International Journal of Pure and Applied Mathematics
Volume 85 No. 5 2013, 879-885
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
url: http://www.ijpam.eu
doi: http://dx.doi.org/10.12732/ijpam.v85i5.6

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BORDER RANK AND SCHEME RANK FOR LINEAR PROJECTIONS OF CURVES

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Abstract: Let $C \subset \mathbb{P}^r$ be an integral and non-degenerate variety. The border rank (resp. scheme rank) of $P \in \mathbb{P}^r$ with respect to C is the minimal integer t such that P is contained in the t-secant variety of C (resp. $P \in \langle Z \rangle$, where $Z \subset X$ is a degree t subscheme and $\langle\ \ \rangle$ denote the linear span). In this note we study the behavior of the border rank and the scheme-rank under linear projections, when C is a linearly normal smooth curve.

AMS Subject Classification: 14N05, 14H50
Key Words: symmetric tensor rank, X-rank, cactus rank, secant variety, linearly normal curve

1. Introduction

Fix an integral and non-degenerate variety $X \subseteq \mathbb{P}^r$ defined over an algebraically closed field $\mathbb{K}$. For any $P \in \mathbb{P}^r$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle\ \ \rangle$ denote the linear span (see [10], [5], [11], [7], [8] and references therein). The *cactus X-rank* or *scheme X-rank* $z_X(P)$ (resp. *smoothable scheme X-rank* $z'_X(P)$) of P is the minimal integer t such that there is a zero-dimensional (resp. zero-dimensional and smoothable) scheme $Z \subset X$ with $P \in \langle Z \rangle$ and $\deg(Z) = t$ ([9]). Obviously $z_X(P) \le z'_X(P) \le r_X(P)$. If X is smooth and $\dim(X) \le 2$, then every zero-dimensional subscheme of X is smoothable. For any integer $t > 0$ the t-secant

Received: February 17, 2013

variety $\sigma_t(X)$ of X is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle S\rangle$ with $S \subset X$ and $\sharp(S) = t$ [1]. The border X-rank $b_X(P)$ of P is the minimal integer t such that $P \in \sigma_t(X)$. When X is smooth and either $b_X(P)$ is very small or $z'_X(P)$ is very small, then $b_X(P) = z'_X(P)$ ([5], Proposition 11, [8], Lemma 2.1.6; see Lemma 1 for the precise statement). Let $V \subset \mathbb{P}^r$ be a linear subspace. Let $\ell_V : \mathbb{P}^r \to \mathbb{P}^n$, $n := r - \dim(V) - 1$, denote the linear projection from V. Let X_V denote the closure of $\ell_V(X \setminus X \cap V)$ in $\mathbb{P}^n$. Assume $P \notin V$. Then $b_{X_V}(\ell_V(P))$, $z_{X_V}(\ell_V(P))$ and $r_{X_V}(\ell_V(P))$ are well-defined. There are some relations between these invariant for P with respect to X and these invariants for $\ell_V(P)$ with respect to X_V (see for the X-rank), but the picture seems to be blurred if $P \in \sigma_{b_X}(P)(P) \cap V$ (Lemmas 1 and 2). In this note we study the relations between $z_X(P)$ and $z_{X_V}(\ell_V(P))$ when V is spanned by the scheme $X \cap V$ and X is a smooth linearly normal curve in a range in which $z_X(P) = b_X(P)$, $z_{X_V}(P) = b_{X_V}$. We prove the following result.

Theorem 1. *Fix an integer $g \geq 0$. Let $C \subset \mathbb{P}^r$, $r \geq 2g + 3$, be a linearly normal embedding of a smooth curve of genus g. Fix an integer $a \in \{1, \dots r - g - 1\}$ and a zero-dimensional scheme $A \subset C$ such that $\deg(A) = a$. Let $\ell : \mathbb{P}^r \setminus \langle A\rangle \to \mathbb{P}^{r-a}$ denote the linear projection from $\langle A\rangle$. Let C_A be the closure $\ell(C \setminus C \cap \langle A\rangle)$ in $\mathbb{P}^{n-a}$. Fix $P \in \mathbb{P}^r \setminus \langle A\rangle$ and set $b := z_C(P)$. Assume $2b \leq r - g + 1$. Let $Z \subset C$ be the only scheme evincing $z_C(P)$. Set $e := \deg(Z \cap A)$ (scheme-theoretic intersection). Assume $a + b - e \leq r - g + 1$.*

(a) There are a zero-dimensional scheme $W \subset Y$ and $O \in \langle A\rangle$ such that $P \in \langle W\rangle \cup \{O\}\rangle$ and $\deg(W) = z_{C_A}(\ell(P))$ if and only if either $Z \cap A = \emptyset$ or for all $Q \in (Z \cap A)_{red}$ the multiplicity of Z at Q is at most the one of A at Q. If this condition is satisfied, then $b_{C_A}(\ell(P)) = b_C(P) - e$.

(b) Assume $Z \cap A = \emptyset$. Then $W := Z$ is the only scheme with that property, while as O we may take any point of $\langle Z\rangle$.

(c) Assume $Z \cap A \neq \emptyset$. Then $W := Z - Z \cap A$ is the only scheme with the claimed property and there is a unique point $O \in \langle A \cap Z\rangle$ such that $P \in \langle\{O\} \cup W\rangle$. Moreover, $Z \cap A$ evinces $z_C(O)$. If $a + 2b - e \leq r - g + 1$, then O is the only point of $\langle A\rangle$ such that $P \in \langle\{O\} \cup (Z - Z \cap A)\rangle$.

The condition " for all $Q \in (Z \cap A)_{red}$ the multiplicity of Z at Q is at most the one of A at Q " made in the statement of Theorem 1 is equivalent to the condition " $A \cap (Z - Z \cap A) = \emptyset$ ". The main point of Theorem 1 is that this condition fails in a very controlled way. If we fix A then we we have the dimension of the set of all $P \in \mathbb{P}^r$ for which this condition is not satisfied. For instanceif $A = aQ$ for some Q and $2b \leq r - g + 1$, then the failing P's is empty if $b \le a$ and has dimension $(b-a-1)+(b-1)$ if $b > a$. Indeed it is given by the set of all $P \in \langle Z\rangle$ such that $P \notin \langle Z'\rangle$ for all $Z' \subset Q$ and Z any degree b effective divisor whose multiplicity at Q is at least $a+1$ (use part (ii) of Lemma 1 to get $\langle Z\rangle \cap \langle E\rangle = \langle Z \cap E\rangle$ for any two degree b zero-dimensional subschemes Z, E of C).

2. The Proofs

For any integral variety $Y \subset \mathbb{P}^r$ let $\rho(Y)$ (resp. $\rho'(Y)$) be the the maximal integer t such that $\dim(\langle Z\rangle) = \deg(Z) - 1$ for every zero-dimensional (resp. zero-dimensional and smoothable) scheme $Z \subset Y$ such that $\deg(Z) \le t$. We recall that any zero-dimensional subscheme of a smooth curve is smoothable. Hence $\rho'(Y) = \rho(Y)$ if Y is a smooth curve. The integer $\rho'(Y)$ is introduced to state the following lemma ([5], Proposition 11, [8], Lemma 2.1.6 and Theorem 1.5.1).

Lemma 1. *Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix an integer b such that $\rho'(Y) \ge b$.*

(i) For every $P \in \sigma_b(Y)$ there is a smoothable zero-dimensional scheme $Z \subset Y$ such that $\deg(Z) \le b$ and $P \in \langle Z\rangle$.

(ii) If $2b \le \rho(Y)$, then there is a unique scheme $W \subset Y$ such that $\deg(W) \le b$, $P \in \langle W\rangle$, and $P \notin \langle W'\rangle$ for any $W' \subsetneq W$.

Part (i) of Lemma 1 is sharp ([4], Example 2.8).

Lemma 2. *Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix an integer $b > 0$ and a linear subspace $V \subset \mathbb{P}^r$ such that $V \cap \sigma_b(Y) = \emptyset$. Then $\ell_V(\sigma_b(Y)) = \sigma_b(\ell_V(Y))$.*

Proof. Since $V \cap \sigma_b(Y) = \emptyset$ and $b > 0$, $\ell_V|\sigma_b(Y)$ and $\ell_V|Y$ are proper morphisms. Hence $\ell_V(\sigma_b(Y))$ is a closed irreducible variety containing all linear spans of b general points of $\ell_V(Y)$. Hence $\ell_V(\sigma_b(Y)) \supseteq \sigma_b(\ell_V(Y))$. For the other inclusion use that for any finite set $S \subset \ell_V(Y)$ there is a finite set $S' \subset Y$ such that $\ell_V(S') = S$ and $\sharp(S') = \sharp(S)$. □

Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate curve. We have $\dim(\sigma_t(Y)) = \min\{r, 2t-1\}$ for all $t > 0$ ([1], Remark 1.6). Hence $b_Y(P) \le \lfloor (r+2)/2 \rfloor$ for all $P \in \mathbb{P}^r$.

Lemma 3. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. Fix an integer $c \in \{0, \dots, n-2\}$ and a c-dimensional linear subspace $V \subset \mathbb{P}^n$. Let*

$\ell : \mathbb{P}^n \setminus V \to \mathbb{P}^{n-c-1}$ denote the linear projection from V. Let $X_V \subseteq \mathbb{P}^{n-c-1}$ denote the closure in $\mathbb{P}^{n-c-1}$ of the quasi-projective variety $\ell(X \setminus X \cap V)$. Fix $P \in \mathbb{P}^n \setminus V$. Take $S \subset X$ (resp. $Z \subset X$) evincing $r_X(P)$ (resp. $z_X(P)$). Take any $S_1 \subseteq S \setminus S \cap V$ such that $\ell(S_1) = \ell(S \setminus S \cap V)$ and $\sharp(S_1) = \sharp(\ell(S \setminus S \cap V))$. Then:

(a) $r_{X_V}(\ell(P)) \le \sharp(S_1) \le r_X(P) - \sharp(S \cap V) \le r_X(P)$.

(b) Assume $V \cap X = \emptyset$. Then $z_{X_V}(\ell(P)) \le z_X(P)$.

(c) Assume $V \cap X = \emptyset$ and that $\ell|X$ is an embedding. Then there are $O, O' \in V$ such that $z_X(P) \le z_X(O) + z_{X_V}(\ell(P))$ and $r_X(P) \le r_X(O') + r_{X_V}(\ell(P))$.

Proof. Since $P \notin V$, we have $S \nsubseteq V$ and $Z \nsubseteq V$. Since $\ell(S_1) = \ell(S \setminus S \cap V)$ and $\ell(P) \in \langle \ell(S \setminus S \cap V) \rangle$, we get part (a).

Now assume $X \cap V = \emptyset$. Notice that $X_V = \ell(X)$. Hence the scheme $\ell(Z)$ is a well-defined zero-dimensional subscheme of X_V and $\ell(P) \in \langle \ell(Z) \rangle$. Hence we get part (b). Now assume that $\ell|X$ is an embedding. Fix any $W \subset X_V$ evincing $z_{X_V}(\ell(P))$. Since $\ell|X$ is proper and an embedding, there is a unique zero-dimensional scheme $M \subset X$ such that $\ell(M) = W$. We have $\deg(M) = \deg(W) = z_{X_V}(\ell(P))$. Since $M \subset X$, we have $V \cap M = \emptyset$. Since $\ell(P) \in \langle \ell(M) \rangle$ and $M \cap V = \emptyset$, we have $P \in \langle V \cup M \rangle$. Hence there is $O \in V$ such that $P \in \langle \{O\} \cup M \rangle$. We have $z_X(P) \le z_X(O) + z_{X_V}(\ell(P))$. Taking $S' \subset X_V$ evincing $r_{X_V}(\ell(P))$ we get $r_X(P) \le z_X(O') + z_{X_V}(\ell(P))$. □

Lemma 4. *Let $Y \subset \mathbb{P}^r$, $r \ge 2$, be an integral and non-degenerate curve such that $h^0(Y, \mathcal{O}_Y(1)) = r + 1$. Fix an integer $a \in \{0, \dots, r-1\}$ and a zero-dimensional scheme $A \subset Y_{reg}$ such that $\deg(A) = a$, $\dim(\langle A \rangle) = a - 1$ and the line bundle $\mathcal{O}_Y(1)(-A)$ is very ample. Let $\ell : \mathbb{P}^r \setminus \langle A \rangle \to \mathbb{P}^{r-a}$ denote the linear projection from $\langle A \rangle$. We have $Y \cap \langle A \rangle = A$ (scheme-theoretic intersection). Let $Y_A \subset \mathbb{P}^{r-a}$ denote the closure of $\ell(Y \setminus A)$ in $\mathbb{P}^{r-a}$. Then $Y_A \cong Y$ and the embedding $Y_A \hookrightarrow \mathbb{P}^{r-a}$ is projectively equivalent to the embedding of Y by the complete linear system $|\mathcal{O}_Y(1)(-A)|$.*

Proof. Since $h^0(Y, \mathcal{O}_Y(1)) = r + 1$, the condition " $\dim(\langle A \rangle) = a - 1$ " is equivalent to " $h^0(Y, \mathcal{O}_Y(1)(-A)) = r + 1 - a$ ". Since $\mathcal{O}_Y(1)(-A)$ is very ample, it is base-point-free. Hence $A = \langle A \rangle \cap Y$ (scheme-theoretic intersection). Since $\langle A \rangle \cap Y \subset Y_{reg}$, the map $\ell|(Y \setminus A)$ extends to a morphism $u : Y \to \mathbb{P}^{r-a}$. Since Y is linearly normal and $h^0(Y, \mathcal{O}_Y(1)(-A)) = r + 1 - a$, u is induced by the complete linear system $|\mathcal{O}_Y(1)(-A)|$. □

Remark 1. Let $Y \subset \mathbb{P}^r$, $r \geq 2$, be an integral and non-degenerate curve such that $h^0(Y, \mathcal{O}_Y(1)) = r+1$. Fix an integer $b > 0$. Fix a zero-dimensional scheme $Z \subset Y$. By Riemann-Roch we have $\dim(\langle Z\rangle) = \deg(Z) - 1$ if and only if $h^1(Y, \mathcal{I}_Z(1)) = h^1(Y, \mathcal{O}_Y(1))$. Hence if $\deg(\mathcal{O}_Y(1)) \geq 2p_a(Y) - 1 + b$, then $\dim(\langle Z\rangle) = \deg(Z) - 1$ for every zero-dimensional scheme $Z \subset Y$ such that $\deg(Z) \leq b$.

Riemann-Roch gives the following result.

Lemma 5. *Fix an integer $g \geq 0$. Let $C \subset \mathbb{P}^r$, $r \geq 2g+3$, be a linearly normal embedding of a smooth curve of genus g. We have $\deg(C) = r+g$ and $\rho(C) \geq \deg(C) - 2g + 1 = r - g + 1$. For any effective divisor $A \subset C$ with $\deg(A) \leq r - g - 1$ the line bundle $\mathcal{O}_C(1)(-A)$ is very ample*

Remark 2. Take the set-up of Theorem 1. We recall that by Lemmas 1, 5 we have $b_C(P) = z_C(P)$ and $b_{C_A}(P) = z_{C_A}(P)$. Fix any $Q \in \langle A\rangle$ and call $A' \subseteq A$ a minimal subscheme of A such that $Q \in \langle A'\rangle$. If $2\cdot\deg(A') \leq r-g+1$, then $b_C(Q) = z_C(Q)$ and A' is the only scheme evincing $z_C(Q)$ (Lemmas 1 and 5).

We may rephrase [3], Lemma 1, in the following way.

Lemma 6. *Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate variety. Fix $P \in \mathbb{P}^r$ and assume the existence of zero-dimensional schemes $A \subset X$, $B \subset X$ such that $P \in \langle A\rangle \cap \langle B\rangle$ and $\deg(A \cup B) \leq \rho(X)$. Then there is a zero-dimensional scheme $Z \subseteq A \cap B$ such that $P \in \langle Z\rangle$.*

Proof of Theorem 1. Since $P \notin \langle A\rangle$, we have $e < b$. Lemma 4 gives $Z \cap \langle A\rangle = Z \cap A$ as schemes. Lemmas 5 and 4 give that $\mathcal{O}_C(1)(-A)$ is very ample and that C_A is the image of C by the complete linear system $|\mathcal{O}_C(1)(-A)|$. Lemma 1 gives $z_C(P) = b_C(P)$, $z_{C_A}(\ell(P)) = b_{C_A}(\ell(P))$ and that Z is the unique scheme evincing $z_C(P)$.

Assume the existence of a zero-dimensional scheme $W' \subset C$ such that $\deg(W') \leq z_{C_A}(\ell(P))$ and $P \in \langle\langle A\rangle \cup W'\rangle$. Set $W'' := W' - W' \cap A$ (as effective Cartier divisors of C). Since $\ell(P) \in \langle \ell(W'')\rangle$, we get $\deg(W') = z_{C_A}(\ell(P))$. Since $\ell|W''$ is a an embedding, we first get $\deg(W') = z_{C_A}(\ell(P))$ and $W' \cap A = \emptyset$, and then get $W' = Z - Z \cap A$. Notice that $(Z - Z \cap A) \cap A = \emptyset$ if and only if for each $Q \in (Z \cap A)_{red}$ the multiplicity of Q is Z is at most the multiplicity of Q in A. Hence to prove part (a) it is sufficient to prove parts (a) and (b). We just proved Part (b).

Now assume $Z \cap A \neq \emptyset$. Since $P \notin \langle A\rangle$, we have $Z \cap A \nsubseteq Z$. Since Z evinces $z_C(P)$, Grassmann's formula gives that $\langle (Z - Z \cap A) \cup \{P\}\rangle \cap \langle Z \cap A\rangle$ is a unique point. Call O this point. We have $O \in \langle Z \cap A\rangle$. Since $Z - Z \cap A \nsubseteq Z$

and Z evinces $z_C(P)$, the scheme $\{P\} \cup (Z - Z \cap A)$ is linearly independent. Since $O \in \langle (Z - Z \cap A) \cup \{P\} \rangle$, we get $P \in \langle (Z - Z \cap A) \cup \{O\} \rangle$. Since $\langle (Z - Z \cap A) \cup \{P\} \rangle \cap \langle Z \cap A \rangle = \{O\}$, O is the only point $O' \in \langle Z \cap A \rangle$ such that $P \in \langle (Z - Z \cap A) \cup \{O'\} \rangle$ and $z_C(O') \leq \deg(Z \cap A)$. Now assume the existence of $O' \in \langle A \rangle \setminus \langle Z \cap A \rangle$ such that $P \in \langle (Z - Z \cap A) \cup \{O'\}$. Take $A_1 \subset Z$ evincing $z_C(P)$. Since $P \in \langle Z \rangle \cap \langle (Z - Z \cap A) \cup A_1 \rangle$ and $\deg(Z) + \deg(Z - Z \cap A) + \deg(A_1) \leq 2b - \deg(Z \cap A) + a \leq r - g + 1 \leq r - g + 1$, Lemma 6 gives the existence of $B \subseteq Z \cap ((Z - Z \cap A) \cup A_1)$ such that $P \in \langle B \rangle$. Since $P \notin Z'$ for any $Z' \subsetneq Z$, we get $Z \cap A \subseteq A_1$. Since $Z \cap A \neq A_1$ and $\deg(A_1) \leq \deg(Z \cap A)$, we obtained a contradiction. □

Remark 3. In part (c) of the statement of Theorem 1 it is sufficient to assume the inequality $a + 2b - e \leq \rho(C)$ instead of the inequality $a + 2b - e \leq r - g + 1$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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