International Journal of Pure and Applied Mathematics
Volume 85 No. 4 2013, 707-718
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i4.7

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THE MINIMUM DOMINATING ENERGY OF A GRAPH

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Abstract: Recently Professor Chandrashekar Adiga et al [3] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Motivated by this, we introduced minimum dominating energy of a graph $E_D(G)$ and computed minimum dominating energies of a star graph, complete graph, crown graph and cocktail graphs. Upper and lower bounds for $E_D(G)$ are established.

AMS Subject Classification: 05C50, 05C69
Key Words: minimum dominating set, minimum dominating matrix, minimum dominating eigenvalues, minimum dominating energy of a graph

1. Introduction

The concept of energy of a graph was introduced by I. Gutman [7] in the year 1978. Let G be a graph with n vertices and m edges and let $A = (a_{ij})$ be the adjacency matrix of the graph. The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of A, assumed in non increasing order, are the eigenvalues of the graph G. As A is real symmetric, the eigenvalues of G are real with sum equal to zero. The energy $E(G)$ of G is

Received: January 25, 2013

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defined to be the sum of the absolute values of the eigenvalues of G. i.e.,$E(G)$ $= \sum_{i=1}^{n} |\lambda_i|$.

For details on the mathematical aspects of the theory of graph energy see the reviews[8], papers [4, 5, 9] and the references cited there in.The basic properties including various upper and lower bounds for energy of a graph have been established in [11, 12], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [6, 10].

2. The Minimum Dominating Energy

Let G be a simple graph of order n with vertex set $V = \{v_1, v_2, ..., v_n\}$ and edge set E. A subset D of V is called a dominating set of G if every vertex of V-D is adjacent to some vertex in D. Any dominating set with minimum cardinality is called a minimum dominating set. Let D be a minimum dominating set of a graph G. The minimum dominating matrix of G is the $n \times n$ matrix defined by $A_D(G) := (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 1 & \text{if } i = j \text{ and } v_i \in D, \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic polynomial of$A_D(G)$ is denoted by $f_n(G, \lambda)$= $det(\lambda I - A_D(G))$. The minimum dominating eigenvalues of the graph G are the eigenvalues of $A_D(G)$. Since $A_D(G)$ is real and symmetric, its eigenvalues are real numbers and we label them in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The minimum dominating energy of G is defined as

$$E_D(G) := \sum_{i=1}^{n} |\lambda_i|.$$

Note that the trace of $A_D(G)$ = Domination Number = k.

Example 1. The possible minimum dominating sets for the following graph G are:

i) D_1=$\{v_1, v_5\}$;

ii) D_2=$\{v_2, v_5\}$ iii) D_3=$\{v_2, v_6\}$.

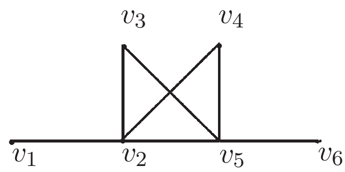

Figure 1

i) $A_{D_1}(G) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Characteristic equation is $\lambda^6 - 2\lambda^5 - 6\lambda^4 + 5\lambda^3 + 7\lambda^2 - 2\lambda = 0$.

Minimum dominating eigen values are $\lambda_1 \approx -1.6473, \lambda_2 \approx -1.1263, \lambda_3 \approx 0, \lambda_4 \approx 0.2546, \lambda_5 \approx 1.3261,\ \lambda_6 \approx 3.1929$.

Minimum dominating energy, $E_{D_1}(G) \approx 7.5471$.

ii) $A_{D_2}(G) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Characteristic equation is $\lambda^6 - 2\lambda^5 - 6\lambda^4 + 2\lambda^3 + 5\lambda^2 = 0$.

Minimum dominating eigen values are $\lambda_1 \approx -1.4495, \lambda_2 \approx -1, \lambda_3 \approx 0, \lambda_4 \approx 0, \lambda_5 \approx 1,\ \lambda_6 \approx 3.4495$.

Minimum domination energy, $E_{D_2}(G) \approx 6.8990$.

Minimum dominating energy depends on the dominating set.

3. Minimum Dominating Energy OF Some Standard Graphs

Definition 3.1. The Cocktail party graph is denoted by $K_{n\times 2}$,is a graph having the vertex set $V = \bigcup_{i=1}^{n} \{u_i, v_i\}$ and the edge set $E = \{u_iu_j, v_iv_j : i \neq$

$j\} \bigcup \{u_i v_j, v_i u_j : 1 \leq i < j \leq n\}$.

Theorem 3.1. *The minimum dominating energy of Cocktail party graph* $K_{n\times 2}$ *is* $(2n-3)+\sqrt{4n^2-4n-9}$.

Proof. Let $K_{n\times 2}$ be the Cocktail party graph with vertex set $V = \bigcup_{i=1}^{n}\{u_i, v_i\}$. The minimum dominating set is $D = \{u_1, v_1\}$. Then

$$A_D(K_{n\times 2}) = \begin{pmatrix} 1 & 0 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & \ldots & 1 & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \ldots & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & 0 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ 0 & \lambda-1 & -1 & -1 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & \lambda & 0 & \ldots & -1 & -1 & -1 & -1 \\ -1 & -1 & 0 & \lambda & \ldots & -1 & -1 & -1 & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & -1 & -1 & \ldots & \lambda & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & \ldots & 0 & \lambda & -1 & -1 \\ -1 & -1 & -1 & -1 & \ldots & -1 & -1 & \lambda & 0 \\ -1 & -1 & -1 & -1 & \ldots & -1 & -1 & 0 & \lambda \end{vmatrix}.$$

Characteristic equation is $\lambda^{n-1}(\lambda-1)(\lambda+2)^{(n-2)}(\lambda^2-(2n-3)\lambda-2n)=0$.
Minimum dominating eigen values are:

$\lambda = 0$ $[(n-1)\text{times}]$, $\lambda = 1$ [one time],

$\lambda = -2[(n-2)$ times$]$, $\lambda = \dfrac{(2n-3)\pm\sqrt{4n^2-4n+9}}{2}$ [one time each].
Minimum dominating energy

$E_D(K_{n\times 2})$

$$= 0+1+|-2|(n-2)+\left|\frac{(2n-3)+\sqrt{4n^2-4n+9}}{2}\right|+\left|\frac{(2n-3)-\sqrt{4n^2-4n+9}}{2}\right|$$
$$= 1+2(n-2)+\sqrt{4n^2+4n-7} = 2n-3+\sqrt{4n^2-4n+9}. \quad \square$$

Theorem 3.2. *For $n \geq 2$, the minimum dominating energy of Star graph $K_{1,n-1}$ is equal to $\sqrt{4n-3}$.*

Proof. Consider the Star graph $K_{1,n-1}$ with vertex set $V = \{v_0, v_1, v_2, ..., v_{n-1}\}$. Minimum dominating set is $D = \{v_0\}$. Then

$$A_D(K_{1,n-1}) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}_{n\times n},$$

Characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & -1 & -1 & \dots & -1 \\ -1 & \lambda & 0 & \dots & 0 \\ -1 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & \lambda \end{vmatrix}.$$

Characteristic equation is $\lambda^{n-2}(\lambda^2-\lambda-(n-1)) = 0$.

The minimum dominating eigen values are:

$\lambda = 0$ [(n-2) times], $\lambda = \dfrac{1 \pm \sqrt{4n-3}}{2}$ [one time each].

Minimum dominating energy is

$$E_D(K_{1,n-1}) = |0|(n-2)+\left|\frac{1+\sqrt{4n-3}}{2}\right|+\left|\frac{1-\sqrt{4n-3}}{2}\right| = \sqrt{4n-3}. \quad \square$$

Theorem 3.3. *For $n \geq 2$, the minimum dominating energy of complete graph K_n is $(n-2)+\sqrt{n^2-2n+5}$.*

Proof. K_n is Complete graph with vertex set $V = \{v_1, v_2, ..., v_n\}$. The minimum dominating set is $D = \{v_1\}$. Then:

$$A_D(K_n) = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ 1 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 0 \end{pmatrix}_{n\times n},$$

Characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & -1 & -1 & \dots & -1 & -1 \\ -1 & \lambda & -1 & \dots & -1 & -1 \\ -1 & -1 & \lambda & \dots & -1 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & -1 & \dots & -1 & \lambda \end{vmatrix}.$$

Characteristic equation is $(\lambda+1)^{n-2}(\lambda^2-(n-1)\lambda-1)=0$.
The minimum dominating eigen values are

$\lambda =$ -1 [(n-2) times], $\lambda = \dfrac{(n-1)\pm\sqrt{n^2-2n+5}}{2}$ [one time each].
Minimum dominating energy is

$$E_D(K_n) = |-1|(n-2)+\left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right|+\left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right|$$
$$= (n-2)+\sqrt{n^2-2n+5}. \quad \square$$

Definition 3.2. The crown graph S_n^0 for an integer $n \geq 2$ is the graph with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ and edge set $\{u_i v_j : 1 \leq i, j \leq n, i \neq j\}$.

S_n^0 coincides with the complete bipartite graph $K_{n,n}$ with horizontal edges removed.

Theorem 3.4. *For $n \geq 2$, the minimum dominating energy of the crown graph S_n^0 is equal to* $2(n-2)+\sqrt{n^2-2n+5}+\sqrt{n^2+2n-3}$.

Proof. For the crown graph S_n^0 with vertex set $V = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$, minimum dominating set is $S = \{u_1, v_1\}$. Then

$$A_D(S_n^0) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 1 & 1 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 0 & 1 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}_{(2n\times 2n)}$$

Characteristic polynomial is

$$\begin{vmatrix} \lambda-1 & 0 & 0 & \dots & 0 & 0 & -1 & -1 & \dots & -1 \\ 0 & \lambda & 0 & \dots & 0 & -1 & 0 & -1 & \dots & -1 \\ 0 & 0 & \lambda & \dots & 0 & -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda & -1 & -1 & -1 & \dots & 0 \\ 0 & -1 & -1 & \dots & -1 & \lambda-1 & 0 & 0 & \dots & 0 \\ -1 & 0 & -1 & \dots & -1 & 0 & \lambda & 0 & \dots & 0 \\ -1 & -1 & 0 & \dots & -1 & 0 & 0 & \lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 & 0 & 0 & 0 & \dots & \lambda \end{vmatrix}$$

Characteristic equation is

$$(\lambda-1)^{n-2}(\lambda+1)^{n-2}(\lambda^2-(n-1)\lambda-1)(\lambda^2+(n-3)\lambda-(2n-3))=0$$

Minimum dominating eigen values are $\lambda = 1[(n-2)\text{times}]$,

$\lambda = -1 \quad [(n-2)\text{times}]$

$\lambda = \dfrac{(n-1) \pm \sqrt{n^2-2n+5}}{2}$, [one time each],

$\lambda = \dfrac{(3-n) \pm \sqrt{n^2+2n-3}}{2}$ [one time each]

Minimum dominating energy

$$\begin{aligned} E_D(S_n^0) &= 1(n-2) + |-1|(n-2) \\ &\quad + \left|\frac{(n-1)+\sqrt{n^2-2n+5}}{2}\right| + \left|\frac{(n-1)-\sqrt{n^2-2n+5}}{2}\right| \\ &\quad + \left|\frac{(3-n)+\sqrt{n^2+2n-3}}{2}\right| + \left|\frac{(3-n)-\sqrt{n^2+2n-3}}{2}\right| \\ &= 2(n-2) + \sqrt{n^2-2n+5} + \sqrt{n^2+2n-3}. \end{aligned}$$ □

4. Properties of Minimum Dominating Eigen Values

Theorem 4.1. *Let G be a simple graph with vertex set $V = \{v_1, v_2, ..., v_n\}$, edge set E and $D = \{u_1, u_2, ..., u_k\}$ be a minimum dominating set. If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigen values of minimum dominating matrix $A_D(G)$ then:*

(i) $\sum_{i=1}^{n} \lambda_i = |D|;$

(ii) $\sum_{i=1}^{n} \lambda_i^2 = 2|E| + |D|.$

Proof. (i) We know that the sum of the eigen values of $A_D(G)$ is the trace of $A_D(G)$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii} = |D| = k.$$

(ii) Similarly the sum of squares of the eigen values of $A_D(G)$ is trace of $[A_D(G)]^2$

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n} (a_{ii})^2 + \sum_{i \neq j} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n} (a_{ii})^2 + 2\sum_{i<j} (a_{ij})^2 \\
&= | D | + 2 | E | .
\end{aligned}$$

□

5. Bounds for Minimum Dominating Energy

Similar to McClelland's [12] bounds for energy of a graph, bounds for $E_D(G)$ are given in the following theorem.

Theorem 5.1. *Let G be a simple graph with n vertices and m edges . If D is the minimum dominating set and $P = |det A_D(G)|$ then*

$$\sqrt{(2m+k) + n(n-1)P^{\frac{2}{n}}} \leq E_D(G) \leq \sqrt{n(2m+k)},$$

where k is domination number.

Proof.

Cauchy Schwarz inequality is $\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$

$$\text{If } a_i = 1, b_i = \mid \lambda_i \mid \text{ then } \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2 \le \Big(\sum_{i=1}^{n} 1\Big)\Big(\sum_{i=1}^{n} \lambda_i^2\Big)$$

$$[E_D(G)]^2 \le n(2m+k) \qquad [\text{Theorem 4.1}]$$

$$\Longrightarrow E_D(G) \le \sqrt{n(2m+k)}$$

Since arithmetic mean is not smaller than geometric mean we have

$$\frac{1}{n(n-1)}\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \ge \Big[\prod_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} \mid \lambda_i \mid^{2(n-1)}\Big]^{\frac{1}{n(n-1)}}$$

$$= \Big[\prod_{i=1}^{n} \mid \lambda_i \mid\Big]^{\frac{2}{n}}$$

$$= \Big|\prod_{i=1}^{n} \lambda_i\Big|^{\frac{2}{n}}$$

$$= |det A_D(G)|^{\frac{2}{n}} = P^{\frac{2}{n}}$$

$$\sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid \ge n(n-1)P^{\frac{2}{n}} \tag{1}$$

$$\text{Now consider,} \quad [E_D(G)]^2 = \Big(\sum_{i=1}^{n} \mid \lambda_i \mid\Big)^2$$

$$= \sum_{i=1}^{n} \mid \lambda_i \mid^2 + \sum_{i\neq j} \mid \lambda_i \mid\mid \lambda_j \mid$$

$$[E_D(G)]^2 \ge (k+2m) + n(n-1)P^{\frac{2}{n}} \qquad [\text{From } (5.1)]$$

$$i.e., \ E_D(G) \ge \sqrt{(k+2m) + n(n-1)P^{\frac{2}{n}}}. \qquad \square$$

Theorem 5.2. *If $\lambda_1(G)$ is the largest minimum dominating eigen value of $A_D(G)$,then $\lambda_1(G) \ge \dfrac{2m+k}{n}$ where k is the domination number.*

Proof. Let X be any nonzero vector .Then by [1],
We have $\lambda_1(A) = \max_{X \neq 0} \left\{ \frac{X'AX}{X'X} \right\}$.
$\lambda_1(A) \geq \frac{J'AJ}{J'J} = \frac{2m+k}{n}$ where J is a unit matrix. □

Similar to Koolen and Moulton's [13] upper bound for energy of a graph,upper bound for $E_D(G)$ is given in the following theorem.

Theorem 5.3. *If G is a graph with n vertices and m edges and $(2m+k) \geq n$ then*

$$E_D(G) \leq \frac{2m+k}{n} + \sqrt{(n-1)\Big[(2m+k) - \Big(\frac{2m+k}{n}\Big)^2\Big]},$$

where k is a domination number.

Proof. Cauchy-Schwartz inequality is

$$\Big[\sum_{i=2}^{n} a_i b_i\Big]^2 \leq \Big(\sum_{i=2}^{n} a_i^2\Big)\Big(\sum_{i=2}^{n} b_j^2\Big)$$

Put $a_i = 1, b_i =\mid \lambda_i \mid$ then$\Big(\sum_{i=2}^{n} \mid \lambda_i \mid\Big)^2 = \sum_{i=2}^{n} 1 \sum_{i=2}^{n} \lambda_i^2$

$$\Rightarrow [E_D(G) - \lambda_1]^2 \leq (n-1)(2m+k-\lambda_1^2)$$

$$\Rightarrow E_D(G) \leq \lambda_1 + \sqrt{(n-1)(2m+k-\lambda_1^2)}$$

Let $f(x) = x + \sqrt{(n-1)(2m+k-x^2)}$

For decreasing function $f'(x) \leq 0 \Rightarrow 1 - \frac{x(n-1)}{\sqrt{(n-1)(2m+k-x^2)}} \leq 0$

$$\Rightarrow x \geq \sqrt{\frac{2m+k}{n}}$$

Since $(2m+k) \geq n$, we have $\sqrt{\frac{2m+k}{n}} \leq \frac{2m+k}{n} \leq \lambda_1$

$f(\lambda_1) \leq f\Big(\frac{2m+k}{n}\Big)$

i.e. $E_D(G) \leq f(\lambda_1) \leq f\Big(\frac{2m+k}{n}\Big)$

$$\text{i.e. } E_D(G) \leq f\Big(\frac{2m+k}{n}\Big)$$
$$\text{i.e. } E_D(G) \leq \frac{2m+k}{n} + \sqrt{(n-1)\Big[2m+k-\Big(\frac{2m+k}{n}\Big)^2\Big]}. \qquad \square$$

Bapat and S.Pati [2]proved that if the graph energy is a rational number then it is an even integer.Similar result for minimum dominating energy is given in the following theorem.

Theorem 5.4. *Let G be a graph with a minimum dominating set D.If the minimum dominating energy $E_D(G)$ is a rational number,then $E_D(G) \equiv \mid D \mid$(mod 2).*

Proof. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be minimum dominating eigen values of a graph G of which $\lambda_1, \lambda_2, ..., \lambda_r$ are positive and the rest are non-positive, then

$$\sum_{i=1}^{n} \mid \lambda_i \mid = (\lambda_1 + \lambda_2 + ... + \lambda_r) \text{ - } (\lambda_{r+1} + ... + \lambda_n)$$
$$\sum_{i=1}^{n} \mid \lambda_i \mid = 2(\lambda_1 + \lambda_2 + ... + \lambda_r) \text{ - } (\lambda_1 + \lambda_2 + ... + \lambda_n)$$
$$\text{i.e. } E_D(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - \sum_{i=1}^{n} \lambda_i$$
$$\text{i.e. } E_D(G) = 2(\lambda_1 + \lambda_2 + ... + \lambda_r) - \mid D \mid$$

$E_D(G) \equiv \mid D \mid (mod 2)$.
Hence the theorem holds true. $\square$

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