \right| = 2n\alpha'(G)$.

Suppose that $\alpha'(F_{m,n} \otimes G) \geq \left| \bigcup_{i=1}^{m} M_i^* \right| = 2n\alpha'(G)$. Then there exist at least two vertices $(u_i, w_j), (u_j, w_i) \in F_{m,n} \otimes G$ with edge $(u_i, w_j), (u_j, w_i) \not\in M_i^*$. But this is not possible.

Hence $\alpha'(F_{m,n} \otimes G) = 2n\alpha'(G)$ where $m = n$.

**case2** $m > n$. We have the matching $\bigcup_{i=1}^{m} M_i^*$. By definition of matching, we get another matching in $G$ be set of edges $M^{**} = \{uv \in E(G) \mid u \text{ is not matched in maximum matching in } G \text{ and } v \in N_G(u)\}$. So $\bigcup_{i=1}^{m} M_i^* \cup (m-n)M^{**}$. 

Figure 2: The case of $m = n$
Hence $\alpha'(F_{m,n} \otimes G) \geq \sum_{i=1}^{m} M_i^* + (m-n)|M^{**}| = 2n\alpha'(G) + (m-n)|M^{**}|$.

Figure 3: The case of $m > n$

Suppose that $\alpha'(F_{m,n} \otimes G) > \sum_{i=1}^{m} M_i^* + (m-n)|M^{**}|$, then there exist a matching $M$ is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in $M$.

Hence $\alpha'(F_{m,n} \otimes G) = 2n\alpha'(G) + (m-n)|M^{**}|$, where $m > n$.

case3 $m > n$ and $n - m = 1$. We have the matching $\bigcup_{i=1}^{m} M_i^*$. By definition of matching, we get another matching in $G$ be set of edges $M^{**} = \{uv \in E(G) | u \text{ is not matched in maximum matching in } G \text{ and } v \in N_G(u)\}$. So $\bigcup_{i=1}^{m} M_i^* \cup M^{**}$.

Hence $\alpha'(F_{m,n} \otimes G) \geq \sum_{i=1}^{m} M_i^* + |M^{**}| = 2m\alpha'(G) + |M^{**}|$.

Suppose that $\alpha'(F_{m,n} \otimes G) > \sum_{i=1}^{m} M_i^* + |M^{**}| = 2m\alpha'(G) + |M^{**}|$, then there exist a matching $M$ is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in $M$.

Hence $\alpha'(F_{m,n} \otimes G) = 2m\alpha'(G) + |M^{**}|$, where $m > n$ and $n - m = 1$.

case4 $m > n$ and $n - m$ is even. We have the matching $\bigcup_{i=1}^{m} M_i^* \cup (n-m)\alpha'(G)$. 
Figure 4: The case of $m > n$ and $n - m = 1$

\[
\text{Hence } \alpha' (F_{m,n} \otimes G) \geq \left| \bigcup_{i=1}^{m} M_i^* \right| + (n - m)\alpha' (G) = 2m\alpha' (G) + (n - m)\alpha' (G) = (m + n)\alpha' (G)
\]

Figure 5: The case of $m > n$ and $n - m$ is even

Suppose that $\alpha' (F_{m,n} \otimes G) > \left| \bigcup_{i=1}^{m} M_i^* \right| + (n - m)\alpha' (G) = 2m\alpha' (G) + (n - m)\alpha' (G) = (m + n)\alpha' (G)$, then there exist a matching $M$ is augmenting path. That is not true because each edges in $F_{m,n} \otimes G$ incident with edge in $M$.

Hence $\alpha' (F_{m,n} \otimes G) = 2m\alpha' (G) + (n - m)\alpha' (G) = (m + n)\alpha' (G)$, where $m > n$ and $n - m$ is even.
case 5 $m > n$ and $n - m$ is odd. We have the matching $\bigcup_{i=1}^{m} M_i^* \cup (n - m - 1)\alpha'(G) \cup M^{**}$, when $M^{**} = \{uv \in E(G) \mid u \text{ is not matched in maximum matching in } G \text{ and } v \in N_G(u)\}$.

Hence $\alpha'(F_{m,n} \otimes G) \geq \bigcup_{i=1}^{m} M_i^* | + (n - m - 1)\alpha'(G) | |M^{**}| = 2m\alpha'(G) + (n - m - 1)\alpha'(G) + |M^{**}| = (m + n - 1)\alpha'(G) + |M^{**}|$

Hence $\alpha'(F_{m,n} \otimes G) = \bigcup_{i=1}^{m} M_i^* | + (n - m - 1)\alpha'(G) | |M^{**}| = (m + n - 1)\alpha'(G) + |M^{**}|$, where $m > n$ and $n - m$ is odd.

3. Edge Covering Number of the Graph of $F_{m,n} \otimes G$

We begin this section by giving Lemma 3.1 that shows a relation of Matching number and Edge covering number.
Lemma 3.1 [2]. Let $G$ be a simple graph with order $n$. Then $\alpha(G) + \beta(G) = n$.

Next, we establish theorem 3.2 for a minimum Edge covering number of $F_{m,n} \otimes G$.

**Theorem 3.2** Let $G$ be connected graph order $p$, then

$$
\beta'(F_{m,n} \otimes G) = \begin{cases} 
2n\beta'(G), & m = n \text{ or } m < n \text{ and } M \text{ is perfect matching} \\
2n\beta'(G) + (m - n)(p - |M^*|), & m > n \\
2m\beta'(G) + (n - m)p - |M^*|, & m < n \text{ and } n - m = 1 \\
(m + n)\beta'(G), & m < n \text{ and } n - m \text{ is even} \\
(m + n - 1)\beta'(G) - |M^*|, & m < n \text{ and } n - m \text{ is odd},
\end{cases}
$$

where $M^* = \{uv \in E(G) \mid u \text{ is not matched in maximum matching in } G \text{ and } v \in N_G(u)\}$.

**Proof.** By theorem 2.3 and lemma 3.1, we can also show that $\alpha'(F_{m,n} \otimes G) + \beta'(F_{m,n} \otimes G) = (m + n)p$.

case 1 $m = n$

$$
\beta'(F_{m,n} \otimes G) = (m + n)p - 2n\alpha'(G) \\
= (n + n)p - 2n\alpha'(G) \\
= 2np - 2n\alpha'(G) \\
= 2n(p - \alpha'(G)) \\
= 2n\beta'(G)
$$

Hence $\beta'(F_{m,n} \otimes G) = 2n\beta'(G)$, where $m = n$.

case 2 $m > n$

$$
\beta'(F_{m,n} \otimes G) = (m + n)p - 2n\alpha'(G) - (m - n)|M^*| \\
= (m - n + 2n)p - 2n\alpha'(G) - (m - n)|M^*| \\
= (m - n)p + 2np - 2n\alpha'(G) - (m - n)|M^*| \\
= 2n(p - \alpha'(G)) + (m - n)(p - |M^*|) \\
= 2n(p - \beta'(G)) + (m - n)(p - |M^*|)
$$

Hence $\beta'(F_{m,n} \otimes G) = 2n(p - \beta'(G)) + (m - n)(p - |M^*|)$, where $m > n$. 
case 3 $m > n$ and $n - m = 1$

$$\beta'(F_{m,n} \otimes G) = (m + n)p - 2m\alpha'(G) - |M^{**}|$$

$$= ((-m + n) + 2m)p - 2m\alpha'(G) - |M^{**}|$$

$$= (n - m)p + 2mp - 2m\alpha'(G) - |M^{**}|$$

$$= 2m(p - \alpha'(G)) + (n - m)p - |M^{**}|$$

$$= 2m\beta'(G) + (n - m)p - |M^{**}|$$

Hence $\beta'(F_{m,n} \otimes G) = 2m\beta'(G) + (n - m)p - |M^{**}|$, where $m > n$ and $n - m = 1$.

case 4 $m > n$ and $n - m$ is even

$$\beta'(F_{m,n} \otimes G) = (m + n)p - (m + n)\alpha'(G)$$

$$= (m + n)(p - \alpha'(G))$$

$$= (m + n)\beta'(G)$$

Hence $\beta'(F_{m,n} \otimes G) = (m + n)\beta'(G)$, where $m > n$ and $n - m$ is even.

case 5 $m > n$ and $n - m$ is odd

$$\beta'(F_{m,n} \otimes G) = (m + n)p - (m + n - 1)\alpha'(G) - |M^{**}|$$

$$= (m + n)p - (m + n)\alpha'(G) + \alpha'(G) - |M^{**}|$$

$$= (m + n)(p - \alpha'(G)) + p - \beta'(G) - |M^{**}|$$

$$= (m + n)\beta'(G) - \beta'(G) + p - |M^{**}|$$

$$= (m + n - 1)\beta'(G) + p - |M^{**}|$$

Hence $\beta'(F_{m,n} \otimes G) = (m + n - 1)\beta'(G) + p - |M^{**}|$, where $m > n$ and $n - m$ is odd.

\[\square\]

References

