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COMMON FIXED POINT FOR THREE NONSELF-MAPS THROUGH AN IMPLICIT RELATION

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Abstract: A common fixed point is obtained for three weakly compatible nonself-maps defined on a nonempty subset of a metric space through an implicit relation and the notion of property E.A. The result of the paper is a generalization of those of Sing and Kumar, and of Akkouchi and Popa, and is an extended generalization of Khan and Dolmo's result for a pair of maps.

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1. Introduction

Let (X, d) be a metric space. Then Sx denotes the image of $x \in X$ under a self-map S on X and SA, the composition of self-maps S and A on X. Self-maps

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S and A on X are compatible [2] if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1.1}$$

whenever $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ax_n = p \quad \text{for some} \quad p \in X. \tag{1.2}$$

Therefore self-maps S and A on X are noncompatible if (1.2) holds good but $\lim_{n\to\infty} d(Sx_n, Ax_n)$ is nonzero or $+\infty$ for at least one $\langle x_n \rangle_{n=1}^{\infty} \subset X$. It can be easily seen that both compatible and noncompatible maps are included in the class of maps with the choice (1.2). Self-maps S and A on X are said to satify the property E.A. [1] if (1.2) holds good for some $\langle x_n \rangle_{n=1}^{\infty} \in X$. On the other hand, if $x_n = x$ for all n, the compatibility implies that $SAx = ASx$ whenever $Ax = Sx$. That is, S and A commute at their coincidence points. Self-maps which commute at their coincidence points are called weakly compatible [3]. However weak compatibility and property E. A. are independent notions of each other [6].

We aim at a common fixed point for three nonself-maps defined on a nonempty subset of a metric space through an implicit relation and the notions of weak compatibility and property E.A. Our contribution is a generalization of those of Singh and Kumar [7], and of Akkouchi and Popa [5]. This is also extended generalization of that of Khan and Dolmo [4], which was given for a pair of maps.

2. Notation and Main Results

In this paper, X denotes a metric space, Y an arbitrary non empty subset of X and $G : \mathbb{R}_+^6 \to \mathbb{R}$, a continuous function such that

(G_1) $G(u, 0, 0, u, u, 0) > 0$ for all $u > 0$,

(G_2) $G(u, u, 0, 0, u, u) \geq 0$ for all $u > 0$,

(G_3) $G(u, 0, u, 0, 0, u) > 0$ for all $u > 0$.

We prove

Theorem 2.1. *Let S, T and $A : Y \to X$ satisfying the implicit contractive-type condition*

$$G(d(Sx,Ty), d(Ax,Ay), d(Ax,Sx), d(Ay,Ty), d(Ax,Ty), d(Ay,Sx)) < 0 \quad for\ all \quad x,y \in Y, \tag{2.1}$$

Suppose that one of the pairs (S, A) and (T, A) satisfies property E.A. on Y, and any one of the following conditions holds good:

1. *$A(Y)$ is closed subspace of Y,*
2. *$\overline{S(Y)} \subset A(Y)$,*
3. *$\overline{T(Y)} \subset A(Y)$.*

Then there is a coincidence point u common to S, T and A in Y. Further if the point of common coincidence of S, T and A with respect to u lies in Y, it will be their unique common fixed point, provided either (S, A) or (T, A) is a weakly compatible pair.

Proof. First suppose that (S, A) satisfies the property E.A. on Y. Then there exists a $\langle p_n \rangle_{n=1}^{\infty} \in Y$ with the choice (1.2). We claim that $q = \lim_{n\to\infty} Tp_n = p$. If possible suppose that $d(p, q) > 0$. Then writing $x = y = p_n$ in (2.1), we find that

$$G(d(Sp_n,Tp_n), d(Ap_n,Ap_n), d(Ap_n,Sp_n), d(Ap_n,Tp_n), d(Ap_n,Tp_n), d(Ap_n,Sp_n)) < 0.$$

Applying the limit as $n \to \infty$ in this and using (1.2), we get

$$G(d(p,q), 0, 0, d(p,q), d(p,q), 0) \leq 0,$$

which would be a contradiction to (G_1). Thus $p = q$ and hence

$$\lim_{n\to\infty} Sp_n = \lim_{n\to\infty} Tp_n = \lim_{n\to\infty} Ap_n = p \quad \text{for some} \quad p \in Y. \tag{2.2}$$

Now let (T, A) satisfy the property E.A. on Y so that

$$\lim_{n\to\infty} Tp_n = \lim_{n\to\infty} Ap_n = p \quad \text{for some} \quad p \in X$$

for some $\langle p_n \rangle_{n=1}^{\infty} \in Y$. If $s = \lim_{n\to\infty} Sp_n \neq p$, using again (2.1) and proceeding as above, we get contradiction to the choice (G_1). This proves (2.2).

Case (a)**:** Suppose that $A(Y)$ is closed subspace of Y. Then $\langle Ap_n\rangle_{n=1}^{\infty}$ converges in $A(Y)$. Hence $p \in A(Y)$ so that $p = Au$ for some $u \in Y$.

First we claim that $Su = Au$. If $d(p, Su) = d(Au, Su) > 0$, from (2.1) with $x = u$ and $y = p_n$, we would get

$$G(d(Su, Tp_n), d(Au, Ap_n), d(Au, Su), d(Ap_n, Tp_n), d(Au, Tp_n), d(Ap_n, Su)) < 0.$$

Applying the limit as $n \to \infty$ and using (2.2), this gives

$$G(d(Su, Au), 0, d(Au, Su), 0, 0, d(Au, Su)) \leq 0,$$

which is a contradiction to (G_3). Thus $d(Au, Su) = 0$ or $Au = Su = p$.

Again if $d(p, Tu) > 0$, from (2.1) with $x = u = y$ it follows that

$$G(d(Su, Tu), d(Au, Au), d(Au, Su), d(Au, Tu), d(Au, Su), d(Au, Tu)) < 0$$

or $G(d(Su, Tu), 0, 0, d(Au, Tu), 0, d(Au, Tu)) < 0$ contradicting (G_1). Thus $d(p, Tu) = 0$, that is $Su = Tu = Au = p$.

Case (b)**:** Now suppose that (2) holds good. Then $\langle Sp_n\rangle_{n=1}^{\infty} \subset S(Y)$ and hence lies in $\overline{S(Y)}$ so that $p \in \overline{S(Y)} \subset A(Y)$ and u is a common coincidence by Case (a).

Case (c)**:** Now suppose that (3) holds good. Then as above we get that $\langle Tp_n\rangle_{n=1}^{\infty}$ converges in $\overline{T(Y)}$ and hence $p \in \overline{S(Y)} \subset A(Y)$. It follows that u is a common coincidence from again Case (a).

Suppose that $p \in Y$. If (S, A) is weakly compatible, we find that

$$SSu = SAu = ASu = AAu \text{ or } Ap = Sp.$$

If $Sp \neq Tp$ so that $d(Sp, Tp) > 0$, then from (2.1), we get

$$G(d(Sp, Tp), d(Ap, Ap), d(Ap, Sp), d(Ap, Tp), d(Ap, Tp), d(Ap, Sp)) < 0$$

or $G(d(Sp, Tp), 0, 0, d(Sp, Tp), d(Sp, Tp), 0) < 0$, which is a contradiction to (G_1). Thus $d(Tp, Sp) = 0$ or $Sp = Ap = Tp$.

On the other hand, if (T, A) is weakly compatible, we find that

$$TTu = TAu = ATu = AAu \text{ or } Ap = Tp.$$

Then (2.1) would give

$$G(d(Sp,Tp),d(Ap,Ap),d(Ap,Sp),d(Ap,Tp),d(Ap,Tp),d(Ap,Sp)) < 0$$

or $G(d(Sp,Tp),0,d(Tp,Sp),0,0,d(Tp,Sp)) < 0$, a contradiction to (G_3) whenever $d(Sp,Tp) > 0$. Thus $d(Tp,Sp) = 0$ or $Sp = Ap = Tp$.

Thus p is also a common coincidence point of S,T and A in Y whenever either (S,A) or (T,A) is weakly compatible.

Now if $p \neq Sp$ so that $d(p,Sp) = d(Su,Tp) > 0$ and from (2.1), we get

$$G(d(Su,Tp),d(Au,Ap),d(Au,Su),d(Ap,Tp),d(Au,Tp),d(Ap,Su)) < 0$$

or $G(d(p,Sp),d(p,Sp),0,0,d(p,Sp),d(Sp,p)) < 0$ which is a contradiction to (G_2). Thus $d(p,Sp) = 0$ and hence p is a common fixed point of S,T and A in Y.

The uniqueness of the common fixed point can be easily obtained from (2.1) and the choice of G. □

We write $G(t_1,t_2,t_3,t_4,t_5,t_6) = t_1 - \max\left\{t_2, \alpha t_3, \alpha t_4, \frac{t_5+t_6}{2}\right\}$ in Theorem 2.1, where $0 < \alpha < 1$. Then

(G_1) $G(u,0,0,u,u,0) = u - \max\left\{0, 0.\alpha, \alpha.u, \frac{u+0}{2}\right\} = (1-k)u > 0$ for all $u > 0$ where $k = \max\{\alpha, 1/2\}$,

(G_2) $G(u,u,0,0,u,u) = u - \max\left\{u, \alpha.0, \alpha.0, \frac{u+u}{2}\right\} = u - u > 0$ for all $u > 0$,

(G_3) $G(u,0,u,0,0,u) = u - \max\left\{0, \alpha.u, \alpha.0, \frac{0+u}{2}\right\} = (1-\alpha)u > 0$ for all $u > 0$.

Therefore we have

Corollary 2.1 (Theorem 2, [7]). *Let S, T and $A : Y \to X$ satisfy the inequality*

$$d(Sx,Ty) < \max\left\{d(Ax,Ay), \alpha d(Ax,Sx), \alpha d(Ay,Ty), \frac{d(Ax,Ty)+d(Sx,Ay)}{2}\right\} \quad for\ all \quad x,y \in X, \tag{2.3}$$

where $0 \leq \alpha < 1$. Suppose that both (2) and (3) of Theorem 2.1 hold good and one of the pairs (S,A) and (T,A) satisfies the property E.A. on Y. Then there is a coincidence point common to S, T and A in Y. Further if $Y = X$ and (S,A) and (T,A) are weakly compatible, then S, T and A will have a unique common fixed point in X.

Remark 2.1. Corollary 2.1 required both the inclusions (2) and (3); weak compatibility of both the pairs (S, A) and (T, A). Also a common fixed point was obtained under the condition that $Y = X$. Our proof suggests that weak compatibility and property E.A. of either pair is sufficient to obtain a common fixed point, even without the condition that $Y = X$.

In extending the property E.A. to more than two self-maps, Akkouchi and Popa [5] defined a class C of self-maps which satisfy property E.A. if there is a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $\lim\limits_{n\to\infty} Sx_n = p$ for each $S \in C$ for some $p \in X$.

Then the following was proved:

Theorem 2.2 (Theorem 2, [5])**.** *Let S, T and A be self maps on X satisfying the inclusions*

$$S(X) \subset A(X) \quad \textit{and} \quad T(X) \subset A(X) \tag{2.4}$$

and the implicit condition (2.1) with $x \neq y$. Suppose that the triplet (S,T,A) satisfies property E.A. and any one of $A(X)$, $S(X)$ and $T(X)$ is a closed subspace of X. If (S, A) and (T, A) are weakly compatible, then S, T and A will have a unique common fixed point.

We claim that Theorem 2.2 is a special case of Theorem 2.1 with the help of the following

Lemma 2.1. *The following three conditions are equivalent under* (2.1)
(a) (S, A) *satisfies property E.A.*
(b) (T, A) *satisfies property E.A.*
(c) *the triad* (S, T, A) *satisfies property E.A.*

Proof. We establish the equivalence through arguments:
$(c) \Rightarrow (a)$, $(c) \Rightarrow (b)$, $(a) \Rightarrow (b) \Rightarrow (c)$ and $(b) \Rightarrow (a) \Rightarrow (c)$.
Note that $(c) \Rightarrow (a)$ and $(c) \Rightarrow (b)$ are obvious.
$(a) \Rightarrow (b) \Rightarrow (c)$: Suppose that (a) holds good. Then there is a $\langle x_n \rangle_{n=1}^{\infty} \in X$ with the choice (1.2). We claim that $q = \lim\limits_{n\to\infty} Tx_n = p$. If possible suppose that $d(p, q) > 0$. Then writing $x = y = x_n$ in (2.1), we see that

$$G(d(Sx_n, Tx_n), d(Ax_n, Ax_n), d(Ax_n, Sx_n), d(Ax_n, Tx_n), d(Ax_n, Tx_n), d(Ax_n, Sx_n)) < 0.$$

Applying the limit as $n \to \infty$ in this and using (1.2), we get a contradiction to (G_1) that $G(d(p, q), 0, 0, d(p, q), d(p, q), 0) \leq 0$. Thus $p = q$ and (b) and hence (c) holds good.

$(b) \Rightarrow (a) \Rightarrow (c)$: If (b) holds good, then

$$\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ay_n = r \quad \text{for some} \quad r \in X \tag{2.5}$$

for some $\langle y_n \rangle_{n=1}^{\infty} \in X$. Let $d(r,s) > 0$ where $s = \lim_{n\to\infty} Sy_n$. Then (2.1) with $x = y = y_n$ gives

$$G(d(Sy_n, Ty_n), d(Ay_n, Ay_n), d(Ay_n, Sy_n), d(Ay_n, Ty_n), d(Ay_n, Ty_n), d(Ay_n, Sy_n)) < 0.$$

Applying the limit as $n \to \infty$ in this and using (2.5), we get

$$F(d(s,r), 0, d(r,s), 0, 0, d(r,s)) \leq 0,$$

which is also a contradiction to (G_2). Thus $s = r$ so that (a) and hence (c) holds good. □

The following example shows that (a) and (b) need not imply (c) without (2.1):

Example 2.1. Let $X = \mathbb{R}_+$ with the usual metric d. Define $S, A : X \to X$ by $Sx = ax + b$, $Tx = (b-a)x + a$ and $Ax = cx + b$ for all $x \in X$, where a, b and c are nonnegative numbers with $a \neq c$ and $b - a - c > 0$. Choose $x_n = \frac{1}{n}$ and $y_n = \frac{b-a}{b-a-c} + \frac{1}{n}$ for all n so that

$$\lim_{n\to\infty} Sx_n = b = \lim_{n\to\infty} Ax_n \text{ and } \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Ay_n = \tfrac{b^2-ab-ac}{b-a-c}.$$

That is, (S, A) and (T, A) satisfy the property E.A. By a routine computation, it can be easily seen that (S, T, A) does not satisfy the property E.A., though the individual pairs satisfy the property E. A.

Now take $a = \frac{1}{2}, b = 1, c = \frac{1}{2}$ and

$$G(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - q \max\left\{t_2, t_3, t_4, \tfrac{t_5+t_6}{2}\right\},$$

where $0 \leq q < 1$. Then (2.1) becomes

$$d(Sx, Ty) \leq q \max\left\{d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), \tfrac{d(Ty,Ax)+d(Sx,Ay)}{2}\right\}$$
$$\textit{for all} \quad x, y \in X \quad \textit{with} \quad x \neq y \tag{2.6}$$

we see that for $x = 2$ and $y = 0$, we see that its LHS is $\frac{3}{2}$ while RHS is $\frac{13}{12}$ so that (2.6) holds good only if $q \geq \frac{18}{13}$ which is against the choice of q. Thus (2.6) fails.

Remark 2.2. In view of Lemma 2.1, Theorem 2.2 follows as a particular case of our result when $Y = X$. Here we note that Theorem 2.2 required weak compatibility of both the pairs (S, A) and (T, A). If $S(X)$ is closed in X, then hence $\overline{S(X)} = S(X) \subset A(X)$, due to (2.4). Similarly if $T(X)$ is closed so that $\overline{T(X)} = T(X) \subset A(X)$, again from (2.4). It is thus interesting to note that the completeness of $A(X)$ is weakened and the inclusions (2.4) and the completeness of $S(X)$ and $T(X)$ are chipped in (2) and (3).

No we set $F(t_1, t_2, t_3, t_4, t_5, t_6) = t_1 - \max\left\{t_2, \beta t_3 + \alpha t_4, \frac{t_5+t_6}{2}\right\}$, where $\beta \geq 0$ and $0 < \alpha < 1$. Then

(F_1) $F(u, 0, 0, u, u, 0) = u - \max\left\{0, 0.\beta + \alpha.u, \frac{u+0}{2}\right\} = (1-k)u > 0$ for all $u > 0$ where $k = \max\{\alpha, 1/2\} < 1$,

(F_2) $F(u, u, 0, 0, u, u) = u - \max\left\{u, 0.\beta + \alpha.0 \frac{u+u}{2}\right\} = u - u > 0$ for all $u > 0$.

With this choice and $T = S$ in Theorem 2.1, we get the following result due to Khan and Dolmo [4]:

Corollary 2.2. *Let $S, A : Y \to X$ satisfying*

$$d(Sx, Sy) < \max\left\{d(Ax, Ay), \beta d(Sx, Ax) + \alpha d(Sy, Ay), \frac{d(Sy, Ax) + d(Sx, Ay)}{2}\right\}$$

$$for\ all \quad x, y \in Y \quad with \quad x \neq y, \tag{2.7}$$

where $\beta \geq 0$ and $0 < \alpha < 1$. Suppose that either $A(Y)$ is a complete subspace of Y or $S(Y)$ is a complete subspace of Y with $S(Y) \subset A(Y)$. Then (S, A) have a coincidence point a in Y. Further if the point of coincidence of S and A with respect to a is in Y, then S and A will have a unique unique common fixed point in Y, provided S and A are weakly compatible.

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