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A GENERAL SOLUTION OF A QUARTIC FUNCTIONAL EQUATION AND ITS STABILITY

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Abstract: A general solution of the functional equation

$$f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x) = 13\left(f(x+y)+f(x-y)\right)+168f(y)$$

is determined over the reals without assuming any regularity conditions, and the stability of this functional equation is established.

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Key Words: additive function, difference operator, Fréchet functional equation, quartic functional equation

1. Introduction

A function $A : \mathbb{R} \to \mathbb{R}$ is said to be *additive*, [1], if $A(x+y) = A(x)+A(y)$. For $n \in \mathbb{N}$, a function $A_n : \mathbb{R}^n \to \mathbb{R}$ that is additive in each of its variable is said to be *n-additive.* If

$$A_n(x_1, x_2, ..., x_n) = A_n(x_{\pi(1)}, x_{\pi(2)}, ..., x_{\pi(n)})$$

for every permutation $\{\pi(1), \pi(2), ..., \pi(n)\}$ of $\{1, 2, ..., n\}$ $(n \in \mathbb{N})$, then A_n is said to be *symmetric.* Denote the diagonal element $A_n(x, x, ..., x)$ by $A^n(x)$, if

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$A_n(x_1, x_2, ..., x_n)$ is an n-additive symmetric function, and denote the resulted function obtained by putting $x_1 = x_2 = ... = x_\ell = x$ and $x_{\ell+1} = x_{\ell+2} = ... = x_n = y$ in $A_n(x_1, x_2, ..., x_n)$ by $A^{\ell,n-\ell}(x, y)$.

For $f : \mathbb{R} \to \mathbb{R}$, the difference operator Δ_h with $h \in \mathbb{R}$ is defined by

$$\Delta_h f(x) = f(x + h) - f(x).$$

The superposition of difference operations is defined by

$$\Delta_{h_1,...,h_n} f = \Delta_{h_1} \circ \Delta_{h_2} \circ \ldots \circ \Delta_{h_n} f, \qquad (n \in \mathbb{N}),$$

where $\Delta_{h_i} \circ \Delta_{h_j}$ denotes composition. If $h_1 = h_2 = ... = h_n = h$, we write

$$\Delta_{\underbrace{h, ..., h}_{n}} f = \Delta_h^n f.$$

For $n \in \mathbb{N} \cup \{0\}$, if f satisfies the functional equation

$$\Delta_h^{n+1} f(x) = 0 \qquad (x, h \in \mathbb{R}), \tag{1.1}$$

then f is called a *polynomial function of order* n. In explicit form (1.1) can be written as

$$\sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} f(x + kh) = 0. \tag{1.2}$$

It is known ([4]) that with functions defined over $\mathbb{R}$, the equation (1.2) is equivalent to the Fréchet functional equation

$$\Delta_{h_1,...,h_{n+1}} f(x) = 0 \qquad (x, h_1, ..., h_{n+1} \in \mathbb{R}). \tag{1.3}$$

We quote three results from [3, pp.71-77, Theorems 9.3, 9.4, 9.6] which are needed in our work here.

Theorem 1.1. *I. If $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function of order n, then*

$$\Delta_{h_1,...,h_{n+1}} f(x) = 0 \quad (x, h_1, ..., h_{n+1} \in \mathbb{R}).$$

II. If $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 0, 1, ..., n)$ are symmetric k-additive functions and if A^k are their diagonalizations, then the function $\sum_{k=0}^n A^k(x)$ is a polynomial function of order n.

III. If $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function of order n, then there exist k-additive symmetric functions $A_k : \mathbb{R}^k \to \mathbb{R}$ $(k = 0, 1, ..., n)$ such that $f(x) = \sum_{k=0}^n A^k(x)$, where A^k are the diagonalizations of A_k.

Our work here is originated from the functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4\left(f(x+y)+f(x-y)+6f(y)\right) \qquad (x,y\in\mathbb{R}), \quad (1.4)$$

which was considered by Chung and Sahoo, [2] in 2003. Clearly, $f(x)=x^4$ is a solution of (1.4), which leads us to call (1.4) a *quartic functional equation*, and a solution of (1.4) is called a quartic function. Chung and Sahoo's results are :

Theorem 1.2. *I. If* $f:\mathbb{R}\to\mathbb{R}$ *satisfies the functional equation* (1.4), *then* f *is a solution of the Fréchet functional equation*

$$\Delta_{x_1,x_2,x_3,x_4,x_5}f(x_0)=0 \qquad (x_0,x_1,x_2,x_3,x_4,x_5\in\mathbb{R}).$$

II. The function $f:\mathbb{R}\to\mathbb{R}$ *satisfies the functional equation* (1.4) *if and only if it is of the form* $f(x)=A^4(x)$, *where* $A^4(x)$ *is diagonal of a 4-additive symmetric function* A_4.

Later in 2004, Sahoo, [5], considered the functional equation

$$f(x+2y)+f(x-2y)+6f(x)=4\left(f(x+y)+f(x-y)\right) \qquad (x,y\in\mathbb{R}), \quad (1.5)$$

and showed that its general solution is of the form

$$f(x)=A^0+A^1(x)+A^2(x)+A^3(x),$$

where $A^n(x)$ is the diagonal of the n-additive symmetric function $A_n:\mathbb{R}^n\to\mathbb{R}$ $(n=1,2,3)$ and A^0 is an arbitrary constant. One year later, he generalized (1.5) to

$$f_1(2x+y)+f_2(2x-y)=f_3(x+y)+f_4(x-y)+f_5(x) \qquad (x,y\in\mathbb{R}), \quad (1.6)$$

and proved that $f_1,f_2,f_3,f_4,f_5:\mathbb{R}\to\mathbb{R}$ satisfy (1.6) if and only if

$$\begin{aligned}
f_1(x)&=A^3(x)+A^2(x)+A^1(x)+A^0+B^2(x)+B^1(x)+B^0,\\
f_2(x)&=A^3(x)+A^2(x)+A^1(x)+A^0-B^2(x)-B^1(x)-B^0,\\
f_3(x)&=2A^3(x)+A^2(x)+A^1(x)+\frac{1}{2}A^0+C^1(x)\\
&\quad+C^0+2B^2(x)+B^1(x)+B^0+D^0,\\
f_4(x)&=2A^3(x)+A^2(x)+A^1(x)+\frac{1}{2}A^0+C^1(x)\\
&\quad+C^0-2B^2(x)-B^1(x)-B^0-D^0,
\end{aligned}$$

$$f_5(x) = 12A^3(x) + 6A^2(x) + 2A^1(x) + A^0 - 2C^1(x) - 2C^0,$$

where A^0, B^0, C^0, D^0 are arbitrary constants, $A^n(x), B^n(x), C^n(x)$ are the diagonals of n-additive symmetric functions A_n, B_n, C_n, respectively.

Recognizing the identity

$$(x+3y)^4 + (x-3y)^4 + (x+2y)^4 + (x-2y)^4 + 22x^4 \tag{1.7}$$
$$= 13\left((x+y)^4 + (x-y)^4\right) + 168y^4 \tag{1.8}$$

which renders a solution $f(x) = x^4$ to the functional equation

$$f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) \tag{1.9}$$
$$= 13\left(f(x+y) + f(x-y)\right) + 168f(y), \tag{1.10}$$

here we find a general solution of (1.9) without assuming any regularity condition and establish its stabilitya stability. Our main results are:

Theorem 1.3. *I. A function $f:\mathbb{R}\to\mathbb{R}$ satisfies the functional equation* (1.9) *if and only if it is of the form*

$$f(x) = A^4(x) \quad (x\in\mathbb{R}), \tag{1.11}$$

where $A^4(x)$ is the diagonal of a 4-additive symmetric function A_4.
II. Let $\phi:\mathbb{R}^2\to[0,\infty)$ be a function such that the series

$$\sum_{i=0}^{\infty}\frac{\phi(2^i x, 2^i y)}{16^i} \tag{1.12}$$

converges for all $x,y\in\mathbb{R}$. If $f:\mathbb{R}\to\mathbb{R}$ is an even function with $f(0)=0$ and satisfies

$$|Df(x,y)| \le \phi(x,y), \tag{1.13}$$

where

$$Df(x,y) := f(x+3y) + f(x-3y) + f(x+2y) + f(x-2y) + 22f(x) - 13f(x+y),$$

then there exists a unique $T:\mathbb{R}\to\mathbb{R}$ satisfying (1.9) *and*

$$|f(y) - T(y)| \le \frac{1}{5376}\times$$

$$\sum_{i=0}^{\infty} \frac{2\phi(3\cdot 2^i y, 2^i y) + 2\phi(2\cdot 2^i y, 2^i y) + \phi(0, 2\cdot 2^i y) + 30\phi(2^i y, 2^i y) + 50\phi(0, 2^i y)}{16^i}, \tag{1.14}$$

where $y \in \mathbb{R}$. Moreover, the function T is given by

$$T(y) = \lim_{n\to\infty} \frac{f(2^n y)}{16^n} \qquad (y \in \mathbb{R}). \tag{1.15}$$

2. Proof of Theorem 1.3 I

In this section, we establish the general solution of the functional equation (1.9). First, we prove an auxiliary lemma.

Lemma 2.1. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation*

$$f(x+4y) - 14f(x+2y) + 35f(x+y) - 35f(x) + 14f(x-y) - f(x-3y) = 0, \tag{2.1}$$

where $x, y \in \mathbb{R}$, then f is a solution of the Fréchet functional equation

$$\Delta_{x_1,x_2,x_3,x_4,x_5,} f(x_0) = 0 \qquad (x_0, x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}).$$

Proof. Substituting

$$x_0 = x+4y, \quad y_1 = x-3y, \quad \text{i.e.,} \quad x = \frac{3}{7}x_0 + \frac{4}{7}y_1, \; y = \frac{1}{7}x_0 - \frac{1}{7}y_1$$

in (2.1), we obtain

$$f(x_0) - 14f\left(\frac{5}{7}x_0 + \frac{2}{7}y_1\right) + 35f\left(\frac{4}{7}x_0 + \frac{3}{7}y_1\right) - 35f\left(\frac{3}{7}x_0 + \frac{4}{7}y_1\right) \tag{2.2}$$

$$+ 14f\left(\frac{2}{7}x_0 + \frac{5}{7}y_1\right) - f(y_1) = 0. \tag{2.3}$$

Replacing x_0 by $x_0 + x_1$ in (2.2) and subtracting (2.2) from the resulting expression, we obtain

$$f(x_0+x_1) - f(x_0) - 14f\left(\frac{5}{7}(x_0+x_1) + \frac{2}{7}y_1\right) + 14f\left(\frac{5}{7}x_0 + \frac{2}{7}y_1\right) \tag{2.4}$$

$$+ 35f\left(\frac{4}{7}(x_0+x_1) + \frac{3}{7}y_1\right) - 35f\left(\frac{4}{7}x_0 + \frac{3}{7}y_1\right) - 35f\left(\frac{3}{7}(x_0+x_1) + \frac{4}{7}y_1\right)$$

$$+35f\left(\frac{3}{7}x_0+\frac{4}{7}y_1\right)+14f\left(\frac{2}{7}(x_0+x_1)+\frac{5}{7}y_1\right)-14f\left(\frac{2}{7}x_0+\frac{5}{7}y_1\right)=0.$$

Putting $y_2=\frac{4}{7}x_0+\frac{3}{7}y_1$, i.e., $y_1=-\frac{4}{3}x_0+\frac{7}{3}y_2$ in (2.4), we see that

$$\begin{aligned}&f(x_0+x_1)-f(x_0)-14f\left(\frac{5}{7}x_1+\frac{1}{3}x_0+\frac{2}{3}y_2\right)+14f\left(\frac{1}{3}x_0+\frac{2}{3}y_2\right)\\&+35f\left(\frac{4}{7}x_1+y_2\right)-35f(y_2)-35f\left(\frac{3}{7}x_1-\frac{1}{3}x_0+\frac{4}{3}y_2\right)\\&+35f\left(-\frac{1}{3}x_0+\frac{4}{3}y_2\right)+14f\left(\frac{2}{7}x_1-\frac{2}{3}x_0+\frac{5}{3}y_2\right)-14f\left(-\frac{2}{3}x_0+\frac{5}{3}y_2\right)=0.\end{aligned}\tag{2.5}$$

Replacing x_0 by x_0+x_2 in (2.5) and subtracting (2.5) from the resulting expression, we obtain

$$\begin{aligned}&f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0)\\&-14f\left(\frac{5}{7}x_1+\frac{1}{3}(x_0+x_2)+\frac{2}{3}y_2\right)+14f\left(\frac{5}{7}x_1+\frac{1}{3}x_0+\frac{2}{3}y_2\right)\\&+14f\left(\frac{1}{3}(x_0+x_2)+\frac{2}{3}y_2\right)-14f\left(\frac{1}{3}x_0+\frac{2}{3}y_2\right)\\&-35f\left(\frac{3}{7}x_1-\frac{1}{3}(x_0+x_2)+\frac{4}{3}y_2\right)+35f\left(\frac{3}{7}x_1-\frac{1}{3}x_0+\frac{4}{3}y_2\right)\\&+35f\left(-\frac{1}{3}(x_0+x_2)+\frac{4}{3}y_2\right)-35f\left(-\frac{1}{3}x_0+\frac{4}{3}y_2\right)\\&+14f\left(\frac{2}{7}x_1-\frac{2}{3}(x_0+x_2)+\frac{5}{3}y_2\right)-14f\left(\frac{2}{7}x_1-\frac{2}{3}x_0+\frac{5}{3}y_2\right)\\&-14f\left(-\frac{2}{3}(x_0+x_2)+\frac{5}{3}y_2\right)+14f\left(-\frac{2}{3}x_0+\frac{5}{3}y_2\right)=0.\end{aligned}\tag{2.6}$$

Letting $y_3=\frac{1}{3}x_0+\frac{2}{3}y_2$, i.e., $y_2=-\frac{1}{2}x_0+\frac{3}{2}y_3$ in (2.6), we see that

$$\begin{aligned}&f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0)\\&-14f\left(\frac{5}{7}x_1+\frac{1}{3}x_2+y_3\right)+14f\left(\frac{5}{7}x_1+y_3\right)\\&+14f\left(\frac{1}{3}x_2+y_3\right)-14f(y_3)-35f\left(\frac{3}{7}x_1-\frac{1}{3}x_2-x_0+2y_3\right)\\&+35f\left(\frac{3}{7}x_1-x_0+2y_3\right)+35f\left(-\frac{1}{3}x_2-x_0+2y_3\right)-35f(-x_0+2y_3)\end{aligned}\tag{2.7}$$

$$+ 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right) - 14f\left(\frac{2}{7}x_1 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right)$$
$$- 14f\left(-\frac{2}{3}x_2 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right) + 14f\left(-\frac{3}{2}x_0 + \frac{5}{2}y_3\right) = 0.$$

Replacing x_0 by $x_0 + x_3$ in (2.7) and subtracting (2.7) from the resulting expression, we obtain

$$f(x_0 + x_1 + x_2 + x_3) - f(x_0 + x_1 + x_2) - f(x_0 + x_1 + x_3) - f(x_0 + x_2 + x_3) \tag{2.8}$$
$$+ f(x_0 + x_1) + f(x_0 + x_2) + f(x_0 + x_3) - f(x_0)$$
$$- 35f\left(\frac{3}{7}x_1 - \frac{1}{3}x_2 - (x_0 + x_3) + 2y_3\right) + 35f\left(\frac{3}{7}x_1 - \frac{1}{3}x_2 - x_0 + 2y_3\right)$$
$$+ 35f\left(\frac{3}{7}x_1 - (x_0 + x_3) + 2y_3\right) - 35f\left(\frac{3}{7}x_1 - x_0 + 2y_3\right)$$
$$+ 35f\left(-\frac{1}{3}x_2 - (x_0 + x_3) + 2y_3\right) - 35f\left(-\frac{1}{3}x_2 - x_0 + 2y_3\right)$$
$$- 35f(-(x_0 + x_3) + 2y_3) + 35f(-x_0 + 2y_3)$$
$$+ 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{3}{2}(x_0 + x_3) + \frac{5}{2}y_3\right) - 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right)$$
$$- 14f\left(\frac{2}{7}x_1 - \frac{3}{2}(x_0 + x_3) + \frac{5}{2}y_3\right) + 14f\left(\frac{2}{7}x_1 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right)$$
$$- 14f\left(-\frac{2}{3}x_2 - \frac{3}{2}(x_0 + x_3) + \frac{5}{2}y_3\right) + 14f\left(-\frac{2}{3}x_2 - \frac{3}{2}x_0 + \frac{5}{2}y_3\right)$$
$$+ 14f\left(-\frac{3}{2}(x_0 + x_3) + \frac{5}{2}y_3\right) - 14f\left(-\frac{3}{2}x_0 + \frac{5}{2}y_3\right) = 0.$$

Putting $y_4 = -x_0 + 2y_3$, i.e., $y_3 = \frac{1}{2}x_0 + \frac{1}{2}y_4$ in (2.8), we see that

$$f(x_0 + x_1 + x_2 + x_3) - f(x_0 + x_1 + x_2) - f(x_0 + x_1 + x_3) - f(x_0 + x_2 + x_3) \tag{2.9}$$
$$+ f(x_0 + x_1) + f(x_0 + x_2) + f(x_0 + x_3) - f(x_0)$$
$$- 35f\left(\frac{3}{7}x_1 - \frac{1}{3}x_2 - x_3 + y_4\right)$$
$$+ 35f\left(\frac{3}{7}x_1 - \frac{1}{3}x_2 + y_4\right) + 35f\left(\frac{3}{7}x_1 - x_3 + y_4\right) - 35f\left(\frac{3}{7}x_1 + y_4\right)$$

$$+35f\left(-\frac{1}{3}x_2-x_3+y_4\right)-35f\left(-\frac{1}{3}x_2+y_4\right)-35f(-x_3+y_4)+35f(y_4)$$
$$+14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)-14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$-14f\left(\frac{2}{7}x_1-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)+14f\left(\frac{2}{7}x_1-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$

$$-14f\left(-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)+14f\left(-\frac{2}{3}x_2-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$+14f\left(-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)-14f\left(-\frac{1}{4}x_0+\frac{5}{4}y_4\right)=0.$$

Replacing x_0 by x_0+x_4 in (2.9) and subtracting (2.9) from the resulting expression, we obtain

$$f(x_0+x_1+x_2+x_3+x_4)-f(x_0+x_1+x_2+x_3) \tag{2.10}$$
$$-f(x_0+x_1+x_2+x_4)$$
$$-f(x_0+x_1+x_3+x_4)-f(x_0+x_2+x_3+x_4)+f(x_0+x_1+x_2)$$
$$+f(x_0+x_1+x_3)+f(x_0+x_1+x_4)+f(x_0+x_2+x_3)+f(x_0+x_2+x_4)$$
$$+f(x_0+x_3+x_4)-f(x_0+x_1)-f(x_0+x_2)-f(x_0+x_3)-f(x_0+x_4)$$
$$+f(x_0)+14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)$$
$$-14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$-14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)+14f\left(\frac{2}{7}x_1-\frac{2}{3}x_2-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$-14f\left(\frac{2}{7}x_1-\frac{3}{2}x_3-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)+14f\left(\frac{2}{7}x_1-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$+14f\left(\frac{2}{7}x_1-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)-14f\left(\frac{2}{7}x_1-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$-14f\left(-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)+14f\left(-\frac{2}{3}x_2-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$+14f\left(-\frac{2}{3}x_2-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)-14f\left(-\frac{2}{3}x_2-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$
$$+14f\left(-\frac{3}{2}x_3-\frac{1}{4}(x_0+x_4)+\frac{5}{4}y_4\right)-14f\left(-\frac{3}{2}x_3-\frac{1}{4}x_0+\frac{5}{4}y_4\right)$$

$$- 14f\left(-\frac{1}{4}(x_0 + x_4) + \frac{5}{4}y_4\right) + 14f\left(-\frac{1}{4}x_0 + \frac{5}{4}y_4\right) = 0.$$

Letting $y_5 = -\frac{1}{4}x_0 + \frac{5}{4}y_4$ in (2.10), we see that

$$\begin{aligned} &f(x_0 + x_1 + x_2 + x_3 + x_4) - f(x_0 + x_1 + x_2 + x_3) - f(x_0 + x_1 + x_2 + x_4) \qquad (2.11)\\ &- f(x_0 + x_1 + x_3 + x_4) - f(x_0 + x_2 + x_3 + x_4) + f(x_0 + x_1 + x_2)\\ &+ f(x_0 + x_1 + x_3) + f(x_0 + x_1 + x_4) + f(x_0 + x_2 + x_3) + f(x_0 + x_2 + x_4)\\ &+ f(x_0 + x_3 + x_4) - f(x_0 + x_1) - f(x_0 + x_2) - f(x_0 + x_3) \end{aligned}$$

$$\begin{aligned} &- f(x_0 + x_4) + f(x_0) + 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{3}{2}x_3 - \frac{1}{4}x_4 + y_5\right)\\ &- 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{3}{2}x_3 + y_5\right) - 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 - \frac{1}{4}x_4 + y_5\right)\\ &+ 14f\left(\frac{2}{7}x_1 - \frac{2}{3}x_2 + y_5\right) - 14f\left(\frac{2}{7}x_1 - \frac{3}{2}x_3 - \frac{1}{4}x_4 + y_5\right)\\ &+ 14f\left(\frac{2}{7}x_1 - \frac{3}{2}x_3 + y_5\right) + 14f\left(\frac{2}{7}x_1 - \frac{1}{4}x_4 + y_5\right)\\ &- 14f\left(\frac{2}{7}x_1 + y_5\right) - 14f\left(-\frac{2}{3}x_2 - \frac{3}{2}x_3 - \frac{1}{4}x_4 + y_5\right)\\ &+ 14f\left(-\frac{2}{3}x_2 - \frac{3}{2}x_3 + y_5\right) + 14f\left(-\frac{2}{3}x_2 - \frac{1}{4}x_4 + y_5\right)\\ &- 14f\left(-\frac{2}{3}x_2 + y_5\right) + 14f\left(-\frac{3}{2}x_3 - \frac{1}{4}x_4 + y_5\right)\\ &- 14f\left(-\frac{3}{2}x_3 + y_5\right) - 14f\left(-\frac{1}{4}x_4 + y_5\right) + 14f(y_5) = 0. \end{aligned}$$

Replacing x_0 by $x_0 + x_5$ in (2.11) and subtracting (2.11) from the resulting expression, we obtain

$$\begin{aligned} &f(x_0 + x_1 + x_2 + x_3 + x_4 + x_5) - f(x_0 + x_1 + x_2 + x_3 + x_4) \qquad (2.12)\\ &- f(x_0 + x_1 + x_2 + x_3 + x_5) - f(x_0 + x_1 + x_2 + x_4 + x_5)\\ &- f(x_0 + x_1 + x_3 + x_4 + x_5) - f(x_0 + x_2 + x_3 + x_4 + x_5)\\ &+ f(x_0 + x_1 + x_2 + x_3) + f(x_0 + x_1 + x_2 + x_4) + f(x_0 + x_1 + x_2 + x_5)\\ &+ f(x_0 + x_1 + x_3 + x_4) + f(x_0 + x_1 + x_3 + x_5) + f(x_0 + x_1 + x_4 + x_5)\\ &+ f(x_0 + x_2 + x_3 + x_4) + f(x_0 + x_2 + x_3 + x_5) + f(x_0 + x_2 + x_4 + x_5) \end{aligned}$$

$$+ f(x_0 + x_3 + x_4 + x_5) - f(x_0 + x_1 + x_2) - f(x_0 + x_1 + x_3) - f(x_0 + x_1 + x_4) - f(x_0 + x_1 + x_5) - f(x_0 + x_2 + x_3) - f(x_0 + x_2 + x_4) - f(x_0 + x_2 + x_5) - f(x_0 + x_3 + x_4) - f(x_0 + x_3 + x_5) - f(x_0 + x_4 + x_5) + f(x_0 + x_1) + f(x_0 + x_2) + f(x_0 + x_3) + f(x_0 + x_4) + f(x_0 + x_5) - f(x_0) = 0.$$

The desired result follows by noting that the left-hand side is indeed $\Delta_{x_1,\ldots,x_5} f(x_0)$. □

We are now ready to prove Theorem 1.3 I.

Replacing x by $x + y$ in (1.9), and subtracting (1.9) from the resulting expression, we obtain

$$f(x + 4y) - 14f(x + 2y) + 35f(x + y) - 35f(x) + 14f(x - y) - f(x - 3y) = 0,$$

which by Lemma 2.1 yields $\Delta_{x_1,\ldots,x_5} f(x_0) = 0$. Invoking upon Theorem 1.1 III, we deduce that

$$f(x) = A^4(x) + A^3(x) + A^2(x) + A^1(x) + A^0 \quad (x \in \mathbb{R}). \tag{2.13}$$

Now, replacing y by $-y$ in (1.9), we get

$$f(x + 3y) + f(x - 3y) + f(x + 2y) + f(x - 2y) + 22f(x) \tag{2.14}$$
$$= 13\,(f(x + y) + f(x - y)) + 168f(-y). \tag{2.15}$$

Subtracting (2.14) from (1.9), we get $f(y) = f(-y)$, i.e., f is even, which forces $A^3(x)$ and $A^1(x)$ to vanish. Since $f(0) = 0$, we have $A^0 \equiv 0$. Putting these into (2.13), we have

$$f(x) = A^4(x) + A^2(x). \tag{2.16}$$

Substituting f from (2.16) into (1.9), we obtain

$$\begin{aligned} &A^4(x + 3y) + A^4(x - 3y) + A^4(x + 2y) + A^4(x - 2y) + A^2(x + 3y) \\ &\quad + A^2(x - 3y) + A^2(x + 2y) + A^2(x - 2y) + 22\left(A^4(x) + A^2(x)\right) \\ &= 13\left(A^4(x + y) + A^4(x - y) + A^2(x + y) + A^2(x - y)\right) \\ &\quad + 168\left(A^4(y) + A^2(y)\right). \end{aligned}$$

Using

$$A^4(x + y) + A^4(x - y) = 2A^4(x) + 12A^{2,2}(x, y) + 2A^4(y)$$

$$\begin{aligned}
A^2(x+y)+A^2(x-y) &= 2A^2(x)+2A^2(y)\\
A^{2,2}(x,3y) &= 9A^{2,2}(x,y)\\
A^4(3y) &= 81A^4(y)\\
A^{2,2}(x,2y) &= 4A^{2,2}(x,y)\\
A^4(2y) &= 16A^4(y)\\
A^2(2y) &= 4A^2(y)\\
A^2(3y) &= 9A^2(y),
\end{aligned}$$

we get

$$\begin{aligned}
&2A^4(x)+108A^{2,2}(x,y)+162A^4(y)+2A^4(x)+48A^{2,2}(x,y)+32A^4(y)\\
&\quad +2A^2(x)+18A^2(y)+2A^2(x)+8A^2(y)+22\left(A^4(x)+A^2(x)\right)\\
&=13\left(2A^4(x)+12A^{2,2}(x,y)+2A^4(y)+2A^2(x)+2A^2(y)\right)\\
&\quad +168\left(A^4(y)+A^2(y)\right),
\end{aligned}$$

which simplifies to $A^2(y)=0$, and so (2.16) yields $f(x)=A^4(x)$.

Conversely, assume that there exists a 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$ such that $f(x)=A^4(x)$. Since

$$\begin{aligned}
A^4(x+y)+A^4(x-y) &= 2A^4(x)+12A^{2,2}(x,y)+2A^4(y),\\
A^{2,2}(x,2y) &= 4A^{2,2}(x,y),\ A^4(2y)=16A^4(y),
\end{aligned}$$

we have

$$\begin{aligned}
&f(x+3y)+f(x-3y)+f(x+2y)+f(x-2y)+22f(x)\\
&=A^4(x+3y)+A^4(x-3y)+A^4(x+2y)+A^4(x-2y)+22A^4(x)\\
&=2A^4(x)+108A^{2,2}(x,y)+162A^4(y)+2A^4(x)+48A^{2,2}(x,y)\\
&\quad +32A^4(y)+22A^4(x)\\
&=26A^4(x)+156A^{2,2}(x,y)+194A^4(y)\\
&=13\left(2A^4(x)+12A^{2,2}(x,y)+2A^4(y)\right)+168A^4(y)\\
&=13\left(f(x+y)+f(x-y)\right)+168f(y).
\end{aligned}$$

3. Proof of Theorem 1.3 II

We first prove an auxialiary result.

Lemma 3.1. *If $T : \mathbb{R} \to \mathbb{R}$ satisfies (1.9), then $T(2^n y) = 16^n T(y)$ for each $n \in \mathbb{N}$.*

Proof. The case $n = 1$ follows at once from Theorem 1.3 I, namely,

$$T(2y) = A^4(2y) = 16A^4(y) = 16T(y).$$

Assume the assertion holds up to k. Since $A^4(y)$ is the diagonal of a 4-additive symmetric function $A_4 : \mathbb{R}^4 \to \mathbb{R}$ and $T(y) = A^4(y)$, using induction we have

$$\begin{aligned} T(2^{k+1}y) &= A^4(2^k y + 2^k y) = A_4(2^k y + 2^k y, 2^k y + 2^k y, 2^k y + 2^k y, 2^k y + 2^k y) \\ &= 16A_4(2^k y, 2^k y, 2^k y, 2^k y) = 16A^4(2^k y) = 16T(2^k y) = 16^{k+1}T(y). \end{aligned}$$

□

Now we proceed to prove theorem 1.3 II. Putting $x = 3y$ in (1.13) and using $f(0) = 0$, we have

$$|f(6y) + f(5y) - 13f(4y) + 22f(3y) - 13f(2y) - 167f(y)| \le \phi(3y, y). \quad (3.1)$$

Putting $x = 2y$ in (1.13), using $f(0) = 0$ and f is even, we have

$$|f(5y) + f(4y) - 13f(3y) + 22f(2y) - 180f(y)| \le \phi(2y, y). \quad (3.2)$$

From (3.1) and (3.2), we obtain

$$|f(6y) - 14f(4y) + 35f(3y) - 35f(2y) + 13f(y)| \le \phi(3y, y) + \phi(2y, y). \quad (3.3)$$

Putting $x = 0$ in (1.13), using $f(0) = 0$ and f is even, we have

$$|f(3y) + f(2y) - 97f(y)| \le \frac{\phi(0, y)}{2}. \quad (3.4)$$

Replacing y by $2y$ in (3.4). we obtain

$$|f(6y) + f(4y) - 97f(2y)| \le \frac{\phi(0, 2y)}{2}. \quad (3.5)$$

From (3.3) and (3.5), we obtain

$$|-15f(4y) + 35f(3y) + 62f(2y) + 13f(y)| \le \phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2}. \quad (3.6)$$

Putting $x = y$ in (1.13), using $f(0) = 0$ and f is even, we have

$$|15f(4y) + 15f(3y) - 180f(2y) - 2175f(y)| \leq 15\phi(y, y). \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$|50f(3y) - 118f(2y) - 2162f(y)| \leq \phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2} + 15\phi(y, y). \tag{3.8}$$

From (3.4), we have

$$|50f(3y) + 50f(2y) - 4850f(y)| \leq 25\phi(0, y). \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$|f(2y) - 16f(y)| \leq \frac{1}{168}\left(\phi(3y, y) + \phi(2y, y) + \frac{\phi(0, 2y)}{2} + 15\phi(y, y) + 25\phi(0, y)\right). \tag{3.10}$$

Dividing by 16 in (3.10), we have

$$\left|\frac{f(2y)}{16} - f(y)\right| \leq \frac{1}{5376}[2\phi(3y, y) + 2\phi(2y, y) + \phi(0, 2y) + 30\phi(y, y) + 50\phi(0, y)]. \tag{3.11}$$

Replacing y by $2y$ in (3.11) and dividing by 16, we obtain

$$\begin{aligned}&\left|\frac{f(2^2y)}{(16)^2} - \frac{f(2y)}{16}\right| \\ &\leq \frac{1}{5376}\left[\frac{2\phi(3(2y), 2y) + 2\phi(2(2y), 2y) + \phi(0, 2(2y)) + 30\phi(2y, 2y) + 50\phi(0, 2y)}{16}\right].\end{aligned} \tag{3.12}$$

From the equations (3.11) and (3.12), we have

$$\begin{aligned}&\left|\frac{f(2^2y)}{(16)^2} - f(y)\right| \\ &\leq \frac{1}{5376}\Big(\frac{2\phi(3(2y), 2y) + 2\phi(2(2y), 2y) + \phi(0, 2(2y)) + 30\phi(2y, 2y) + 50\phi(0, 2y)}{16} \\ &\quad + 2\phi(3y, y) + 2\phi(2y, y) + \phi(0, 2y) + 30\phi(y, y) + 50\phi(0, y)\Big).\end{aligned} \tag{3.13}$$

By induction, (3.13) extends to

$$\begin{aligned}
&\left|\frac{f(2^n y)}{16^n} - f(y)\right| \qquad (3.14)\\
&\leq \frac{1}{5376}\sum_{i=0}^{n-1}\frac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+30\phi(2^i y,2^i y)+50\phi(o,2^i y)}{16^i}\\
&\leq \frac{1}{5376}\sum_{i=0}^{\infty}\frac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+30\phi(2^i y,2^i y)+50\phi(o,2^i y)}{16^i}\\
&\qquad (y\in\mathbb{R}, n\in\mathbb{N}).
\end{aligned}$$

Next, we show that $\{f(2^n y)/16^n\}$ is a Cauchy sequence. For $m,n\in\mathbb{N}$, we have

$$\begin{aligned}
&\left|\frac{f(2^{n+m}y)}{16^{n+m}} - \frac{f(2^m y)}{16^m}\right| = \frac{1}{16^m}\left|\frac{1(2^n 2^m y)}{16^n} - f(2^m y)\right| \qquad (3.15)\\
&\leq \frac{1}{5376}\sum_{i=0}^{\infty}\frac{1}{16^{i+m}}(2\phi(3(2^{i+m}y),2^{i+m}y)+2\phi(2(2^{i+m}y),2^{i+m}y)\\
&\quad + \phi(0,2(2^{i+m}y))+30\phi(2^{i+m}y,2^{i+m}y)+50\phi(0,2^{i+m}y))\\
&\to 0 \qquad (n\to\infty),
\end{aligned}$$

i.e., the sequence $\{f(2^n y)/16^n\}$ is a Cauchy sequence. Since $\mathbb{R}$ is complete, the limit function $T(y)=\lim_{n\to\infty} f(2^n y)/16^n$ exists for all $y\in\mathbb{R}$. From (3.14), we obtain

$$\begin{aligned}
&|f(y)-T(y)| = \left|\lim_{n\to\infty}\frac{f(2^n y)}{16^n} - f(y)\right| \qquad (3.16)\\
&\leq \frac{1}{5376}\sum_{i=0}^{\infty}\frac{2\phi(3(2^i y),2^i y)+2\phi(2(2^i y),2^i y)+\phi(0,2(2^i y))+30\phi(2^i y,2^i y)+50\phi(o,2^i y)}{16^i}.
\end{aligned}$$

Next, we show that T satisfies the equation (1.9). Consider

$$\begin{aligned}
D_T(x,y) &= T(x+3y)+T(x-3y)+T(x+2y)+T(x-2y)+22T(x)\\
&\quad - 13T(x+y)-13T(x-y)-168T(y)\\
&= \lim_{n\to\infty}\frac{1}{16^n}(f(2^n(x+3y))+f(2^n(x-3y))+f(2^n(x+2y))\\
&\quad + f(2^n(x-2y))+22f(2^n x)-13f(2^n(x+y))-13f(2^n(x-y))\\
&\quad - 168f(2^n y))\\
&= \lim_{n\to\infty}\frac{D_f(2^n x,2^n y)}{16^n}.
\end{aligned}$$

Thus,

$$|D_T(x,y)| \leq \lim_{n\to\infty} \frac{1}{16^n} |D_f(2^n x, 2^n y)| \leq \lim_{n\to\infty} \frac{\phi(2^n x, 2^n y)}{16^n} = 0 \quad (x, y \in \mathbb{R}),$$

i.e., T satisfies (1.9). To prove the uniqueness of T, suppose that there exists $S : \mathbb{R} \to \mathbb{R}$ satisfying (1.9) and (1.14). Lemma 3.1 gives

$$\begin{aligned} |T(y) - S(y)| &= \left| \frac{T(2^n y)}{16^n} - \frac{S(2^n y)}{16^n} \right| \\ &\leq \frac{1}{16^n} |T(2^n y) - f(2^n y)| + \frac{1}{16^n} |f(2^n y) - S(2^n y)| \\ &\leq \quad \frac{2}{5376} \sum_{i=0}^{\infty} \frac{1}{16^{i+n}} (2\phi(3(2^{i+n}y), 2^{i+n}y) + 2\phi(2(2^{i+n}y), 2^{i+n}y) \\ &\quad + \phi(0, 2(2^{i+n}y)) + 30\phi(2^{i+n}y, 2^{i+n}y) + 50\phi(0, 2^{i+n}y)) \\ &\to 0 \qquad (n \to \infty), \end{aligned}$$

which completes the proof.

It is worth remarking that with a much longer proof the conditions that f is even and $f(0) = 0$ can be omitted.

Corollary 3.2. *Let $\varepsilon > 0$. If $f : \mathbb{R} \to \mathbb{R}$ is even with $f(0) = 0$ and satisfies $|D_f(x,y)| \leq \varepsilon$, then there exists a unique $T : \mathbb{R} \to \mathbb{R}$ such that T satisfies* (1.9) *and*

$$|f(y) - T(y)| \leq \frac{17\varepsilon}{1008} \qquad (y \in \mathbb{R}).$$

Proof. Taking $\phi(x,y) = \varepsilon$, the condition (1.12) holds and Theorem (1.3) II shows that there exists a unique $T : \mathbb{R} \to \mathbb{R}$ satisfying (1.9) and

$$|f(y) - T(y)| \leq \frac{1}{5376} \sum_{i=0}^{\infty} \frac{2\varepsilon + 2\varepsilon + \varepsilon + 30\varepsilon + 50\varepsilon}{16^i} = \frac{17\varepsilon}{1008}.$$

□

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