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RELAXATION METHOD FOR TWO STAGES RIDGE REGRESSION ESTIMATOR

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Abstract: This paper introduces a new Estimator for multicollinearly matrix data and autocorrelated errors. We purpose two Stages Ridge Estimator(TR) for the multiple linear regression model, which suffers from both problems autocorrelation (AR(1)) and multicollinearity. After adjusting this with the ordinary ridge regression estimator (ORR), we use a mixed method to apply the two stages least squares procedure (TS). We also derive some statistical properties of this biased estimator and the paper is achieved by an application example.

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1. Introduction

The ordinary least squares method is considered as one of the most important ways of estimating the parameters of the general linear model because of its ease and simplicity and rationality of the results that obtained when the specific assumptions are achieved regarding the general linear model about error term and explanatory variables which are supposed to be orthogonal.

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Yet if these assumptions are not verified, the ordinary least squares method will give undesirable results, and there appears the problem of inaccurate estimation , one of which is associated with the autocorrelation of errors which occurs when the value of the error term in any particular period is correlated with its own preceding value or values, where the two stages method is used to deal with it. Multicollinearity is another significant problem, this occurs when the explanatory variables are correlated with each other, where the ridge regression method is used to deal with it. See for example [8].

Suppose there is a linear relation between dependent variable Y_j for $j = 1, \ldots, n$ and explanatory variables X_i for $i = 1, \ldots, p$ and error term U_i, where this relation is written in matrix form as follows:

$$Y = X\beta + U, w \tag{1}$$

Where Y is an $(n \times 1)$ dimensional vector observation of the dependent variable, X is the $(n \times (p+1))$ matrix of explanatory variables, β is the $((p+1) \times 1)$ vector of regression coefficients and U refers the $(n \times 1)$ vector of errors with proprieties $E(U) = 0$, $Cov(U) = \sigma^2 I_n$ and I_n represents the n dimensional identity matrix. The ordinary least square (OLS) estimator of β is

$$b_{_{OLS}} = (X'X)^{-1}X'Y. \tag{2}$$

Both the OLS estimator and its covariance matrix heavily depend on the characteristics of the matrix. If $X'X$ is ill-conditioned, i.e. the column vectors of X are linearly dependent, the OLS estimators are sensitive to a number of errors. For example, some of the regression coefficients may be statistically insignificant or have the wrong sign, and they may result in wide confidence intervals for individual parameters.

With ill-conditioned matrix, it is difficult to make valid statistical inferences about the regression parameters. One of the most popular estimator dealing with multi-collinearity is the ordinary ridge regression (ORR) estimator proposed by Hoerl and Kennard [6, 7] and defined as:

$$b_{_{ORR}} = \left(X'X + CI_n\right)^{-1}X'Y = \left(I_p + C(X'X)^{-1}\right)^{-1}b_{_{OLS}}, \tag{3}$$

where C is a constant, such that $0 < C \leq 1$.

The Liu estimator(LE) b_d is defined, see for example [9, 10], as follows

$$b_d = \left(X'X + I_n\right)^{-1}\left((X'X + dI_n\right)^{-1}b_{_{OLS}}, \tag{4}$$

Where d is the biasing parameter with real entries ($d_i \in \mathbb{R}$, for $i = 1, \ldots, p$), see for instance [1]. The advantage of the Liu over the ORR is that the Liu is a linear function of d, so it is easy to choose d than to choose C in the ORR estimator.

Since the matrix $X'X$ is symmetric, it exists an orthogonal matrix $V = [V_1, V_2, \ldots, V_p]$, such that $V'(X'X)V = diag\big(\lambda_1, \ldots, \lambda_p\big)$, where the λ_i is the i^{th} eigenvalue of $X'X$, and the columns of V are normalized eigenvectors associated with eigenvalues . Thus, model $Y = X\beta + U$ can be written in the canonical form as:

$$Y = Za + U, \tag{5}$$

where $Z = XV$ and $a = V'\beta$. The OLS, ORR and Liu estimator for (5) are respectively given as:

$$\begin{aligned} \hat{a}_{OLS} &= \lambda^{-1} Z'Y \\ \hat{a}_{ORR} &= \Big(I_p + C\lambda^{-1}\Big)^{-1} \hat{a}_{OLS} \qquad \text{and} \\ \hat{a}_d &= \Big(\lambda + I_n\Big)^{-1} \Big(\lambda + dI_n\Big) \hat{a}_{OLS} \end{aligned}$$

The model with first-order autoregressive process $AR(1)$ has the form, see for instance [8]:

$$U_t = \rho U_{t-1} + W_t, \qquad t = 2, \ldots, n$$

where ρ is the autocorrelation parameter(coefficient) ($|\rho| < 1$), W_t is a normal distributed random variable, which satisfies

$$W_t \sim N(0, \sigma^2), \qquad E(W_t W_{t-s}) = \begin{cases} \sigma^2, & \text{if } s = 0 \\ 0, & \text{else.} \end{cases}$$

Luis in [5] concluded that the Ridge Regression estimators which take the autocorrelation into account can perform better than the other methods . Hussein in [4] has used the two stage procedure to deal with autocorrelation and the biased Estimation ridge regression , Principle Component and Latent Roots to deal with multi-collinearity. The generalize least squares (GLS) is given as:

$$b = \Big(X'V^{-1}X\Big)^{-1} X'V^{-1}Y, \tag{6}$$

Trenker in [13] propose a relaxation-like method for improving the ridge estimator of β as:

$$b_c = \Big(X'V^{-1}X + CI_n\Big)^{-1} X'V^{-1}Y, \tag{7}$$

Because b_c is a biased estimator, Ozkale in [12] proposed a jackknife ridge estimator to reduce the bias of b_c. Moreover, Kaciranlar in [10] combined liu estimator of equation (4) with GLS of equation (6) to obtain:

$$b_c = \Big(X'V^{-1}X + I_n\Big)^{-1}\Big(X'V^{-1}X + dI_n\Big)b_{OLS}. \tag{8}$$

Using the canonical form, we can rewrite (8) as follows:

$$\hat{a}_d = \Big(\Gamma + I_n\Big)^{-1}\Big(1-d\Big)\hat{a} \tag{9}$$

Alheethy et al. in [2] construct the AUL estimator as follows:

$$\hat{a}_d^* = \Big(I_n + (\Gamma + I_n)^{-1}(1-d)\Big)\hat{a}_d = \Big(I_n + (\Gamma + I_n)^{-2}(1-d)^2\Big)\hat{a}$$

In this paper, we introduce a new estimator TR by mixing the Two Stages procedure TS with the Ordinary Ridge regression estimators ORR to deal simultaneously with both problems. Moreover, we present some interesting statistical characteristics of the estimator TR. In the following, we use the notation A^{-2} for the square of the inverse od a matrix A, such that $A^{-2} = \Big(A^{-1}\Big)^2$.

2. The Two Stages Ridge Estimator TR

First, we can reform the Two Stages procedure. By Pre multiply equation (1) with ρ, we obtain:

$$\begin{aligned} \rho Y &= \rho X\beta + \rho U \quad \text{which is equivalent to} \\ Y^* &= X^*\beta + U^*, \end{aligned} \tag{10}$$

where $E(U^*) = 0$ and $Cov(U^*) = \sigma^2 I_n$. Therefore, the OLS estimator for the model (10) is:

$$b = \Big(X^{*'}X^*\Big)^{-1}X^{*'}Y^*$$

where

$$Y^* := \rho Y = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ \vdots \\ Y_n \end{pmatrix}$$

$$X^* := \rho X = \begin{pmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 \\ -\rho & 1 & 0 & \cdots & 0 \\ 0 & -\rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -\rho & 1 \end{pmatrix} \begin{pmatrix} 1 & X_{11} & X_{12} & \cdots & X_{1k} \\ 1 & X_{21} & X_{22} & \cdots & X_{2k} \\ \vdots & \vdots & \cdots & \cdots & \vdots \\ 1 & \vdots & \vdots & \vdots & \vdots \\ 1 & X_{n1} & \cdots & \cdots & X_{nk} \end{pmatrix}.$$

Note that $X^{*'}X = X'\rho'\rho X = X'\Omega X$ and $X^{*'}Y = X'\rho'\rho Y = X'\Omega Y$, where

$$\Omega = \rho'\rho := \begin{pmatrix} 1 & -\rho & 0 & \cdots & \cdots & 0 \\ -\rho & 1+\rho^2 & -\rho & \ddots & & \vdots \\ 0 & -\rho & 1+\rho^2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1+\rho^2 & -\rho \\ 0 & \cdots & \cdots & 0 & -\rho & 1 \end{pmatrix} \tag{11}$$

Thus, the two stages estimator TS is given as

$$b_{TS} = \left(X'\Omega X\right)^{-1} X'\Omega Y. \tag{12}$$

To Estimate the linear model with both multi-collinearity and autocorrelation AR(1) simultaneously, we purpose the mixed estimator, which is developed by mixing equation (3) with (12). Therefore,The tow stages ridge estimator TR is

$$b_{TR} = \left(X'\Omega X + CI_n\right)^{-1} X'\Omega Y, \tag{13}$$

where C is a constant and Ω as defined in (11).

In the following, we derive the properties for the Estimator TR:

Lemma 1. *The Estimator TR is a bias Estimator and its expectation is given as*

$$E(b_{TR}) = \beta - CI_n\left(X'\Omega X + CI_n\right)^{-1}\beta \tag{14}$$

Proof. Recall that $b_{TR} = \left(X'\Omega X + CI_n\right)^{-1} X'\Omega Y$, we substitute Y in above equation by its value in (1) we obtain:

$$b_{TR} = \left(X'\Omega X + CI_n\right)^{-1} X'\Omega X\beta + \left(X'\Omega X + CI_n\right)^{-1} X'\Omega U.$$

Taking the expectation of the above equation and since $E(U) = 0$, we get

$$E(b_{TR}) = \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega X\beta$$

By adding and subtracting CI_n from the matrix $X'\Omega X$, the formula above leads to the expectation of TR, which is given as:

$$E(b_{TR}) = \Big(X'\Omega X + CI_n\Big)^{-1}\Big[(X'\Omega X + CI_n) - CI_n\Big]\beta$$
$$= \beta - CI_n\Big(X'\Omega X + CI_n\Big)^{-1}\beta.$$

where the term $CI_n\Big(X'\Omega X + CI_n\Big)^{-1}\beta$ represents the bias of the Estimator TR.

Lemma 2. *The Variance of TR is given as:*

$$Var(b_{TR}) = \sigma^2\Big(X'\Omega X + CI_n\Big)^{-2} X'\Omega\Omega' X. \tag{15}$$

Proof. We directly compute the variance of b_{TR}, this is

$$\begin{aligned} Var(b_{TR}) &= Var\Big[\Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega Y\Big] \\ &= \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega\Omega' X\Big(X'\Omega X + CI_n\Big)^{-1} Var(Y) \\ &= \sigma^2\Big(X'\Omega X + CI_n\Big)^{-2} X'\Omega\Omega' X. \end{aligned}$$

Lemma 3. *The Mean squares Error of TR is given as:*

$$MSE(b_{TR}) = \Big(X'\Omega X + CI_n\Big)^{-2} C^2\beta'\beta + \sigma^2\Big(X'\Omega X + CI_n\Big)^{-2} X'\Omega\Omega' X. \tag{16}$$

Proof. We know that:

$$MSE(b_{TR}) = E(b_{TR} - \beta)'(b_{TR} - \beta).$$

Therefore $b_{TR} - \beta = \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega Y - \beta$. Substitute Y in above equation by its value in the standard regression model, we obtain:

$$\begin{aligned} b_{TR} - \beta &= \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega(X\beta + U) - \beta \\ &= \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega X\beta + \Big(X'\Omega X + CI_n\Big)^{-1} X'\Omega U - \beta \end{aligned}$$

$$= \left[\left(X'\Omega X + CI_n\right)^{-1} X'\Omega X - I_n\right]\beta + \left(X'\Omega X + CI_n\right)^{-1} X'\Omega U.$$

Adding and subtracting CI_n from the first term of the above equation, we get:

$$\begin{aligned}
b_{TR} - \beta &= \left[\left(X'\Omega X + CI_n\right)^{-1} X'\Omega X + CI_n - CI_n - I_n\right]\beta \\
&\quad + \left(X'\Omega X + CI_n\right)^{-1} X'\Omega U \\
&= \left[(X'\Omega X + CI_n)^{-1}(X'\Omega X + CI_n) - CI_n(X'\Omega X + CI_n)^{-1} - I_n\right]\beta \\
&\quad + (X'\Omega X + CI_n)^{-1} X'\Omega U \\
&= \left[I_n - CI_n(X'\Omega X + CI_n)^{-1} - I_n\right]\beta + (X'\Omega X + CI_n)^{-1} X'\Omega U \\
&= -CI_n(X'\Omega X + CI_n)^{-1}\beta + (X'\Omega X + CI_n)^{-1} X'\Omega U \\
&= \mathcal{H}.
\end{aligned}$$

Therefore $MSE(b_{TR}) = E\left(\mathcal{H}'\mathcal{H}\right)$. Since $E(UU') = \sigma^2$ and the cross-product is zero because of $E(U) = 0$, we conclude that Mean squares Error of TR is given as

$$MSE(b_{TR}) = \left(X'\Omega X + CI_n\right)^{-2} C^2\beta'\beta + \sigma^2\left(X'\Omega X + CI_n\right)^{-2} X'\Omega\Omega' X.$$

where the first term represents $Bias^2(b_{TR})$ and the second one refers $Var(b_{TR})$ and it's the same as in equation (15).

Lemma 4. *The dependence between b_{TS} and b_{TR} is given in the following formula:*

$$b_{TR} = \left(I_n + C\left(X'\Omega X\right)^{-1}\right)^{-1} b_{TS} = \Psi b_{TS}. \tag{17}$$

Proof. Recall that

$$b_{TR} = \left(X'\Omega X + CI_n\right)^{-1} X'\Omega Y, \tag{18}$$

$$b_{TS} = \left(X'\Omega X\right)^{-1} X'\Omega Y. \tag{19}$$

Pre multiply the two sides of above equation (19) by $X'\Omega X$, we obtain:

$$X'\Omega Y = (X'\Omega X)b_{TS}.$$

Substitute $X'\Omega Y$ in equation (18) by its value in above equation we get:

$$\begin{aligned} b_{TR} &= \Big(X'\Omega X + CI_n\Big)^{-1}(X'\Omega X)b_{TS} \\ &= \Big(I_n + C\Big(X'\Omega X\Big)^{-1}\Big)^{-1} b_{TS} \\ &= \Psi b_{TS}. \end{aligned}$$

Remark. Since Ψ is deterministic, the lemma above mention that the b_{TR} estimators are linear combination of the b_{TS} ones.

3. Some interesting transforms of TR Estimator

In this section, we use some proprieties of symmetrical matrices to improve the results above by using eigenvalues and eigenvectors. Recall that $X'\Omega X$ is a symmetric matrix (correlation form), therefore it exists an orthogonal matrix Q such that

$$Q'\big(X'\Omega X\big)Q = \Gamma = diag(\gamma_1, \ldots, \gamma_p),$$

where γ_i is the i^{th} eigenvalue of the matrix $X'\Omega X$, the column of Q are normalized eigenvectors associated with eigenvalues. Thus, we can rewrite the two Stages estimator b_{TS} of equation (12) and the two Stages Ridge estimator b_{TR} of equation (13) respectively as follows:

$$b_{TS} = Q\Gamma^{-1}Q' r_{X^*Y^*} = \sum_{i=1}^{p} \gamma_i^{-1} Q_j Q_j' r_{X^*Y^*}, \tag{20}$$

$$b_{TR} = Q\big(\Gamma^{-1} + CI_p\big)Q' r_{X^*Y^*} = \sum_{i=1}^{p} (\gamma_i + C)^{-1} Q_j Q_j' r_{X^*Y^*}, \tag{21}$$

where $r_{X^*Y^*}$ is a correlation matrix between X^* and Y^* and Q_j represents the j^{th} column of the orthogonal matrix Q. Moreover, the model (10) can be written in the following canonical form:

$$Y^* = Wa^* + U^*, \tag{22}$$

where $W = X^*Q$ and $a^* = Q'\beta$. Thus, the TS estimator of equation (12) is given as

$$\hat{a}^* = (W'W)^{-1}W'Y^* = \Gamma^{-1}W'Y^*. \tag{23}$$

Lemma 5. *The dependence between the TR and TS estimators is given as:*

$$\hat{a}^*_{TR} = \left(I_p + C\Gamma^{-1}\right)^{-1} \hat{a}^*_{TS} \tag{24}$$

Proof. We can rewrite TR using model (22) as follows:

$$\hat{a}^*_{TR} = \left(W'W + CI_p\right)^{-1} W'Y^*. \tag{25}$$

Pre multiply the two sides of the formula (23) by $W'W$, we obtain $W'Y^* = W'W\hat{a}^*_{TS}$. Substitute $W'Y^*$ in equation (25) by its value in above equation we get

$$\begin{aligned}\hat{a}^*_{TR} &= \left(W'W + CI_p\right)^{-1} W'W\hat{a}^*_{TS} \\ &= \left(I_p + CI_p(W'W)^{-1}\right)^{-1} \hat{a}^*_{TS} \\ &= \left(I_p + C\Gamma^{-1}\right)^{-1} \hat{a}^*_{TS}.\end{aligned}$$

Lemma 6. *The expectation of TS is given as:*

$$E(\hat{a}^*_{TS}) = a^*. \tag{26}$$

Proof. Substitute Y^* in equation (23) by its value in equation (22) we obtain:

$$\hat{a}^*_{TS} = \Gamma^{-1} W'\left(Wa^* + U^*\right) = \Gamma^{-1} W'Wa^* + \Gamma^{-1} W'U^*.$$

Since $W = X^*Q$ and $W'W = Q'X'^*X^*Q = \Gamma$, then $\hat{a}^*_{TS} = a^* + \Gamma^{-1}W'U^*$. We compute the expectation for the two sides of above equation and since $E(U^*) = 0$, we get

$$E(\hat{a}^*_{TS}) = E(a^* + \Gamma^{-1}W'U^*) = a^* + \Gamma^{-1}W'E(U^*) = a^*.$$

Lemma 7. *The expectation of TR is given as:*

$$E(\hat{a}^*_{TR}) = a^* - CI_p(\Gamma^{-1} + CI_p)a^*. \tag{27}$$

Proof. Recall that $a^*_{TR} = (W'W + CI_p)^{-1}W'Y^*$. Substitute Y^* in above equation by its value in (22) we obtain:

$$\begin{aligned}a^*_{TR} &= (W'W + CI_p)^{-1} W'(Wa^* + U^*) \\ &= (W'W + CI_p)^{-1} W'Wa^* + (W'W + CI_p)^{-1} W'U^*.\end{aligned}$$

Take the expectation for the two sides of above equation and using the fact that U^* is centered, we get

$$E(a^*_{TR}) = E\left((W'W + CI_p)^{-1} W'Wa^*\right) + E\left((W'W + CI_p)^{-1} W'U^*\right)$$

$$= (W'W + CI_p)^{-1}W'Wa^*.$$

By adding and subtracting CI_p from the matrix $W'W$, we obtain the expectation of TR:

$$\begin{aligned}
E(a^*_{TR}) &= (W'W + CI_p)^{-1}\Big((W'W + CI_p) - CI_p\Big)a^* \\
&= \Big[I_p - CI_p(W'W + CI_p)^{-1}\Big]a^* \\
&= a^* - CI_p\Big(\Gamma^{-1} + CI_p\Big)^{-1}a^*,
\end{aligned}$$

where, the term $CI_p(\Gamma^{-1} + CI_p)a^*$ represents the bias of TR.

Lemma 8. *The Variance of TS is given as :*

$$Var(\hat{a}^*_{TS}) = \sigma^2\Gamma^{-1}. \tag{28}$$

Proof. The variance can be computed as follows:

$$\begin{aligned}
Var(\hat{a}^*_{TS}) &= Var(\Gamma^{-1}W'Y^*) \\
&= \Gamma^{-2}W'WVar(Y^*) \\
&= \sigma^2\Big(\Gamma^{-2}Q'X'^*QX^*\Big) \\
&= \sigma^2\Big(\Gamma^{-2}\Gamma\Big) \\
&= \sigma^2\Gamma^{-1}.
\end{aligned}$$

Theorem 1. *The Variance of TR is given as:*

$$Var(\hat{a}^*_{TR}) = \sigma^2\Big(\Gamma^{\frac{1}{2}} + C\Gamma^{-\frac{1}{2}}\Big)^{-2}. \tag{29}$$

Proof. The variance is

$$\begin{aligned}
Var(\hat{a}^*_{TR}) &= Var\left((I_p + C\Gamma^{-1})^{-1}\hat{a}^*_{TR}\right) \\
&= (I_p + C\Gamma^{-1})^{-2}Var\left(\hat{a}^*_{TR}\right) \\
&= \sigma^2\Gamma^{-1}(I_p + C\Gamma^{-1})^{-2} \\
&= \sigma^2\Big(\Gamma^{\frac{1}{2}} + C\Gamma^{-\frac{1}{2}}\Big)^{-2}.
\end{aligned}$$

Theorem 2. *The Mean Squares Error of TR is given as:*

$$MSE(\hat{a}^*_{TR}) = \sigma^2\Big(\Gamma^{\frac{1}{2}} + C\Gamma^{-\frac{1}{2}}\Big)^{-2} + C^2(\Gamma^{-1} + CI_p)^{-2}\hat{a}^{*2}. \tag{30}$$

Proof. We know that: $MSE(\hat{a}^*_{TR}) = E(\hat{a}^*_{TR} - a^*)'(\hat{a}^*_{TR} - a^*)$ and

$$(\hat{a}^*_{TR} - a^*) = (W'W + CI_p)^{-1}W'Y^* - a^*.$$

Substitute Y^* in above equation by its value in (22), we obtain:

$$\begin{aligned}\hat{a}^*_{TR} - a^* &= (W'W + CI_p)^{-1}W'(Wa^* + U^*) - a^* \\ &= (W'W + CI_p)^{-1}W'Wa^* + (W'W + CI_p)^{-1}W'U^* - a^* \\ &= \Big[(W'W + CI_p)^{-1}W'W - I_p\Big]a^* + (W'W + CI_p)^{-1}W'U^*.\end{aligned}$$

Adding and subtracting CI_p from the first term of the above equation, we get:

$$\begin{aligned}\hat{a}^*_{TR} - a^* &= \Big[(W'W + CI_p)^{-1}W'W + CI_p - CI_p - I_p\Big]a^* \\ &+ (W'W + CI_p)^{-1}W'U^* \\ &= \Big[(W'W + CI_p)^{-1}(W'W + CI_p) - CI_p(W'W + CI_p) - I_p\Big]a^* \\ &+ (W'W + CI_p)^{-1}W'U^* \\ &= \Big[I_p - CI_p(W'W + CI_p)^{-1} - I_p\Big]a^* + (W'W + CI_p)^{-1}W'U^* \\ &= -CI_p(W'W + CI_p)^{-1}a^* + (W'W + CI_p)^{-1}W'U^* \\ &= \mathcal{K}\end{aligned}$$

Thus,

$$\begin{aligned}MSE(\hat{a}^*_{TR}) &= E(\mathcal{K}'\mathcal{K}) \\ &= (W'W + CI_p)^{-2}C^2a^{*'}a^* + \sigma^2(W'W + CI_p)W'W \\ &= (\Gamma^{-1} + CI_p)^{-2}C^2a^{*'}a^* + \sigma^2(\Gamma^{-1} + CI_p)^{-2}\Gamma^{-1} \\ &= (\Gamma^{-1} + CI_p)^{-2}C^2a^{*2} + \sigma^2(\Gamma^{-1} + CI_p)^{-2}\Gamma^{-1}.\end{aligned}$$

Remark. Note that the first term of the MSE in equation (30) represents the $Bias^2(\hat{a}^*_{TR})$ which is increasing function of C, and the second one is $Var(\hat{a}^*_{TR})$ which is decreasing function of C, this means we accepted some bias in order to decrease the variance.

4. Application example

The data in this example represents the product in the manufacturing sector, the imported intermediate and capital commodities and imported raw materials in the period from 1960 to 1990. Consider the following linear model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon \tag{31}$$

where Yrepresents the product value in the manufacturing sector and X_i for $i = 1, 2$ and 3 refers, respectively, the values of the imported intermediate commodities, imported capital commodities and the value of imported raw materials.
The estimated model is:

$$Y = 208.88 + 0.611X_1 + 1.256X_2 - 1.217X_3, \tag{32}$$

where the statistical outputs are given in the following table:

α	n	p	$F_{(3,27)}$	R^2	$\hat{\sigma}^2$	dl	du	DW	VIF_1	VIF_2	VIF_3
0.05	31	3	860.4	0.9896	20851.36	1.23	1.63	0.905	128.29	103.43	70.87

Where VIF_i for $i = 1, 2, 3$ represent the Variance Inflation Factors. The $X'X$ (correlation form) is:

$$r_X = \begin{pmatrix} 1 & 0.9947 & 0.9923 \\ 0.9947 & 1 & 0.9905 \\ 0.9923 & 0.9905 & 1 \end{pmatrix}$$

Since $DW < dl$, the model suffer from positive first order autoregressive scheme and since all $VIF's > 4$, the model suffer from Multicollinearity. Thus, using the relation between d and ρ to estimate the coefficient of autocorrelation. We found that $\hat{\rho} = 0.547$. The estimated model using the two stages estimator TS is:

$$Y = 243.2884 + 0.6076X_1 + 1.2402X_2 - 1.4731X_3, \tag{33}$$

where the statistical outputs are given in the following table:

α	n	p	$F_{(3,27)}$	R^2	$\hat{\sigma}^2$	dl	du	DW	VIF_1	VIF_2	VIF_3
0.05	31	3	481.46	0.9861	14730.36	1.23	1.63	1.699	26.82	38.32	16.89

Remark that $DW > du$, therefore the model does not suffer from first order autoregressive scheme. But, since all $VIF's > 4$, the model still suffer from multicollinearity. The corresponding $X'\Omega X$ (correlation form) is:

$$r_{X'\Omega X} = \begin{pmatrix} 1 & 0.9809 & 0.9562 \\ 0.9809 & 1 & 0.9695 \\ 0.9562 & 0.9695 & 1 \end{pmatrix}$$

The following table **??** summarizes these interesting comparison of the estima-

	$\mathbf{b_{ORR}}$			$\mathbf{b_{TR}}$		
C	VAR	Bias^2	MSE	VAR	Bias^2	MSE
0	1848.9533	0	1848.9533	889.0630	0	889.0630
0.01	1845.6878	0.0341	1845.7220	887.9317	0.0061	887.9379
0.01	1845.6878	0.0341	1845.7220	887.9317	0.0061	887.9379
0.03	1839.1829	0.3062	1839.4891	885.6756	0.0555	885.7311
0.04	1835.9433	0.5434	1836.4867	884.5508	0.0985	884.6494
0.05	1832.7123	0.8476	1833.5599	883.4281	0.1538	883.5820
0.06	1829.4898	1.2184	1830.7083	882.3076	0.2212	882.5288
0.07	1826.2758	1.6555	1827.9314	881.1892	0.3007	881.4899
0.08	1823.0703	2.1585	1825.2289	880.0729	0.3923	880.4652
0.09	1819.8732	2.7271	1822.6004	878.9587	0.4959	879.4546
0.1	1816.6846	3.3609	1820.0455	877.8467	0.6114	878.4582
0.2	1785.2545	13.2107	1798.465	866.8416	2.4150	869.2567
0.3	1754.6342	29.2136	1783.8479	856.0426	5.3661	861.4087
0.4	1724.7961	51.05092	1775.8471	845.4444	9.4215	854.8659
0.5	**1695.7139**	**78.4200**	**1774.1339**	835.0421	14.5396	849.5818
0.6	1667.3621	111.0339	1778.3960	824.8310	20.6806	845.5117
0.7	1639.7165	148.6197	1788.3363	814.8063	27.8060	842.6123
0.8	1612.7539	190.9186	1803.6725	804.9636	35.8785	840.8422
0.9	1586.4520	237.6842	1824.1363	**795.2985**	**44.8626**	**840.1611**
1	1560.7893	288.6828	1849.4722	785.8067	54.7237	840.5305

Table 1: Statistical values of the estimators $\mathbf{b_{ORR}}$ and $\mathbf{b_{TR}}$

Estimator	Variance	Bias^2	MSE	
b_{OLS}	1848.953	0	1848.953	
b_{ORR}	1695.71	78.42	1774.13	$C = 0.5$
b_{TS}	889.06	0	889.06	
b_{TR}	795.29	44.86	840.16	$C = 0.9$

Table 2: Direct comparison of the estimators.

tors subject of our study We remak that the computed values shows clearly the good result of our improved estimator. Moreover, we constat that (see figure 1) the $Bias^2(\hat{a}^*_{TR})$ increases exponentially in function of C, where the $Var(\hat{a}^*_{TR})$ decreases linearly.

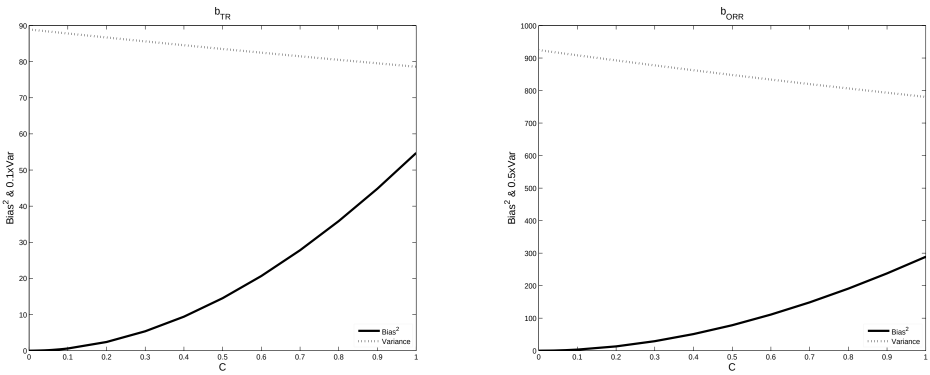

Figure 1: Plot of the variance and the bias square in dependence of C, where the solid lines refer the bias and the dotted ones represent the variance, (Left) the $b_{_{TR}}$ and (Right) $b_{_{ORR}}$.

5. Concluding remarks

In this work, we mainly have introduced a new formula for the TR estimator for singular data matrix and discrete auto correlated errors. Moreover, we concluded that TR estimator is an unbiased and provide a better (smaller) MSE, Variance and Bias than the ORR one. To improve the idea presented in this work, we will study in a future paper, in one hand the case of higher order autoregressive schemes, such as second autocorrelated errors. In other hand, we derive a new estimator by mixing two stages method with generalized ridge regression, where the constant C will be a diagonal matrix with different entries.

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