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?REDUCIBLE CURVES AND THEIR ASSOCIATED RANK: TWO EXTREMAL BEHAVIORS

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Abstract: Let $Y \subset \mathbb{P}^n$ be a reduced curve with $\langle Y \rangle = \mathbb{P}^n$, where $\langle\ \rangle$ denotes the linear span. For each $P \in \mathbb{P}^n$ the Y-rank $r_Y(P)$ is the minimal cardinality of $S \subset Y$ such that $P \in \langle S \rangle$. Here we study two extremal behaviors: " existence of P with $r_Y(P) = n$ " or " $r_Y(P) \le \lceil (n+1)/2 \rceil$ for all $P \in \mathbb{P}^n$ ".

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1. Introduction

Let $Y \subset \mathbb{P}^n$ be any projective set spanning $\mathbb{P}^n$. For each $P \in \mathbb{P}^n$ the Y-rank of P is the minimal cardinality of a set $S \subset Y$ such that $P \in \langle S \rangle$, where $\langle\ \rangle$ denotes the linear span. In this note we consider the case in which Y is a reducible curve spanning $\mathbb{P}^n$. If Y is an integral curve, then $r_Y(P) \le n$ for all $P \in \mathbb{P}^n$ ([2], Proposition 5.1). The same is true if Y is connected by [1], Lemma 8.1. A dimensional count with joins gives $r_Y(P) \ge \lceil (n+1)/2 \rceil$ for a general $P \in \mathbb{P}^n$. In this case we explore two extremal cases (" existence of $P \in \mathbb{P}^n$ with $r_Y(P) = n$ " or " $r_Y(P) \le \lceil (n+1)/2 \rceil$ for all $P \in \mathbb{P}^n$ ", often with Y a union of lines. We mainly considered non-degenerate reduced and connected curves with degree n, i.e. the generalization of rational normal curves. In the

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following cases we characterize all $P \in \mathbb{P}^n$ with $r_Y(P) = n$.

Proposition 1. *Let $Y \subset \mathbb{P}^n$, $n \geq 5$, be a reduced and connected curve with two irreducible components C, T with $\deg(C) = n-1$, $\deg(T) = 1$ and $\langle C \cup T \rangle = \mathbb{P}^n$. Set $O := C \cap T$. C is a rational normal curve in its linear span and Y is nodal at O. Let J be the tangent line of C at O. Take $P \in \mathbb{P}^n$. We have $r_Y(P) = n$ if and only if $P \in \langle T \cup J \rangle \setminus T \cup J$.*

Proposition 2. *Let $Y \subset \mathbb{P}^n$, $n \geq 6$, be a reduced and connected curve with 3 irreducible components C, T, D with $\deg(C) = n-2$, $\deg(D) = \deg(T) = 1$, $\langle Y \rangle = \mathbb{P}^n$ and $D \cap T = \emptyset$. Then $r_Y(P) < n$ for all $P \in \mathbb{P}^n$.*

Proposition 3. *Fix integers n, a such that $3 \leq a \leq n-3$. Let $Y \subset \mathbb{P}^n$ be a reduced and connected curve with two irreducible components, say $Y = A \cup B$, such that $\langle Y \rangle = \mathbb{P}^n$, $\deg(A) = a$ and $\deg(B) = n - a$. The set $\langle A \rangle \cap \langle B \rangle$ is a single point $O \in A \cap B$ and A, B are rational normal curves in their linear span. Let J (resp. T) be the tangent line of A (resp. B) at O. Fix $P \in \mathbb{P}^n$. We have $r_Y(P) = n$ if and only if either there is $Q \in A$ such that $P \in \langle T_Q A \cup T \rangle \setminus \{Q, O\}$ or there is $E \in B$ such that $P \in \langle T_E B \cup J \rangle \setminus \{E, O\}$.*

Definition 1. Fix an odd integer $n \geq 3$. Let $Y \subset \mathbb{P}^n$ be a reduced curve spanning $\mathbb{P}^n$. A *spread* of Y is a subcurve $E \subseteq Y$ such that $\deg(E) = (n+1)/2$ and $\langle E \rangle = \mathbb{P}^n$.

Definition 2. Fix an even integer $n \geq 2$. Let $Y \subset \mathbb{P}^n$ be a finite union of lines such that $\langle Y \rangle = \mathbb{P}^n$. We say that Y has a *weak spread* if there is $E \subset Y$ such that $\deg(E) = 1 + n/2$ and $\langle E \rangle = \mathbb{P}^n$. Now assume that Y is connected. Let $\mathcal{F}(Y)$ be the set of all final lines of Y. We say that Y has a *quasi-spread* if there is $D \in \mathcal{F}$ such that $\langle \overline{Y \setminus D} \rangle$ is a hyperplane and the curve $\overline{Y \setminus D}$ has a spread with respect to the hyperplane $\langle \overline{Y \setminus D} \rangle$.

Proposition 4. *Fix an odd integer $n \geq 3$. Let $Y \subset \mathbb{P}^n$ be a connected union of n lines such that $\langle Y \rangle = \mathbb{P}^n$. The following conditions are equivalent:*

(i) $r_Y(P) \leq (n+1)/2$ *for all* $P \in \mathbb{P}^n$;

(ii) $r_Y(P) = (n+1)/2$ *for a general* $P \in \mathbb{P}^n$;

(iii) $r_Y(P) \leq (n+1)/2$ *for a general* $P \in \mathbb{P}^n$;

(iv) Y *has a spread.*

As an immediate corollary we get the following result.

Corollary 1. *Fix an even integer $n \geq 2$. Let $Y \subset \mathbb{P}^n$ be a union of finitely many lines. If Y contains a quasi-spread, then $r_Y(P) \leq (n+2)/2$ for every $P \in \mathbb{P}^n$ and equality holds for a general $P \in \mathbb{P}^n$.*

A *pencil* of $\mathbb{P}^n$ (or with embedding dimension n) is a reduced union $Y \subset \mathbb{P}^n$ of n lines of $\mathbb{P}^n$ spanning $\mathbb{P}^n$ and through a common point. Notice that a connected union $Y \subset \mathbb{P}^n$ of n lines with $\langle Y \rangle = \mathbb{P}^n$ (i.e. $p_a(Y) = 0$) is a pencil if and only if it has a unique singular point.

Definition 3. Let $Y \subset \mathbb{P}^n$, $n \geq 2$, be a union of lines with $\langle Y \rangle = \mathbb{P}^n$. Let $\rho(Y)$ be the minimal degree of a subcurve F of Y such that $\langle F \rangle = \mathbb{P}^n$.

Proposition 5. *Let $Y \subset \mathbb{P}^n$, $n \geq 3$, be a connected union of n lines and such that $\langle Y \rangle = \mathbb{P}^n$. Y is a pencil if and only if $\rho(Y) = n$. If Y is a pencil, then $r_Y(P) = n$ for all $P \in \mathbb{P}^n$ outside n hyperplanes.*

We work over an algebraically closed base field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

2. The Proofs

Definition 4. Let $A, B \subseteq \mathbb{P}^n$ be closed algebraic subsets. The *join* $[A; B]$ of A and B is the closure in $\mathbb{P}^n$ of the union of all lines $\langle \{P, Q\} \rangle$ with the conventions $[A; B] = \emptyset$ if either $A = \emptyset$ or $B = \emptyset$ and $[A; B] = A$ if $A = B$ and A is a single point. For any integer $s \geq 3$ and any closed algebraic subets $A_i \subseteq \mathbb{P}^n$ define recursively their join $[A_1; \cdots ; A_s]$ by the formula $[A_1; \cdots ; A_s] := [[A_1; \cdots , A_{s-1}]; A_s]$. Write $\sigma_1(A) := A$. For each integer $a \geq 2$ define recursively the algebraic set $\sigma_a(A) \subseteq \mathbb{P}^n$ by the formula $\sigma_a(A) := [\sigma_{a-1}(A); A]$.

Lemma 1. *Fix an odd integer $n \geq 3$. Let $Y \subset \mathbb{P}^n$ be a reduced curve spanning $\mathbb{P}^n$. Assume that Y has a spread E. Then E is formed by $(n+1)/2$ distinct lines and $\dim(\langle F \rangle) = 2 \cdot \deg(F) - 1$ for each subcurve $F \subseteq E$.*

Proof. Let $F_1, \ldots, F_s$ be the irreducible components of E. We have

$$\dim(\langle F_i \rangle) \leq \deg(F_i)$$

and hence $n = \dim(\langle E \rangle) \leq \deg(E) + s - 1$. Hence $s = (n+1)/2$, i.e. each F_i is a line. By induction on the integer $(n+1)/2$ we immediately get the second assertion. □

Example 1. Fix $O \in \mathbb{P}^n$, $n \geq 3$ and an integer $e \geq n$. Let $Y \subset \mathbb{P}^n$ be a union of e lines spanning $\mathbb{P}^n$. It is easy to check that $\dim(\sigma_a(Y)) = \min\{n, a\}$ for all $a \geq 1$.

Lemma 2. *Let $A, B \subset \mathbb{P}^n$ be closed algebraic subsets, each of them a non-empty finite union of linear spaces of positive dimension and with* $\dim(B) = 1$. *Then $[A; B]$ is the union of all lines spanned by a point of A and a different point of B. Moreover, $[A; B]$ is a finite union of linear spaces, each of them either of the form L with L an irreducible component of both A and B or spanned by an irreducible component of A and a different irreducible component of B (all such linear spaces are contained in $[A; B]$).*

Proof. Since $[A_1 \cup A_2; E] = [A_1; E] \cup [A_2; E]$ for all algebraic sets $A_1, A_2, E \subset \mathbb{P}^n$, we reduced to the case in which A is a positive -dimensional linear space and B is a line. If $A \supseteq B$, then use that $[A; B] = A$ is spanned by $P \in A$ and $Q \in B \setminus \{P\} \cap B$. If $A \supsetneq B$ and $A \cap B \neq \emptyset$, then use that $\langle A \cup B \rangle$ is a $(\dim(A) + 1)$-dimensional linear space and that each $O \in \langle A \cup B \rangle$ is contained in a line spanned by a point of A and a different point of $B \setminus A \cap B$. If $A \cap B = \emptyset$, then the result is true for arbitrary algebraic sets; since B is a line, we have $\dim([A; B]) = \dim(A) + 2$. □

Since $\mathbb{P}^n$ is irreducible, the distributive property for joins gives the first part of the following lemma, the second part being true, because $\mathbb{P}^n$ is irreducible.

Lemma 3. *Let $A \subset \mathbb{P}^n$ be a union of finitely many lines.*

(i) We have $\dim(\sigma_a(A)) \leq \min\{2a - 1, n\}$ *for all $a \geq 1$.*

(ii) Assume that Y spans $\mathbb{P}^n$ and for each linear subspace $M \subset \mathbb{P}^n$ with $\dim(M) \leq n-2$ *there is a line $L \subset A$ such that $L \cap M = \emptyset$. Then* $\dim(\sigma_a(A)) = \min\{2a - 1, n\}$ *for all $a \geq 1$*

Proof. Part (i) is true for any finite union A of arbitrary integral curves. Hence it is sufficient to check part (ii). Since $\sigma_1(A) = A$, the lemma is true for $a = 1$. Hence we may assume $a \geq 2$, $2a \leq n + 2$ and that $\sigma_{a-1}(A)$ has dimension $2a - 3$. Take a component E of $\sigma_{a-1}(A)$ of dimension $2a - 1$. Take an irreducible component E of $\sigma_{a-1}(A)$ with dimension $2a - 3$. Lemma 2 gives that E is a linear space. Since $E \neq \mathbb{P}^n$, E is not a cone with vertex spanning $\mathbb{P}^n$. Hence there is a line $L \subset Y$ such that E is not a cone with vertex L. Hence $\dim([E; L]) \geq 1 + \dim(E)$. □

If $2 \leq a \leq n/2$, then $\sigma_a(A)$ may have components of different dimension, we only claim that $\sigma_a(A)$ has at least one component of dimension $2a - 1$.

Proof of Proposition 4. Obviously $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iii)$. First assume that Y has a spread E. Call $L_1, \dots, L_{(n+1)/2}$ the lines of E. Fix $P \in \mathbb{P}^n$. Since $\langle E \rangle = \mathbb{P}^n$ and each L_i is a linear subpace, there are $O_i \in L_1$, $1 \leq i \leq (n + 1)/2$

such that $P \in \{O_1, \dots, O_{(n+1)/2}\}$. Hence $r_Y(P) \le r_E(P) \le (n+1)/2$. Hence (i) is true. Since $\dim(\sigma_a(Y)) \le 2a-1$ for all $a \le (n+1)/2$, we also get that (iv) implies (ii). Notice that $\dim(\sigma_{(n+1)/2}(E)) = n$. Hence $\dim(\sigma_a(Y)) = 2a-1$ for all $a \le (n+1)/2$. Since $\dim(\sigma_{(n+1)/2}(Y) = n > \dim(\sigma_{(n-1)/2}(Y)$, (ii) is satisfied. Assume (iii). To conclude the proof it is sufficient to prove that Y has a spread. Since $\dim(\sigma_{(n+1)/2}(Y)) = \mathbb{P}^n$ and $\mathbb{P}^n$ is irreducible, there are lines $L_i \subset Y$, $1 \le i \le (n+1)/2$, such that $\mathbb{P}^n = [L_1; \cdots : L_{(n+1)/2}]$. Set $E := \cup_{i=1}^{(n+1)/2} L_i$. Since $[L; L] = L$ for every line, we get $L_i \ne L_j$ for all $i \ne j$. Hence $E \subset Y$. Since $\dim([L_1; \cdots : L_{(n+1)/2}]) = n$, E is a spread of Y. □

Proof of Proposition 5. If Y is a pencil, then $r_Y(P) = n$ for all $P \in \mathbb{P}^n$ outside the n hyperplanes spanned by $n-1$ of the lines of Y. Now assume that Y is not a pencil. Hence there are different lines T_1, T_2, T_3 such that $T_1 \cap T_2 \cap T_3 = \emptyset$. First assume $\dim(\langle T_1 \cup T_2 \cup T_3) \ge 3$. Since $T_1 \cap T_2 \cap T_3) = \emptyset$, there are $i, j \in \{1,2,3\}$ such that $\dim(\langle T_i \cup T_j \rangle) = 3$. Take a minimal $F \subseteq Y$ such that $T_i \cup T_j \subseteq F$ and $\langle F \rangle = \mathbb{P}^n$. Since $\deg(F) \le n-1$, we get $\rho(Y) \le n-1$. Now assume that T_1, T_2, T_3 are coplanar. Since $T_3 \subset \langle T_1 \cup T_2 \rangle$ and $\langle Y \rangle = \mathbb{P}^n$, we have $\rho(Y) \le n-1$. □

Proof of Proposition 1. Since $r_Y(Q) \le n$ for any $Q \in \mathbb{P}^n$ ([1], Lemma 8.1), we have $r_Y(Q) = n$ if and only if $r_Y(Q) \ge n$. For any $Q \in \langle C \rangle$ we have $r_Y(Q) \le r_C(Q) \le n-1$ ([2], Proposition 5.1, or [1], Lemma 8.1). For any $Q \in T$ we have $r_Y(Q) = 1$. Take $P \in \mathbb{P}^n \setminus (M \cup T)$ and assume $r_Y(P) = n$. There are $O_1 \in M$ and $O_2 \in T$ such that $P \in \langle \{O_1, O_2\} \rangle$. Since $r_Y(P) \le 1 + r_C(O_1)$, we get $r_C(O_1) = n-1$, i.e. there is $A \in C$ such that $O_1 \in T_A C \setminus \{A\}$. Assume $A \ne O$. Let $Z \subset C$ be the degree two effective divisor on C with A as its support. We have $\langle Z \rangle = T_A C$. Since C is a rational normal curve in its linear span and $n-1 \ge 3$, any degree 4 effective divisor of C is linearly independent. The point O_1 is not uniquely determined by P. For all $O_1' \in \langle \{O_1, O\} \rangle \setminus \{O\}$ we have $P = \langle \{O_1', O_2'\} \rangle$ for some $O_2' \in T$. Taking $O_1' \ne O_1$ we again get $A' \in C$ such that $O_1' \in T_{A'} C$. We may also assume $A_1' \ne O$. Since $P \notin \langle C \rangle$, we have $P \notin T_A C$. Hence $A' \ne A$. Let $Z' \subset C$ be the effective degree 2 divisor with A' as support. Since $O \in \langle \{O_1, O_1'\} \rangle$, we have $O \in \langle Z \cup Z' \rangle$. Hence the degree 5 divisor $Z \cup Z' \cup \{O\}$ of C is not linearly independent. Since $n-1 \ge 4$ and C is a degree $n-1$ rational normal curve, we have $\dim(\langle Z \cup Z' \cup \{O\} \rangle) = 4$, a contradiction.

Take $P \in \langle T \cup J \rangle \setminus T \cup J$ and take $B_1 \in T$ and $B_2 \in J$ such that $P \in \langle \{B_1, B_2\}$. Since $P \notin T$, we have $J = \langle \{B_2, O\} \rangle$. We have $r_C(B_3) = n-1$ for all $B_3 \in J \setminus \{O\}$ ([2], Theorem 4.1). Take $S \subset Y$ evincing $r_Y(P)$ and assume $\sharp(S) \le n-1$. Since $P \notin \langle C \rangle$, we have $S \cap (T \setminus \{O\}) \ne \emptyset$. Assume for the moment either $O \in S$ or $\sharp(S \cap (T \setminus \{O\})) \geq 2$, i.e. assume $T \subset \langle S \rangle$. Set $S' := S \setminus S \cap T$. Since $\langle C \rangle \cap T = \{O\}$, we have $\langle C \rangle \cap \langle T \cup B_2 \rangle = J$. Since $P \in \langle S \rangle$, we have $P \in \langle S' \cup T \rangle$. Since $P \neq O$, we get $\langle S' \rangle \cap (J \setminus \{O\}) \neq \emptyset$. Hence there is $B_3 \in J \setminus \{O\}$ with $r_C(B_3) \leq n-3$, a contradiction. If $O \notin S$ and $\sharp(S \cap T) = 1$ we get the existence of $B_4 \in J \setminus \{O\}$ with $r_C(B_4) \leq n-2$, a contradiction. □

Proof of Proposition 2. Since Y is a reduced and connected, $\deg(Y) = n$ and $\langle Y \rangle = \mathbb{P}^n$, we have $p_a(Y) = 0$. Since $D \cap T = \emptyset$, we get that Y is nodal, $D \cap C$ is a single point, O, $C \cap T$ is a single point, Q, $\dim(\langle E \rangle) = \deg(E)$ for every connected curve $E \subset Y$, C is a rational normal curve of $\langle C \rangle$, $T \cap \langle C \cup D \rangle = \{Q\}$ and $D \cap \langle C \cup T \rangle = \{O\}$. Let $J \subset \langle C \rangle$ be the tangent line of C at O. Let $J' \subset \langle C \rangle$ be the tangent line to C at Q. Fix $P \in \mathbb{P}^n$ and assume $r_Y(P) \geq n$. There are $P_1 \in T$ and $P_2 \in \langle C \cup D \rangle$ such that $P \in \langle \{P_1, P_2\} \rangle$. Since $r_Y(P) \geq n$, we have $r_{C \cup D}(P_2) \geq n-1$. Since $n-1 \geq 5$, Proposition 1 gives $P_2 \in \langle J' \cup D \rangle$. Hence $P \in \langle J' \cup D \cup T \rangle$. Exchanging the role of T and D we get $P \in \langle J \cup D \cup T \rangle$. Grassmann's formula gives $\langle D \cup T \rangle \cap \langle C \rangle = \langle \{O, Q\} \rangle$. Hence $\langle J \cup D \cup T \rangle \neq \langle J' \cup D \cup D$. Hence $\langle J \cup D \cup T \rangle \cap \langle J' \cup D \cup T \rangle = \langle D \cup T \rangle$. Hence $P \in \langle D \cup T \rangle$. Hence $r_Y(P) \leq r_{D \cup T}(P) \leq 2$. □

Lemma 4. *Let $C \subset \mathbb{P}^n$ be a rational normal curve. Fix $O \in C$ and $P \in T_O C \setminus \{O\}$.*

(a) Let $S \subset C$ be a subset evincing $r_C(P)$, i.e. with $\sharp(S) = n$ and $P \in \langle S \rangle$. Then $O \notin S$.

(b) Assume $n \geq 3$. Fix $Q \in C \setminus \{O\}$. Then there is $S \subset C$ evincing $r_C(P)$ and with $Q \in S$.

Proof. Fix $S \subset C$ evincing $r_C(P)$. Let $Z \subset C$ be the degree 2 effective divisor with O as its support. Since $P \in \langle Z \rangle \cap \langle S \rangle$ and $Z \nsubseteq S$, we have $h^1(C, \mathcal{I}_{Z \cup S}(1)) > 0$. Hence $\deg(Z \cup S) \geq n+2$, i.e. $O \notin S$.

Now we check part (b). Take a general hyperplane $H \subset \mathbb{P}^n$ containing $\{Q, P\}$. Let $W \subset C$ be the degree two effective divisor of C with Q as its support. Since $n \geq 3$, any degree 4 zero-dimensional scheme. Hence $\langle Z \rangle$ and $\langle W \rangle$ are the only tangent lines to C intersecting the line $\{Q, P\}$. Hence Bertini's theorem gives that $H \cap C$ is reduced. We have $Q \in H \cap C$ and $P \in \langle H \cap C \rangle$. □

Proof of Proposition 3. Fix $P \in \mathbb{P}^n$ with $r_Y(P) \geq n$. Take $P_1 \in \langle A \rangle$ and $P_2 \in \langle B \rangle$ such that $P \in \langle \{P_1, P_2\} \rangle$. Since $\dim(\langle A \rangle) = a$ and $\dim(\langle B \rangle) = n-a$, we have $r_A(P_1) \leq n-a$ and $r_B(P_2) \leq n-b$. Hence $r_A(P_1) = a$ and $r_B(P_2) = n-a$. Hence there are $Q \in A$ and $E \in B$ such that $P_1 \in T_Q A$

and $P_2 \in T_EB$. To prove the " only if " part it is sufficient to prove that $O \in \{Q, E\}$. Assume $O \neq Q$ and $O \neq E$. Lemma 4 gives the existence of $S_1 \subset A$, $S_2 \subset B$ such that $P_i \in \langle S_i \rangle$, $i = 1, 2$ and $O \in S_1 \cap S_2$. Since $P \in \langle S_1 \cup S_2 \rangle$ and $\sharp(S_1 \cup S_2) = n - 1$, we get a contradiction. Now assume the existence of $Q \in A$ such that $P \in \langle T_QA \cup T \rangle \setminus \{Q, O\}$. Take $S \subset Y$ evincing $r_Y(P)$. Set $S' := S \cap A$ and $S'' := S \cap B$. First assume $O \in S$. Since $\langle A \rangle \cap \langle B \rangle = \{O\}$, we get $\langle S \rangle \cap \langle B \rangle = \langle S' \rangle$. Hence there is $P_1 \in \langle S' \rangle$ and $P_2 \in \langle S'' \rangle$ such that $P \in \langle \{P_1, P_2\} \rangle$. Take any $Q_1 \in \langle T_QA \rangle$ and any $Q_2 \in \langle B \rangle$ such that $P \in \langle \{Q_1, Q_2\} \rangle$. The points P_1, Q_1, O (resp. P_2, Q_2, O) are collinear. Since $O \in S'$, we get $Q_1 \in \langle S' \rangle$. Hence $\sharp(S') \geq a$. Part (a) of Lemma 4 gives $\sharp(S'' \setminus \{O\}) \geq n - a$. Hence $\sharp(S) \geq n$, i.e. $r_Y(P) \geq n$. Now assume $O \notin S$. Since B is a rational normal curve in its linear span, any $n - a + 1$ points of B are linearly independent. Hence $O \notin \langle S'' \rangle$. Hence $\langle A \rangle \cap \langle S \rangle = \langle S' \rangle$ and we conclude as above. The case $P \in \langle T_EB \cup J \rangle \setminus \{E, O\}$ for some $E \in B$ is similar. □

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References

[1] J. Buczynski, J.M. Landsberg, Ranks of tensors and a generalization of secant varieties, *Linear Algebra Appl.*, **438**, No. 2 (2013), 668-689.

[2] J.M. Landsberg, Z. Teitler, On the ranks and border ranks of symmetric tensors, *Found. Comput. Math.*, **10**, No. 3 (2010), 339-366.