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ANALOGOUS TO RAMANUJAN'S REMARKABLE PRODUCT OF THETA FUNCTION OF DEGREE 5 AND THEIR EXPLICIT EVALUATION

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Abstract: In this paper, We study the Analogous of Ramanujan's Remarkable product of theta-function of degree 5 of and their explicit values.

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1. Introduction

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad | ab |< 1,$$

$$= (-a;ab)_{\infty}(-b;ab)_{\infty}(ab;ab)_{\infty}, \quad (1)$$

where $(a;q)_0 = 1$ and $(a;q)_{\infty} = (1-a)(1-aq)(1-aq^2)\cdots$.

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Following Ramanujan, we defined

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty \tag{4}$$

and

$$\chi(q) := (-q;q^2)_\infty. \tag{5}$$

On page 338 in his first notebook [11, p.338], Ramanujan defines

$$a_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{6}$$

He then, on pages 338 and 339, offers a list of eighteen particular values. All these eighteen values have been established by Berndt, Chan and Zhang [1]. An account of these can be found in Berndt's book [3], M. S. Mahadeva Naika and B. N. Dharmendra [4], also established some general theorems for explicit evaluations of the product of $a_{m,n}$ and found some new explicit values therefrom. Further results on $a_{m,n}$ can be found by Mahadeva Naika, Dharmendra and K. Shivashankar [6], and Mahadeva Naika and M. C. Mahesh Kumar [7].

In [8], Mahadeva Naika et al. defined the product

$$b_{m,n} = \frac{n e^{\frac{-(n-1)\pi}{4}\sqrt{\frac{m}{n}}} \psi^2(-e^{-\pi\sqrt{mn}})\varphi^2(-e^{-2\pi\sqrt{mn}})}{\psi^2(-e^{-\pi\sqrt{\frac{m}{n}}})\varphi^2(-e^{-2\pi\sqrt{\frac{m}{n}}})}. \tag{7}$$

They established general theorems for explicit evaluation of $b_{m,n}$ and obtained some particular values. Mahadeva Naika et al. [9] established general formulas for explicit values of Ramanujan's cubic continued fraction $V(q)$ in terms of the products $a_{m,n}$ and $b_{m,n}$ defined above, where

$$V(q) := \frac{q^{1/3}}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots, \quad |q|<1, \tag{8}$$

and found some particular values of $V(q)$

In [10], Nipen Saikia defined the product of theta-fuctions $I_{m,n}$ as

$$I_{m,n} = \frac{\psi(-q)\varphi(q^m)}{q^{(m-1)/8}\psi(-q^m)\varphi(q)}; \quad e^{-\pi\sqrt{\frac{n}{m}}}, \tag{9}$$

where m and n are positive real numbers. We establish several properties of the product $I_{m,n}$. They prove general formulas for explicit evaluations of evaluation of $I_{m,n}$ and find its explicit values.

In this paper, we establish several General Theorems and explicit evaluation of $I_{5,n}$.

Now we define a modular equation in brief. The ordinary hypergeometric series
${}_2F_1(a, b; c; x)$ is defined by

$$ {}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n, $$

where $(a)_0 = 1, (a)_n = a(a+1)(a+2)\cdots(a+n-1)$ for any positive integer n, and $| x |< 1$.
Let

$$z := z(x) := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \tag{10}$$

and

$$q := q(x) := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \tag{11}$$

where $0 < x < 1$.
Let r denote a fixed natural number and assume that the following relation holds:

$$r\frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}. \tag{12}$$

Then a modular equation of degree r in the classical theory is a relation between α and β induced by (12). We often say that β is of degree r over α and $m := \dfrac{z(\alpha)}{z(\beta)}$ is called the multiplier. We also use the notations $z_1 := z(\alpha)$ and $z_r := z(\beta)$ to indicate that β has degree r over α.

2. Preliminary Results

Lemma 1. *If* $P = \dfrac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}$ *and* $Q = \dfrac{\varphi(q)}{\varphi(q^5)}$, *then*

$$P^2 + P^2Q^2 = 5 + Q^2. \tag{13}$$

Lemma 2. *[5] If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^2)}{\varphi(q^{10})}$, *then*

$$\begin{aligned}&\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + (PQ)^2 + \left(\frac{5}{PQ}\right)^2 + 16\left(\frac{P}{Q} - \frac{Q}{P}\right)\\ &= 2\left(P^2 + \frac{5}{P^2}\right) + 2\left(Q^2 + \frac{5}{Q^2}\right) + 4.\end{aligned} \tag{14}$$

Lemma 3. *[3, Ch. 25, Entry 66, p.233] If* $P = \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q = \dfrac{\varphi(q^3)}{\varphi(q^{15})}$, *then*

$$PQ + \frac{5}{PQ} = -\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + 3\left(\frac{P}{Q} + \frac{Q}{P}\right). \tag{15}$$

Lemma 4. *[5] If* $P := \dfrac{\varphi(q)\varphi(q^4)}{\varphi(q^5)\varphi(q^{20})}$ *and* $Q := \dfrac{\varphi(q)\varphi(q^{20})}{\varphi(q^5)\varphi(q^4)}$, *then*

$$\begin{aligned}&Q^4 + \frac{1}{Q^4} - 112\left(Q^3 + \frac{1}{Q^3}\right) + 1440\left(Q^2 + \frac{1}{Q^2}\right)\\ &- 3184\left(Q + \frac{1}{Q}\right) + 7316 = 8\left(P + \frac{1}{P}\right)\left[22\left(Q^2 + \frac{1}{Q^2}\right)\right.\\ &\left.-31\left(Q + \frac{1}{Q}\right) + 170\right] - 2\left(P^2 + \frac{5^2}{P^2}\right)\left[3\left(Q^2 + \frac{1}{Q^2}\right)\right.\\ &\left.+24\left(Q + \frac{1}{Q}\right) + 64\right] + 4\left(P^3 + \frac{5^3}{P^3}\right)\left[\left(Q + \frac{1}{Q}\right) + 4\right].\end{aligned} \tag{16}$$

Lemma 5. *[5] If* $P := \dfrac{\varphi(q)}{\varphi(q^5)}$ *and* $Q := \dfrac{\varphi(q^5)}{\varphi(q^{25})}$, *then*

$$\begin{aligned}&\frac{Q^3}{P^3} - \frac{5Q^2}{P^2} - \frac{15Q}{P} + 5\left(PQ + \frac{5}{PQ}\right) + 5\left(Q^2 + \frac{5}{P^2}\right)\\ &= P^2Q^2 + \frac{5^2}{P^2Q^2} + 15.\end{aligned} \tag{17}$$

Lemma 6. *[10]*

$$I_{m,1} = 1 \tag{18}$$

Lemma 7. *[10]*

$$I_{m,n} I_{m,1/n} = 1 \tag{19}$$

3. General Theorems and Explicit Evaluations of $I_{5,n}$

Theorem 8. *If $X := I_{5,n}$ and $Y := I_{5,4n}$ then*

$$\begin{aligned}
&(41 - 19080y^6 - 477576y^{10} + 5832y^{14} + 59049y^{16} - 3588y^4 \\
&+ 456y^2 + 190134y^8 + 378108y^{12})x^{16} + (809280y^{14} + 5832y^{16} \\
&- 1621792y^4 - 10888000y^6 - 1394976y^{12} - 63680y^2 + 456 \\
&- 7002432y^{10} - 15707216y^8)x^{14} + (378108y^{16} + 12465808y^{12} \\
&- 29385584y^4 - 135695328y^6 - 3588 - 12611040y^{10} - 1621792y^2 \\
&- 15502360y^8 - 1394976y^{14})x^{12} + (295520720y^8 - 7002432y^{14} \\
&- 12611040y^{12} - 29291200y^{10} - 477576y^{16} - 19080 \\
&- 315184320y^6 - 10888000y^2 - 135695328y^4)x^{10} + (295520720y^6 \\
&- 15502360y^{12} + 190134y^{16} + 295520720y^{10} - 15707216y^2 \\
&+ 190134 - 15707216y^{14} + 1364493764y^8 - 15502360y^4)x^8 \\
&+ (295520720y^8 - 315184320y^{10} - 29291200y^6 - 7002432y^2 \\
&- 135695328y^{12} - 477576 - 19080y^{16} - 12611040y^4 - 10888000y^{14})x^6 \\
&+ (378108 - 15502360y^8 - 1394976y^2 - 29385584y^{12} + 12465808y^4 \\
&- 12611040y^6 - 135695328y^{10} - 1621792y^{14} - 3588y^{16})x^4 \\
&+ (456y^{16} - 15707216y^8 + 809280y^2 - 10888000y^{10} + 5832 \\
&- 63680y^{14} - 1621792y^{12} - 7002432y^6 - 1394976y^4)x^2 \\
&+ 59049 - 19080y^{10} + 5832y^2 + 456y^{14} - 477576y^6 \\
&+ 41y^{16} + 378108y^4 + 190134y^8 - 3588y^{12} = 0.
\end{aligned} \tag{20}$$

Proof. Employing the definition of $I_{m,n}$ (9) with $m = 5$, we obtain

$$I_{5,n} = \frac{\psi(-q)\varphi(q^5)}{q^{(1/2}\psi(-q^5)\varphi(q)}; \quad e^{-\pi\sqrt{\frac{n}{5}}} \tag{21}$$

By using lemma (1), we obtain

$$P = \sqrt{\frac{Q^2+5}{Q^2+1}}. \tag{22}$$

The above two equations (21) and (22) can be written as

$$I_{5,n} = \frac{Q^2+5}{Q^4+Q^2}, \tag{23}$$

where $Q = \dfrac{\varphi(q)}{\varphi(q^5)}$, the above the equation (23) can be written as

$$Q^2 = \frac{(1-a)+\sqrt{a^2+18a+1}}{2a}, \tag{24}$$

where $a = I_{5,n}$.

Employing the equation (24) in (14), we obtain (8) □

Corollary 9. *We have*

$$I_{5,2} = \left[\frac{(19-8\sqrt{5})(29+20\sqrt{2})}{41}\right]^{1/2}, \tag{25}$$

$$I_{5,1/2} = \left[\frac{(19+8\sqrt{5})(29-20\sqrt{2})}{41}\right]^{1/2}. \tag{26}$$

$$\begin{aligned} I_{5,4} &= \left[\frac{7711\left(17511+8280\sqrt{5}-106d+4\sqrt{38555}k\right)}{4017413}\right]^{1/2}, \\ I_{5,1/4} &= \left[\frac{7711\left(17511+8280\sqrt{5}-106d-4\sqrt{38555}k\right)}{4017413}\right]^{1/2}. \end{aligned} \tag{27}$$

where, $k := \sqrt{(5837+2760\sqrt{5})(10193698+4580131\sqrt{5}-92532d)}$,
$d := \sqrt{24371+10909\sqrt{5}}$

Proof. Setting $n = 1/2$ in Theorem (8) and using the Lemma (7), we obtain

$$(41-2204I_{5,2}+4726I_{5,2}^2-2204I_{5,2}^3+41I_{5,2}^4)(I_{5,2}+1)^4(I_{5,2}^2+18I_{5,2}+1)^4 = 0 \tag{28}$$

Since the root of the second and third factors are imaginary and $I_{5,2} < 10$ we deduce that

$$41 - 2204I_{5,2} + 4726I_{5,2}^2 - 2204I_{5,2}^3 + 41I_{5,2}^4 = 0 \tag{29}$$

On solving the above equation (29), we arrive at the equations (25) and (26).

Setting $n = 1$ in Theorem (8) and using the Lemma (6), we obtain

$$\begin{aligned}&(521 - 140088I_{5,4}^2 - 716292I_{5,4}^4 - 842376I_{5,4}^6 + 7396470I_{5,4}^8 - 842376I_{5,4}^{10}\\ &- 716292I_{5,4}^{12} - 140088I_{5,4}^{14} + 521I_{5,4}^{16})(59I_{5,4}^8 + 6284I_{5,4}^6 - 8286I_{5,4}^4\\ &+ 6284I_{5,4}^2 + 59)^2(I_{5,4}^{16} - 88I_{5,4}^{14} + 8668I_{5,4}^{12} - 38248I_{5,4}^{10} + 102598I_{5,4}^8\\ &- 38248I_{5,4}^6 + 8668I_{5,4}^4 - 88I_{5,4}^2 + 1)^2 = 0\end{aligned} \tag{30}$$

Since the root of the second and third factors are imaginary and $I_{5,4} > 0$ we deduce that

$$\begin{aligned}&(521 - 140088I_{5,4}^2 - 716292I_{5,4}^4 - 842376I_{5,4}^6 + 7396470I_{5,4}^8\\ &- 842376I_{5,4}^{10} - 716292I_{5,4}^{12} - 140088I_{5,4}^{14} + 521I_{5,4}^{16}) = 0\end{aligned} \tag{31}$$

by above equation (31) can be written as

$$521z^4 - 140088z^3 - 718376z^2 - 422112z + 8830096 = 0, \; z = I_{5,4}^2 + I_{5,4}^{-2}. \tag{32}$$

On solving the above equation (32), we arrive at the equations (27). □

Theorem 10. *If $X := I_{5,n}$ and $Y := I_{5,9n}$ then*

$$\begin{aligned}&(4908y^2 + 209 + 2940y^6 + 9y^8 - 7362y^4)x^{16} + (4908 + 189552y^8\\ &- 317359y^6 + 144339y^2 + 35670y^4 - 9y^{14} - 2994y^{12} - 9819y^{10})x^{14}\\ &+ (35670y^2 - 2994y^{14} + 5149482y^{10} - 944110y^{12} + 8930322y^6\\ &- 7362 - 3476010y^4 - 10022406y^8)x^{12} + (2940 + 43492872y^8\\ &+ 8930322y^4 - 317359y^2 - 25684089y^{10} - 30624573y^6\\ &+ 5149482y^{12} - 9819y^{14})x^{10} + (43492872y^6 - 68614774y^8\\ &- 10022406y^4 + 189552y^2 + 9 - 10022406y^{12} + 43492872y^{10}\\ &+ 189552y^{14} + 9y^{16})x^8 + (2940y^{16} - 317359y^{14} - 9819y^2\\ &+ 43492872y^8 - 25684089y^6 - 30624573y^{10} + 5149482y^4\\ &+ 8930322y^{12})x^6 + (-7362y^{16} - 3476010y^{12} - 10022406y^8\\ &- 944110y^4 + 35670y^{14} - 2994y^2 + 8930322y^{10} + 5149482y^6)x^4\end{aligned}$$

$$+ (35670y^{12} + 144339y^{14} - 2994y^4 + 189552y^8 + 4908y^{16}$$
$$- 9819y^6 - 317359y^{10} - 9y^2)x^2 - 7362y^{12} + 4908y^{14} + 9y^8$$
$$+ 209y^{16} + 2940y^{10} = 0. \tag{33}$$

Proof. Employing the equation (24) in (15), we obtain (33) □

Corollary 11. *We have*

$$I_{5,3} = \frac{(7+\sqrt{5})\sqrt{11}}{22}, \tag{34}$$

$$I_{5,1/3} = \frac{(7-\sqrt{5})\sqrt{11}}{22}. \tag{35}$$

$$I_{5,9} = \left[\frac{(15-8\sqrt{3})6\sqrt{5}+23(8-3\sqrt{5})}{41}\right]^{1/2}. \tag{36}$$

$$I_{5,1/9} = \left[\frac{(15+8\sqrt{3})6\sqrt{5}-23(8+3\sqrt{5})}{41}\right]^{1/2}. \tag{37}$$

Proof. Employing Theorem (10), Lemma (7) and (6), solving the resulting equation for $I_{5,3}$, $I_{5,9}$ and nothing that $I_{5,3} < 1$ and $I_{5,9} < 1$, we arrive (34)-(37) □

Theorem 12. *If $X := I_{5,n}$ and $Y := I_{5,25n}$ then*

$$(5 - 330y^4 + 840y^2 + 420y^6 - 291y^8 + 60y^{10})x^{20} + (-376735y^{10}$$
$$- 60y^{20} + 840 + 132000y^{12} - 384091y^6 - 309y^{18} + 6390y^{16}$$
$$- 31185y^{14} + 803920y^4 + 152720y^2 + 481790y^8)x^{18} + (803920y^2$$
$$- 29291516y^4 + 130285890y^{10} - 931795y^{20} + 63203590y^6$$
$$- 117602260y^8 + 67420470y^{14} - 33495204y^{16} - 97728055y^{12}$$
$$- 330 + 8886970y^{18})x^{16} + (63203590y^4 + 420 + 11199050y^{16}$$
$$+ 3713550y^{12} - 1530065y^{18} + 184185180y^{10} - 384091y^2$$
$$+ 62340y^{20} + 88653490y^6 - 228679750y^8 - 35882274y^{14})x^{14}$$
$$+ (9481830y^{14} - 46268690y^{10} + 74820y^{18} - 21431094y^{12}$$
$$+ 481790y^2 - 117602260y^4 - 291 + 234609170y^8 - 8730y^{20}$$
$$- 228679750y^6 - 1223835y^{16})x^{12} + (60 - 340650y^{16} + 11025y^{18}$$
$$- 376735y^2 + 4320y^{20} + 130285890y^4 + 1274110y^{12} - 3446698y^{10}$$

$$
\begin{aligned}
&+184185180y^6-46268690y^8+681020y^{14})x^{10}+(132000y^2\\
&+132000y^2-771150y^{14}-21431094y^8-891y^{20}-270y^{18}\\
&+103410y^{16}+3713550y^6-97728055y^4+1274110y^{10}\\
&+2458150y^{12})x^8+(681020y^{10}-891y^{18}+103410y^{14}\\
&+9481830y^8-31185y^2+67420470y^4-35882274y^6\\
&-270y^{16}-771150y^{12})x^6+(103410y^{12}-340650y^{10}\\
&-891y^{16}-33495204y^4+6390y^2+11199050y^6-1223835y^8\\
&-270y^{14})x^4+(74820y^8-1530065y^6+8886970y^4+11025y^{10}\\
&-891y^{14}-891y^{12}-309y^2-270y^{12})x^2-931795y^4+4320y^{10}\\
&-8730y^8+62340y^6-60y^2=0. \qquad (38)
\end{aligned}
$$

Proof. Employing the equation (24) in (17), we obtain (38) □

Corollary 13. *We have*

$$I_{5,5}=\sqrt{\frac{13+6\sqrt{5}}{11}}, \qquad (39)$$

$$I_{5,1/5}=\sqrt{6\sqrt{5}-13}. \qquad (40)$$

Proof. Employing Theorem (12) and Lemma (7), solving the resulting equation for $I_{5,3}$ and nothing that $I_{5,3}<1$, we arrive (39) and (40) □

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