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ON THE MAXIMAL SYMMETRIC TENSOR RANK FOR MULTIVARIATE HOMOGENEOUS POLYNOMIALS

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Abstract: For all positive integers m, d, b let $\rho(m,d)$ (resp. $\rho(m,d,b)$) be the maximal symmetric tensor rank of any $f \in \mathbb{C}[x_0,\dots,x_n] \setminus \{0\}$ homogeneous of degree d (resp. and with border rank $\le b$). Here we prove that $\rho(m,d) \le \binom{m+d}{m} - m$ for all $m \ge 2$ and $d \ge 2$ (only by 1 better than a far more general result of Landsberg and Teitler), that $\rho(m,d,b) \le \rho(b-1,d,b)$ if $2 \le b \le m$ and $d \ge b-1$ and that $\rho(m,d,b) \le d \cdot \lceil \binom{m+d}{m}/(m+1) \rceil$ if $2 \le b \le m$ and $3 \le d \le b-2$.

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1. Introduction

Let $\mathbb{K}$ be an algebraically closed base field such that $\mathrm{char}(\mathbb{K}) = 0$. Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate subvariety. For each $P \in \mathbb{P}^r$ the Y-rank $r_Y(P)$ of Y. For all integers $m \ge 1$, $d > 0$ let $\mathbb{K}[x_0,\dots,x_m]_d$ denote the set of all homogeneous polynomials with degree d. This set is a $\mathbb{K}$-vector space with dimension $\binom{m+d}{m}$. We have $H^0(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)) = \mathbb{K}[x_0,\dots,x_m]_d$. Hence

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$\mathbb{K}[x_0,\dots,x_m]_d$ induces the order d Veronese embedding $\nu_d : \mathbb{P}^m \to \mathbb{P}^{\binom{m+d}{m}-1}$. Set $X_{m,d} := \nu_d(\mathbb{P}^m)$ (the m-dimensional Veronese variety of order d). For any $f \in \mathbb{K}[x_0,\dots,x_m]_d \setminus \{0\}$ the *symmetric tensor rank* $sr_{m,d}(f)$ of f is the minimal integer $s > 0$ such that $f = \sum_{i=1}^{s} \ell_i^d$ for some $\ell_i \in \mathbb{K}[x_0,\dots,x_m]$. Obviously $sr_{m,d}(f) = sr_{m,d}(\lambda f)$ for all $\lambda \in \mathbb{K}\setminus\{0\}$. The non-zero polynomial f corresponds to a unique $P \in \mathbb{P}^{\binom{m+d}{m}-1}$. We set $sr_{m,d}(P) := sr_{m,d}(f)$ and call it the *symmetric tensor rank* of P. Conversely, any $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ corresponds to a unique set $\{\lambda g\}_{\lambda \in \mathbb{K}\setminus\{0\}}$ for some $g \in \mathbb{K}[x_0,\dots,x_m]_d \setminus \{0\}$. Hence the symmetric tensor rank is defined for all points of $\mathbb{P}^{\binom{m+d}{m}-1}$. The definition of Veronese embedding gives $sr_{m,d}(P) = r_{X_{m,d}}(P)$ for all $P \in \mathbb{P}^{\binom{m+d}{m}-1}$. Let $\rho(m,d)$ be the maximal of all $sr_{m,d}(P)$, $P \in \mathbb{P}^{\binom{m+d}{m}-1}$.

We first improve (but only by 1) the upper bound $\rho(m,d) \le \binom{m+d}{m} - m + 1$ for all positive integers m,d ([9], Corollary 5.2). Notice that this result by Landsberg and Teitler is just a very particular case of [9], Proposition 5.1.

Theorem 1. *Fix integers $m \ge 2$ and $d \ge 2$. Then $\rho(m,d) \le \binom{m+d}{d} - m$.*

Remark 1. Take $m = d = 2$ and let $Y := X_{2,2} \subset \mathbb{P}^5$ be the Veronese surface. Theorem 1 says that $r_Y(P) \le 3$ for all $P \in \mathbb{P}^5$. Since the secant variety of $X_{2,2}$ has dimension 4, we have $r_Y(P) \ge 3$ for a general $P \in \mathbb{P}^5$. Hence $\rho(2,2) = 3$.

Theorem 1 is just a particular case of the following result.

Proposition 1. *Let $Y \subset \mathbb{P}^r$ be an integral and non-degenerate m-dimensional variety, $r-2 \ge m \ge 1$, such that* $\dim(Sing(Y)) \le m-2$, *with the convention* $\dim(\emptyset) = -1$. *Fix $P \in \mathbb{P}^r$ and set $E := \{O \in Y_{reg} : P \in T_O Y\}$. Assume* $\dim(E) \le m-2$. *Then $r_Y(P) \le r - m$.*

Now we add a constraint: the border rank. Fix any integral and non-degenerate variety $Y \subset \mathbb{P}^r$. For any integer $b > 0$ the b-secant variety $\sigma_b(Y) \subseteq \mathbb{P}^r$ is (by definition) the closure in $\mathbb{P}^r$ of of the union of all linear spaces $\langle S \rangle$, where S is a subset of Y with cardinality b. The algebraic set $\sigma_b(Y)$ is an integral variety and $\dim(\sigma_b(Y)) \le \min\{r, (b+1)\cdot \dim(Y) - 1\}$ for all r, Y, b. The dimensions of all varieties $\sigma_b(X_{m,d})$ are known by a famous theorem of J. Alexander and A. Hirschowitz ([1], [2], [7]). The points $P \in \sigma_b(X_{m,d}) \setminus \sigma_{m-1}(X_{m,d})$ are said to have border rank b. Hence $\sigma_b(X_{m,d})$ is the set of all $P \in \mathbb{P}^{\binom{m+d}{m}-1}$ with border rank $\le b$. For all positive integers m,d,b let $\rho(m,d,b)$ be the maximal integer $sr_{m,d}(P)$, where $P \in \sigma_b(X_{m,b})$. Of course, $\rho(m,d,1) = 1$ for all m,d, $\rho(m,1,b) = 1$ for all m and b, $\rho(m,d,b) \le \rho(m,d)$ and and $\rho(m,d,b) \le \rho(m,d,b+1)$ for all m,d,b. A theorem of Sylvester says that $\rho(1,d,b) = d$ for all $b \geq 2$ ([5], [8], [9], Theorem 4.1). In this note we prove the following result.

Theorem 2. *Fix integers m, d, b such that $2 \leq b \leq m$.*

(i) If $d \geq b-1$, then $\rho(m,d,b) \leq \rho(m-1,d,b)$.

(ii) If $3 \leq d \leq b-2$, then $\rho(m,d,b) \leq d \cdot \lceil \binom{m+d}{m}/(m+1) \rceil$.

2. The Proofs

Proof of Proposition 1. If $P \in Y$, then $r_Y(P) = 1$. Hence we may assume $P \notin Y$. The case $m = 1$ is the main result of [3] (Notice that we use that $r \geq m+2 = 3$ in this case). Hence we may assume $m \geq 2$. Let $V \subset \mathbb{P}^r$ be a general linear subspace of codimension $m-1$ containing P. Since $P \notin Y$, the restriction to Y of the linear system of all hyperplanes through P has no base points. Hence Bertini's theorem implies $E \cap V = \emptyset$, $V \cap \mathrm{Sing}(Y) = \emptyset$ and that $C := Y \cap V$ is an integral and smooth curve spanning V. Since $E \cap V = \emptyset$, P is not contained in a tangent line of C. Hence $r_C(P) \leq \dim(V) - 1 = r - m$ ([3], Theorem 1) (again, we use that $r \geq m+2$). Since $r_Y(P) \leq r_C(P)$, we get $r_Y(P) \leq r - m$. □

Proof of Theorem 1. Fix $P \in \mathbb{P}^r$, $r := \binom{m+d}{m} - 1$. The variety $X_{m,d} \subset \mathbb{P}^r$ is smooth and non-degenerate. Let $E := \{O \in X_{m,d} : P \in T_O X_{m,d}\}$. By Proposition 1 Theorem 1 is true for P if $\dim(E) \leq m-2$. In particular Theorem 1 is true for P if $E = \emptyset$. If $E \neq \emptyset$, then $sr_{m,d}(P) = d$ by [5], Theorem 32 (the reader may check the proof even in the case $d = 2$, i.e. in the case in which E is the image of a line of $\mathbb{P}^m$). If $E = \emptyset$, then Proposition 1 gives $sr_{m,d}(P) \leq \binom{m+d}{m} - m$. □

Proof of Theorem 2. First assume $d \geq b-1$. In this case there is a zero-dimensional scheme $W \subset \mathbb{P}^m$ such that $\deg(W) = b$ and $P \in \langle \nu_d(W) \rangle$ ([5], Proposition 11, [6], Lemma 2.1.6). Set $M := \langle W \rangle \subseteq \mathbb{P}^m$. Since $\deg(W) = b$, we have $a := \dim(M) \leq b-1$. Since $P \in \langle \nu_d(W) \rangle$, we have $P \in \langle \nu_d(M) \rangle$. We have $sr_{m,d}(P) = r_{\nu_d(M)}(P)$ ([10], Proposition 3.1, for the non-symmetric case; see [8], Exercise 3.2.2.2, for a stronger statement). Hence $sr_{m,d}(P) \leq \rho(a,d)$. Notice that [8], Exercise 3.2.2.2, implies $\rho(a,d) \leq \rho(b-1,d)$. To conclude it is sufficient to prove that $P \in \sigma_b(M)$. By the proof of [5], Proposition 11, W is smoothable inside $\mathbb{P}^m$. By [6], Proposition 2.1.5, W is smoothable inside M. Hence $P \in \sigma_b(M)$ ([6], Lemma 2.1.6).

Now assume $3 \leq d \leq b-2$. Since $d \geq 3$, we have $\rho(m,d) \leq d \cdot \lceil \binom{m+d}{m}/(m+1) \rceil$ ([4], Theorem 1). $\square$

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