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GENERALIZED VOLTERRA SPACES

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Abstract: In this paper we deals with a general concept of classification of subsets of a topological space X with respect to a given nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$. Using system $\mathcal{E}$, the notions of $\mathcal{E}$-Volterra and weakly $\mathcal{E}$-Volterra spaces are introduced which covers classical Volterra and weakly Volterra spaces as well as irresolvable spaces.

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1. Motivation and Basic Definitions

In the sequel, X is a nonempty topological space. By $\overline{A}$, A° we denote the closure, the interior of A, respectively and $\mathcal{U}(z)$ is a base of the open neighborhoods of a point z.

By [1], a topological space X is weakly Volterra (Volterra), if for any G_δ-sets A_1 and A_2 which are dense in X, $A_1 \cap A_2$ is nonempty (dense in X).

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It is clear, that any Baire space is Volterra and any space of second category is weakly Volterra. Recall that a space X is Baire (of second category) if the intersection of any countably many dense open subsets is dense in X (nonempty). In general, these four classes of spaces are all distinct and relevant examples can be found in [3], [4], [5], [6].

There are a few possibilities how to generalize the definitions mentioned above. Firstly, we can change the G_δ-sets by the sets from a priori given system which are understood as "big". As for the systems, we can take into account many types of sets. For example: the nonempty open sets, the sets which are not nowhere dense, the sets of second category, the sets of second category with the Baire property, the perfect sets, the Borel sets, uncountable sets, the sets which are not from a given ideal, respectively.

The "bigness" of a set will be derived from the well known Kuratowski operator. Any nonempty system $\mathcal{E} \subset 2^X \setminus \{\emptyset\}$ will be called a cluster system in X. If any nonempty open subset of a nonempty open set G contains a set from $\mathcal{E}$, then $\mathcal{E}$ is called a π-network in G. For a cluster system $\mathcal{E}$ and a subset A of a topological space X, we can define the set $\mathcal{E}(A)$ of all points $x \in X$ such that for any neighborhood U of x, the intersection $U \cap A$ contains a set form $\mathcal{E}$. Roughly speaking, A is $\mathcal{E}$-big at a point x. Apart from, we will be interested in the complement of A, namely whether $X \setminus A$ is $\mathcal{E}$-big at a point x or not. Our observation is based on the idea: A system $\{A_t\}_{t\in T}$ of pairwise disjoint sets is "small" near a point x if $x \in \cap_{t\in T}\mathcal{E}(A_t)$ (there are "many" pairwise disjoint sets which are $\mathcal{E}$-big at x). Similarly, if $A \subset \cap_{t\in T}\mathcal{E}(A_t)$ ($\mathcal{E}(A) \subset \cap_{t\in T}\mathcal{E}(A_t)$), then the system $\{A_t\}_{t\in T}$ is "small" near A ("small" near points in which A is $\mathcal{E}$-big). Further, we can ask and investigate some further qualitative properties of the intersection $\cap_{t\in T}A_t$. For example, it can be nonempty, dense in X, $\mathcal{E}(\cap_{t\in T}A_t) \neq \emptyset$ (it is $\mathcal{E}$-big at some point) or $\mathcal{E}(\cap_{t\in T}A_t)$ can have an appropriate topological property.

In this paper we focus on the research of the basic properties of weak $\mathcal{E}$-Volterra and $\mathcal{E}$-Volterra spaces which correspond with the known results of the weak Volterra, Volterra and irresolvable spaces. Other alternative systems mentioned above as well as other properties of $\cap_{t\in T}A_t$ are left for further research. We introduce the next definition.

Definition 1. A set A is called weakly $\mathcal{E}$-Volterra, if for any two sets A_1 and A_2, such that $\mathcal{E}(A) \subset \mathcal{E}(A_i)$, $i = 1, 2$, $A_1 \cap A_2$ is nonempty. Moreover, if $A \neq \emptyset$ and $\overline{A_1 \cap A_2} \supset A$, i.e., $A_1 \cap A_2$ is dense in A, then A is called $\mathcal{E}$-Volterra.

If $\mathcal{E}(A) = \emptyset$, then A is not weakly $\mathcal{E}$-Volterra (see Remark 1 item (4)).

In the next example, we will give a few special cluster systems. Two of them (item (2) and (3)) describe irresolvable and weakly Volterra spaces. Their deeper investigation will be delivered in the next papers.

Example 1.

(1) Consider a cluster system $\mathcal{E} = \{G : G \text{ is nonempty open}\}$. Then $\mathcal{E}(A) = \overline{A^\circ}$ and $\mathcal{E}(A) = \emptyset$ if and only if $A^\circ = \emptyset$. Let $\emptyset \neq \mathcal{E}(A) \subset \mathcal{E}(A_i)$, $i = 1, 2$. Then $\emptyset \neq A^\circ \subset \overline{A^\circ} \subset \overline{A_1^\circ} \cap \overline{A_2^\circ}$, so A_1° and A_2° are open and dense in A°. That means, $A_1^\circ \cap A_2^\circ$ is dense in A°, so $A_1 \cap A_2$ is dense in A°. Hence $A_1 \cap A_2 \neq \emptyset$. Moreover, if A is open, then $A_1 \cap A_2$ is dense in A. We have proven A is weakly $\mathcal{E}$-Volterra if and only if $A^\circ \neq \emptyset$ (A is not a boundary set) and any nonempty open set is $\mathcal{E}$-Volterra.

(2) By [2], a space X is called resolvable if there are two disjoint subsets of X which are dense in X and X is irresolvable if it is not resolvable. A subset A of X is resolvable (irresolvable) if A as a subspace is resolvable (irresolvable). Finally, a space X is hereditarily irresolvable if any nonempty subset of X is irresolvable.

Put $\mathcal{E}_0 = 2^X \setminus \{\emptyset\}$. Then $\mathcal{E}_0(A) = \overline{A}$. Let $\mathcal{E}_0(X) \subset \mathcal{E}_0(A_i)$, $i = 1, 2$. Then $X = \overline{X} = \mathcal{E}_0(X) \subset \mathcal{E}_0(A_i) = \overline{A_i}$, so A_1 and A_2 are dense in X. That means, X is irresolvable (resolvable) if and only if $A_1 \cap A_2 \neq \emptyset$ ($A_1 \cap A_2 = \emptyset$) for any (for some) A_1 and A_2 which are dense in X or equivalently, X is weakly $\mathcal{E}_0$-Volterra (not weakly $\mathcal{E}_0$-Volterra). Similarly, if $\mathcal{E}_0^A = 2^A \setminus \{\emptyset\}$, $\emptyset \neq A \subset X$, then A is irresolvable (resolvable) if and only if A is weakly $\mathcal{E}_0^A$-Volterra (not weakly $\mathcal{E}_0^A$-Volterra). Finally, a space X is hereditarily irresolvable if and only if any nonempty subset A of X is weakly $\mathcal{E}_0^A$-Volterra. Note, if $A^\circ = \emptyset$, then A is not weakly $\mathcal{E}_0$-Volterra ($A \subset \overline{A}$, $A \subset \overline{X \setminus A}$ and $A \cap (X \setminus A) = \emptyset$). For example, a singleton in the real line with usual topology is irresolvable but it is not weakly $\mathcal{E}_0$-Volterra. Moreover, if X is resolvable, then any subset of X is not weakly $\mathcal{E}_0$-Volterra (if X_1, X_2 are disjoint and dense in X, then for any A we have $\overline{A} = \mathcal{E}_0(A) \subset \mathcal{E}_0(X_i) = \overline{X_i} = X$, $i = 1, 2$ and $X_1 \cap X_2 = \emptyset$).

(3) Let E_δ be a cluster system containing all G_δ-sets and not nowhere dense. Then X is weakly Volterra (Volterra) if and only if X is weakly $\mathcal{E}_\delta$-Volterra ($\mathcal{E}_\delta$-Volterra).

(4) Consider $X = \{0, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$ with the usual topology, $\mathcal{E}_1 = \{A : A \text{ is infinite}\}$ and $\mathcal{E}_2 = \{A : A \text{ is cofinite}\}$. Then $\mathcal{E}_1(X) = \mathcal{E}_2(X) = \{0\}$ and any subset of X is not weakly $\mathcal{E}_1$-Volterra.

On the other hand, $A \subset X$ is weakly $\mathcal{E}_2$-Volterra if and only if A is cofinite. The sets $A_1 = \{0, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots\}$ and $A_2 = \{\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\}$ are not weakly $\mathcal{E}_2$-Volterra but $A_1 \cup A_2 = X$ is weakly $\mathcal{E}_2$-Volterra. There is no set which is $\mathcal{E}_2$-Volterra.

(5) A set A is called preopen if there is an open set G such that $A \subset G$ and A is dense in G ($G \subset \overline{A}$), see [2]. Let $\mathcal{E}_p$ be a cluster system containing all nonempty preopen sets. Then A is nowhere dense if and only if $\mathcal{E}_p(A) = \emptyset$ and $A \setminus \mathcal{E}_p(A)$ is nowhere dense. A space X is weakly $\mathcal{E}_p$-Volterra ($\mathcal{E}_p$-Volterra) if and only if for any two dense sets in X their intersection is nonempty, i.e., X is irresolvable (dense in X).

2. The Main Results

We start with the basic properties of the operator $\mathcal{E}(A)$ and weakly $\mathcal{E}$-Volterra and $\mathcal{E}$-Volterra spaces. We omit or only outline the trivial proofs in the following remark.

Remark 1.

(1) $\mathcal{E}(A)$ is closed, $\mathcal{E}(A) \subset \overline{A}$ and if $A_1 \subset A_2$, then $\mathcal{E}(A_1) \subset \mathcal{E}(A_2)$,

(2) Let G be nonempty open. Then $\mathcal{E}$ is a π-network in G if and only if $\mathcal{E}(G) = \mathcal{E}(\overline{G}) = \overline{G}$,

(3) If A is weakly $\mathcal{E}$-Volterra ($\mathcal{E}$-Volterra), then $A_1 \cap A$ is nonempty ($A_1 \cap A$ is dense in A), for any A_1 such that $\mathcal{E}(A) \subset \mathcal{E}(A_1)$,

(4) If $\mathcal{E}(A) = \emptyset$ or $A = \emptyset$, then A is not weakly $\mathcal{E}$-Volterra (for $A_1 = A_2 = \emptyset$, we have $\mathcal{E}(A) = \emptyset \subset \mathcal{E}(A_1) = \mathcal{E}(A_2) = \emptyset$ and $A_1 \cap A_2 = \emptyset$). Consequently, if A is weakly $\mathcal{E}$-Volterra, then $\mathcal{E}(A)$ and A are nonempty,

(5) If A is weakly $\mathcal{E}$-Volterra (not weakly $\mathcal{E}$-Volterra), then any $B \supset A$ ($B \subset A$) is weakly $\mathcal{E}$-Volterra (not weakly $\mathcal{E}$-Volterra),

(6) If A is $\mathcal{E}$-Volterra, then $\overline{A}$ is $\mathcal{E}$-Volterra. (Let A_1, A_2, $i = 1, 2$, be such that $\mathcal{E}(\overline{A}) \subset \mathcal{E}(A_i)$. Since $\mathcal{E}(A) \subset \mathcal{E}(\overline{A})$ and A is $\mathcal{E}$-Volterra, $A_1 \cap A_2$ is dense in A, so it is dense in $\overline{A}$,

(7) If $\mathcal{E}$ is a π-network in a nonempty open set G, then G is weakly $\mathcal{E}$-Volterra if and only if $\overline{G}$ is weakly $\mathcal{E}$-Volterra. (”$\Rightarrow$” follows from item (5). ”$\Leftarrow$”

Let A_i be such that $\mathcal{E}(A_i) \supset \mathcal{E}(G)$, $i = 1, 2$. Since $\mathcal{E}(G) = \mathcal{E}(\overline{G})$ (see item (2)) and $\overline{G}$ is weakly $\mathcal{E}$-Volterra, $A_1 \cap A_2 \neq \emptyset$.),

(8) If $\mathcal{E}$ is a π-network in a nonempty open set G, then G is $\mathcal{E}$-Volterra if and only if $\overline{G}$ is $\mathcal{E}$-Volterra. ("$\Rightarrow$" follows from item (6). "$\Leftarrow$" Let A_i be such that $\mathcal{E}(A_i) \supset \mathcal{E}(G)$, $i = 1, 2$. Since $\mathcal{E}(G) = \mathcal{E}(\overline{G})$ (see item (2)) and $\overline{G}$ is $\mathcal{E}$-Volterra, $A_1 \cap A_2$ is dense in $\overline{G}$, so it is dense in G),

(9) If A is not weakly $\mathcal{E}$-Volterra, then $\mathcal{E}(A)$ is also not weakly $\mathcal{E}$-Volterra (if $A_1 \cap A_2 = \emptyset$ and $\mathcal{E}(A) \subset \mathcal{E}(A_i)$, $i = 1, 2$, then by item (1), $\mathcal{E}(\mathcal{E}(A)) \subset \overline{\mathcal{E}(A)} = \mathcal{E}(A) \subset \mathcal{E}(A_i)$).

Theorem 1. (for nonempty weakly Volterra open subspase see [1], Theorem 2.3) Let $\mathcal{E}$ be a π-network in a nonempty open set G. If G is weakly $\mathcal{E}$-Volterra, then for any two sets A_1 and A_2 such that $\mathcal{E}(G) \subset \mathcal{E}(A_i)$, $i = 1, 2$, $A_1 \cap A_2 \cap G$ is not nowhere dense. Consequently, if $\mathcal{E}$ is a π-network in X and X is weakly $\mathcal{E}$-Volterra, then $A_1 \cap A_2$ is not nowhere dense.

Proof. First, we will show that $A_1 \cap A_2$ is not nowhere dense. Let $\mathcal{E}(G) = \overline{G} \subset \mathcal{E}(A_i)$ and $X_i := A_i \setminus A_1 \cap A_2$, $i = 1, 2$. Suppose $A_1 \cap A_2$ is nowhere dense. We will show $\mathcal{E}(X_i) \supset \mathcal{E}(G)$, $i = 1, 2$. Let $x \in \mathcal{E}(G) = \overline{G}$ and $U \in \mathcal{U}(x)$. Then $U \cap G$ is nonempty, so there is a nonempty open set $H \subset U \cap G$ and $H \cap A_1 \cap A_2 = \emptyset$. Since $H \subset \overline{G} \subset \mathcal{E}(A_i)$, there is a set $E \in \mathcal{E}$, $E \subset H$ and $E \subset A_i \setminus A_1 \cap A_2$, so $x \in \mathcal{E}(X_i)$. G is weakly $\mathcal{E}$-Volterra, then $\emptyset \neq X_1 \cap X_2 = (A_1 \setminus A_1 \cap A_2) \cap (A_2 \setminus A_1 \cap A_2) = \emptyset$, contradiction. Since for $B_i := G \cap A_i$ the inclusion $\mathcal{E}(G) \subset \mathcal{E}(B_i)$ holds ($i = 1, 2$), $B_1 \cap B_2 = G \cap A_1 \cap A_2$ is not nowhere dense. □

By [1] (Lemma 3.3, Lemma 3.5), X is Volterra if and only if any nonempty open subspace is weakly Volterra and the union of any family nonempty open non weakly Volterra subspace is not weakly Volterra. Similar results hold for a weakly $\mathcal{E}$-Volterra and $\mathcal{E}$-Volterra space.

Theorem 2. (for characterization of Volterra space see [1], Lemma 3.3 or [4]) If $\mathcal{E}$ is a π-network in an open set X_0, then X_0 is $\mathcal{E}$-Volterra if and only if any nonempty open subset of X_0 is weakly $\mathcal{E}$-Volterra.

Proof. Let X_0 be $\mathcal{E}$-Volterra. We will show that any nonempty open set $G \subset X_0$ is weakly $\mathcal{E}$-Volterra. Let $\mathcal{E}(G) \subset \mathcal{E}(A_i)$, $i = 1, 2$. Since X_0 is $\mathcal{E}$-Volterra, there are two sets X_1, X_2, such that $\mathcal{E}(X_1) \supset \mathcal{E}(X_0)$, $\mathcal{E}(X_2) \supset \mathcal{E}(X_0)$

and $X_1 \cap X_2$ is dense in X_0. Let

$$B_i = \big(G \cap A_i\big) \cup \big((X_0 \setminus \overline{G}) \cap X_i\big), i = 1, 2.$$

We will show $\mathcal{E}(B_i) \supset \mathcal{E}(X_0), i = 1, 2$. Let $x \in \mathcal{E}(X_0) = \overline{X_0} \subset \mathcal{E}(X_i)$. If x is not from $\overline{G}$, then for any $U \in \mathcal{U}(x)$, $U \cap \overline{G} = \emptyset$, there is a nonempty open set $H \subset U \cap X_0 \subset \overline{X_0} \subset \mathcal{E}(X_i)$. So there is a set $E \in \mathcal{E}$, $E \subset U \cap X_0 \cap X_i$. Since $E \cap \overline{G} = \emptyset$, $E \subset (X_0 \setminus \overline{G}) \cap X_i \subset B_i$, so $x \in \mathcal{E}(B_i)$. If $x \in \overline{G} = \mathcal{E}(G) \subset \mathcal{E}(A_i)$, then for any $U \in \mathcal{U}(x)$, $H := U \cap G$ is nonempty and from inclusion $H \subset \overline{G} = \mathcal{E}(G) \subset \mathcal{E}(A_i)$, there is a set $E \in \mathcal{E}$ such that $E \subset H \cap A_i = U \cap (G \cap A_i) \subset B_i$, so $x \in \mathcal{E}(B_i)$. X_0 is $\mathcal{E}$-Volterra, then $B_1 \cap B_2$ is dense in X_0, so $B_1 \cap B_2 \cap G \neq \emptyset$. Then $\emptyset \neq B_1 \cap B_2 \cap G = G \cap A_1 \cap A_2$, so $A_1 \cap A_2 \neq \emptyset$.

Suppose any nonempty open subset of X_0 is weakly $\mathcal{E}$-Volterra. Let $\mathcal{E}(A_i) \supset \mathcal{E}(X_0), i = 1, 2$ and $G \subset X_0$ be nonempty open. Put $B_i := A_i \cap G, i = 1, 2$. We will show $\mathcal{E}(B_i) \supset \mathcal{E}(G)$, $i = 1, 2$. Let $x \in \mathcal{E}(G) = \overline{G}$ and $U \in \mathcal{U}(x)$ be arbitrary. Then $U \cap G \subset X_0$ is nonempty. Since any nonempty open subset of X_0 contains a set from $\mathcal{E}$, $U \cap G \subset \mathcal{E}(X_0) \subset \mathcal{E}(A_i)$. That means $U \cap G \cap A_i$ ($\subset B_i$) contains a set from $\mathcal{E}$, so $\mathcal{E}(B_i) \supset \mathcal{E}(G)$. Since G is weakly $\mathcal{E}$-Volterra, $B_1 \cap B_2 = A_1 \cap A_2 \cap G$ is nonempty, so $A_1 \cap A_2$ is dense in X_0. □

Theorem 3. Let $\mathcal{E}$ be a π-network in a nonempty open subset X_0. If $\{G_t\}_{t \in T}$ is a family of open disjoint non weakly $\mathcal{E}$-Volterra subsets of X_0, then $\cup_{t \in T} G_t$ is not weakly $\mathcal{E}$-Volterra.

Proof. Since G_t is not weakly $\mathcal{E}$-Volterra, there are two subsets A_t, B_t such that $\mathcal{E}(A_t) \supset \mathcal{E}(G_t), \mathcal{E}(B_t) \supset \mathcal{E}(G_t)$ and $A_t \cap B_t = \emptyset$. We will show $\mathcal{E}(A_t \cap G_t) \supset \mathcal{E}(G_t)$ and $\mathcal{E}(B_t \cap G_t) \supset \mathcal{E}(G_t)$. Let $x \in \mathcal{E}(G_t) = \overline{G_t}$ (see Remark 1 item (2)) and U be an arbitrary open neighborhood of x. Since $\mathcal{E}(A_t) \supset \mathcal{E}(G_t) = \overline{G_t} \supset G_t \cap U$ and $U \cap G_t \neq \emptyset$, there is a set $E \in \mathcal{E}, E \subset A_t \cap U \cap G_t$, so $x \in \mathcal{E}(A_t \cap G_t)$. Similarly for $\mathcal{E}(B_t \cap G_t) \supset \mathcal{E}(G_t)$. Let $C_1 := \cup_{t \in T}(A_t \cap G_t)$, $C_2 := \cup_{t \in T}(B_t \cap G_t)$. We will show $\mathcal{E}(C_i) \supset \mathcal{E}(\cup_{t \in T} G_t)$, $i = 1, 2$. Let $x \in \mathcal{E}(\cup_{t \in T} G_t)$ and U be an arbitrary neighborhood of x. Then there is t_0 such that $U \cap G_{t_0} \neq \emptyset$. Let $i = 1$. Since $\mathcal{E}(A_{t_0} \cap G_{t_0}) \supset \mathcal{E}(G_{t_0}) = \overline{G_{t_0}} \supset U \cap G_{t_0}$, there is $E \in \mathcal{E}$, $E \subset U \cap A_{t_0} \cap G_{t_0} \subset \cup_{t \in T}(A_t \cap G_t) = C_1$. Hence $x \in \mathcal{E}(C_1)$. Similarly for $i = 2$. We have two sets C_1 and C_2 for which $\mathcal{E}(C_i) \supset \mathcal{E}(\cup_{t \in T} G_t)$, $i = 1, 2$ and $C_1 \cap C_2 = \emptyset$, hence $\cup_{t \in T} G_t$ is not weakly $\mathcal{E}$-Volterra. □

Remark 2. Let $\{G_t\}_{t \in T}$ be a family of nonempty open subsets of X. Using Zorn' lemma, there is a family $\mathcal{G}$ of pairwise disjoint open sets such that for each set $G \in \mathcal{G}$ there is a set G_t such that $G \subset G_t$ and $\cup_{t \in T}\{G_t\} \subset \overline{\cup \mathcal{G}}$.

Theorem 4. (for weakly Volterra open subspaces see [1], Lemma 3.5) Let $\{G_t\}_{t\in T}$ be a family of open non weakly $\mathcal{E}$-Volterra subsets of X_0. If $\mathcal{E}$ is a π-network in X_0, then $\overline{\cup_{t\in T} G_t}$ is not weakly $\mathcal{E}$-Volterra.

Proof. Let $\mathcal{G}$ be from Remark 2. By Theorem 3, $\cup\mathcal{G}$ is not weakly $\mathcal{E}$-Volterra. By Remark 1 item (7), $\overline{\cup\mathcal{G}}$ is not weakly $\mathcal{E}$-Volterra. Since $\overline{\cup_{t\in T}\{G_t\}} \subset \overline{\cup\mathcal{G}}$, $\overline{\cup_{t\in T}\{G_t\}}$ is not weakly $\mathcal{E}$-Volterra, by Remark 1 item (5). ☐

In [1] (Lemma 3.6), the decomposition of X was given. Namely there are two open disjoint subspaces X_{NV}, X_V, such that $X = \overline{X_{NV}} \cup \overline{X_V}$, any nonempty open subspace of X_{NV} is not weakly Volterra and any nonempty open subspace of X_V is Volterra. Moreover, X is Volterra if and only if $X_{NV} = \emptyset$ and X is weakly Volterra if and only if $X_V \neq \emptyset$. The similar theorem holds for $\mathcal{E}$-Volterra and weakly $\mathcal{E}$-Volterra space.

Theorem 5. For any topological space X there is a family $\mathcal{G}$ of pairwise disjoint nonempty open subsets of $X \setminus \mathcal{E}(X)$ (if $\mathcal{E}(X) = X$, then $\mathcal{G} = \emptyset$) and there are two disjoint open subsets X_0, X_1 of $\mathcal{E}(X)$ (both of them can be empty, see Example 1 item (4)) such that

(1) for any $G \in \mathcal{G}$, G does not contain a set from $\mathcal{E}$ (that means, G is not weakly $\mathcal{E}$-Volterra) and $\cup\mathcal{G}$ is dense in $X \setminus \mathcal{E}(X)$,

(2) $\overline{X_0}$ is not weakly $\mathcal{E}$-Volterra, so any subset of $\overline{X_0}$ is not weakly $\mathcal{E}$-Volterra,

(3) if $X_1 \neq \emptyset$, then $\overline{X_1}$ is $\mathcal{E}$-Volterra and any nonempty open subset of $\overline{X_1}$ is $\mathcal{E}$-Volterra,

(4) $\mathcal{E}(X) \setminus (X_0 \cup X_1)$ is nowhere dense, $\overline{X_0} \cap X_1 = X_0 \cap \overline{X_1} = \emptyset$.

Moreover, if $\mathcal{E}(X) = X$, then

(5) X is $\mathcal{E}$-Volterra if and only if $\overline{X_1} = X$ or $X_0 = \emptyset$,

(6) X is weakly $\mathcal{E}$-Volterra if and only if $X_1 \neq \emptyset$,

(7) Denote $F := \overline{X_0}$ and $G := X \setminus F$. Then $X = F \cup G$ is a decomposition of X, where F is closed non weakly $\mathcal{E}$-Volterra (so any subset of F is not weakly $\mathcal{E}$-Volterra) and if $G \neq \emptyset$, then any nonempty open subset of G is $\mathcal{E}$-Volterra.

Proof. (1) If $x \in X \setminus \mathcal{E}(X)$, then there is an open neighborhood G of x, $G \subset X \setminus \mathcal{E}(X)$, such that G does not contain a set from $\mathcal{E}$. Let $\{G_t\}_{t\in T}$ be a family of all nonempty open subsets of $X \setminus \mathcal{E}(X)$ such that for any $t \in T$, G_t does not contains a set from $\mathcal{E}$. It is clear, $\cup_{t\in T} G_t = X \setminus \mathcal{E}(X)$. Let $\mathcal{G}$ be from Remark 2. Then $\mathcal{G}$ is a family of pairwise disjoint open sets, any open set G from $\mathcal{G}$ does not contain a set from $\mathcal{E}$ (so $\mathcal{E}(G) = \emptyset$ and by Remark 1 item (4), G is not weakly $\mathcal{E}$-Volterra) and $\cup\mathcal{G}$ is dense in $\cup_{t\in T} G_t = X \setminus \mathcal{E}(X)$, so item (1) holds.

Let $\mathcal{S}_0 = \{G : G$ be an open non weakly $\mathcal{E}$-Volterra subset of $\mathcal{E}(X)\}$ and $\mathcal{S}_1 = \{G : G$ be an open $\mathcal{E}$-Volterra subset of $\mathcal{E}(X)\}$. Put $X_0 = \cup\mathcal{S}_0$ and $X_1 = \cup\mathcal{S}_1$. Note, any nonempty open subset of X_i contains a set from $\mathcal{E}$, so $\mathcal{E}$ is a π-network in X_i, $i = 1, 2$. Moreover, $X_0 \cap X_1 = \emptyset$. If not, there are two nonempty open sets $G_0 \in \mathcal{S}_0$ and $G_1 \in \mathcal{S}_1$ such that $\emptyset \neq G_0 \cap G_1$. Then $G_0 \cap G_1$ is not weakly $\mathcal{E}$-Volterra (as a subset of non weakly $\mathcal{E}$-Volterra set G_0, by Remark 1 item (5)) as well as weakly $\mathcal{E}$-Volterra (as a nonempty open subset of G_1 which is $\mathcal{E}$-Volterra, by Theorem 2), contradiction. So $X_0 \cap X_1 = \emptyset$.

(2) From Theorem 4, $\overline{X_0}$ is not weakly $\mathcal{E}$-Volterra.

(3) Let $X_1 \neq \emptyset$ and G be a nonempty open subset of $\overline{X_1}$. We will show G is $\mathcal{E}$-Volterra. Let H be a nonempty open subset of G. Since $\overline{X_1} \setminus X_1$ is not nowhere dense, $H \cap X_1 \neq \emptyset$ and $H \cap X_1$ is weakly $\mathcal{E}$-Volterra (if not, $H \cap X_1 \subset X_0$, that means $X_1 \cap X_2 \neq \emptyset$, contradiction). That means H is weakly $\mathcal{E}$-Volterra (see Remark 1 item (5)). We have proven any nonempty open subset H of G is weakly $\mathcal{E}$-Volterra, so by Theorem 2, G is $\mathcal{E}$-Volterra.

Since any nonempty open subset of $\overline{X}_1$ is $\mathcal{E}$-Volterra, X_1 is also $\mathcal{E}$-Volterra. Consequently, $\overline{X_1}$ is $\mathcal{E}$-Volterra, by Remark 1 item (6).

(4) Suppose $\mathcal{E}(X) \setminus (X_0 \cup X_1)$ is not nowhere dense. Then there is a nonempty open set $G \subset \mathcal{E}(X) \setminus (X_0 \cup X_1)$. We will show G is $\mathcal{E}$-Volterra. Let $H \subset G$ be nonempty open set. Then H is weakly $\mathcal{E}$-Volterra (if not, $H \subset X_0$, contradiction), so G is $\mathcal{E}$-Volterra, by Theorem 2. Hence $G \subset X_1$, contraditcion. Let $\overline{X_0} \cap X_1 \neq \emptyset$ or $X_0 \cap \overline{X_1} \neq \emptyset$. Then $X_0 \cap X_1 \neq \emptyset$, contradiction.

Suppose $\mathcal{E}(X) = X$.

(5) If $\overline{X_1} = X$, then X is $\mathcal{E}$-Volterra, by item (3). If $X_0 = \emptyset$, then (by item (4)) $\mathcal{E}(X) \setminus (X_0 \cup X_1) = X \setminus X_1$ is nowhere dense, so X_1 is dense in X and by item (3), X is $\mathcal{E}$-Volterra.

On the other hand, if X is $\mathcal{E}$-Volterra, by Theorem 2, any nonempty open subset is weakly $\mathcal{E}$-Volterra, so any nonempty open subset is $\mathcal{E}$-Volterra. That means, $X_1 = X$ and $X_0 = \emptyset$.

(6) If $X_1 = \emptyset$, then (by item (4)) $\mathcal{E}(X) \setminus (X_0 \cup X_1) = X \setminus X_0$ is nowhere

dense, so X_0 is dense in X and by item (2), X is not weakly $\mathcal{E}$ Volterra.

On the other hand, if $X_1 \neq \emptyset$, by item (3), X_1 is $\mathcal{E}$-Volterra, so X_1 is weakly $\mathcal{E}$-Volterra. By Remark 1 item (5), X is weakly $\mathcal{E}$-Volterra.

(7) By item (2), F is not weakly $\mathcal{E}$-Volterra. We will show, if $G \neq \emptyset$, then any nonempty open subset of G is $\mathcal{E}$-Volterra. Let $H \subset G$ be nonempty open. Then $H \setminus X_1$ as a subset of $X \setminus (X_0 \cup X_1)$ is nowhere dense, by item (4). So $H \cap X_1$ is dense in H. By item (3), $H \cap X_1$ is $\mathcal{E}$-Volterra. Since $\overline{H \cap X_1} = \overline{H}$, H is $\mathcal{E}$-Volterra, by Remark 1 item (8). □

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