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UNION OF LINES IN PROJECTIVE SPACES AND THEIR INTERSECTION WITH QUADRICS AND HYPERPLANES

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Abstract: A tree $Y \subset \mathbb{P}^r$ is a connected nodal curve with lines as components and $p_a(Y) = 0$. Here we investigate the existence of certain trees with nice intersection with a quadric Q or a hyperplane H. E.g. we find Y with prescribed $Y \cap H$ or with $h^1(Q, \mathcal{I}_{Q\cap Y}(t)) = 0$.

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1. Introduction

Let $Y \subset \mathbb{P}^r$ be a reduced and connected curve. We say that Y is a *tree* if it is nodal, the irreducible components of Y are lines and $p_a(Y) = 0$. We look at the intersection of " sufficiently general " trees with hyperplanes and quadric hypersurfaces. To explain the words " sufficiently general " we need to remember that the set of all trees of fixed degree and fixed " topological type " or " combinatorial type " in $\mathbb{P}^r$, $r \geq 3$, is irreducible. We explain it now, calling " type " the topological (or combinatorial) type. A type τ for an integer $d \geq 2$ is a function $\tau : \{2, \dots d\} \to \{1, \dots, d-1\}$ such that $\tau(i) < i$ for all i. Fix a type τ. Let $Z(r, d, \tau)$ be the set of all degree d trees $Y \subset \mathbb{P}^r$ such that there is an ordering L_i, $1 \leq i \leq d$, of the irreducible components of Y with $L_i \cap L_{\tau(i)} \neq \emptyset$ for all $i = 2, \dots, d$. Let $Z(r, 1, \tau)$ denote the Grassmannian

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of lines in $\mathbb{P}^r$. A tree of degree $d \geq 4$ may have several types, i.e. we may have $Z(r,d,\tau) = Z(r,d,\tau')$ for some $\tau \neq \tau'$. Each set $Z(r,d,\tau)$ is a non-empty irreducible subset of the Hilbert scheme of $\mathbb{P}^r$ and it makes sense to speak about the general $Y \in Z(r,d,\tau)$. A tree $Y \subset \mathbb{P}^r$ of degree $d \geq 2$ is said to be a *comb* if the constant function $\eta : \{2,\dots,d\} \to \{1\}$ is its type; if $Y = L_1 \cup \cdots \cup L_d$ is an ordering of the lines of Y with η as its type, then L_i, $2 \leq i \leq d$, are the *tooth* of Y, while L_1 is the *spine* of Y. A *bamboo* of degree $d \geq 2$ is a degree d tree $Y = L_1 \cup \cdots \cup L_d$ which admits the function $i \mapsto i-1$ as a type. Any line will be called a comb and a bamboo of degree 1.

We will prove the following results, whose proof is a sharp version of a method used in the first version (not the published one) of [1].

Theorem 1. *Fix integers $a \geq 0$, $b \geq 0$, $d > 0$ such that $(a,b,d) \neq (1,1,2)$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Then there is a degree d comb $Y \subset \mathbb{P}^3$ intersecting transversally Q and such that either $h^0(Q, \mathcal{I}_{Y\cap Q}(a,b)) = 0$ (case $2d \geq (a+1)(b+1)$) or $h^1(Q, \mathcal{I}_{Y\cap Q}(a,d)) = 0$ (case $2d \leq (a+1)(b+1)$).*

Theorem 2. *Fix integers $r \geq 3$, $e \in \{0,\dots,r\}$, $d > 0$ and $t > 0$. Let $Q \subset \mathbb{P}^r$ be a rank e quadric hypersurface. Then there is a degree d tree $Y \subset \mathbb{P}^r$ such that Q contains no line of Y, $h^0(Q, \mathcal{I}_{Y\cap Q}(t)) = \max\{0, \binom{r+t}{r} - \binom{r+t-2}{r} - 2d\}$ and $h^1(Q, \mathcal{I}_{Y\cap Q}(t)) = \max\{0, 2d - \binom{r+t}{r} + \binom{r+t-2}{r}\}$.*

As an immediate corollary of Proposition 2 we get the following result.

Corollary 1. *Fix a hyperplane $H \subset \mathbb{P}^r$, $r \geq 3$, a finite set $S \subset H$ and $P \in S$. Let $E \subset H$ be the union of all lines containing at least 3 points of S. There is a comb $Y \subset \mathbb{P}^r$ such that $Y \cap H = S$ if and only if $S \nsubseteq E$.*

2. Proof of Theorems 1 and 2

Proof of Theorem 1. Fix a line $L_1 \subset Q$ intersecting transversally Q. Notice that the two points of $L_1 \cap Q$ are not contained in a line of Q. In each case we will take as Y a comb with L_1 as its spine. Write $\{O, O'\} := L_1 \cap Q$.

(i) Fix (a,b) with $(a,b) \neq (1,1)$ and assume that we need to prove Proposition 1 for the integer d. Set $t := \lfloor (a+1)(b+1)/2 \rfloor$ and $w := \lceil (a+1)(b+1)/2 \rceil$. Assume for the moment to have proved Theorem 1 for the triple (a,b,t) and fix any comb $L_1 \cup \cdots \cup L_t$ solving it. For any integer $d < t$ the comb $L_1 \cup \cdots \cup L_d$ is a solution of Theorem 1 for the triple (a,b,d). Assume for the moment to have proved Theorem 1 for the triple (a,b,w) and fix any comb E solving it. For any integer $d > w$ any comb containing E and intersecting transversally Q

is a solution of Theorem 1 for the triple (a, b, d).

(ii) First assume that either $a = 0$ or $b = 0$. In this case it is sufficient to observe the existence of a degree d comb $Y \subset \mathbb{P}^3$ intersecting transversally Q, with L_1 as its spine and such that no two of the $2d$ points of $Y \cap Q$ are contained in a line of Q.

(iii) Assume $a = b = 1$. If $d = 1$, then we only need that L_1 is transversal to Q. We excluded the case $d = 2$ and $a = b = 1$. If $d \geq 3$ it is sufficient to take as L_i, $2 \leq i \leq d$, general lines intersecting L_1, because if L_2 and L_3 are general, then the 6 points $Q \cap (L_1 \cup L_2 \cup L_3)$ are not contained in a plane.

(iii) Now assume $a = 1$ and $b \geq 2$. Let $C \subset Q$ be a smooth curve of type $(1, b)$ containing O', but not containing O. We also assume that the tangent developable of C does not contain L_1 (either take C general in $|\mathcal{O}_Q(1, b)|$ or fix any C through O' and then take as L_1 a general line through O'). Fix b general points $O_1, \dots O_b \in L_1$. Since $O_i \notin Q$, for any line T_i through O_i Bezout theorem gives $\deg(T_i \cap C) \leq 2$. Since $b \geq 2$, the linear projection from O_i does not induce an embedding of C into a plane. Hence there is a line $R_i \subset \mathbb{P}^3$ such that $O_i \in R_i$ and $\deg(R_i \cap C) \geq 2$. We saw that $\deg(R_i \cap C) \leq 2$. Hence $\deg(R_i \cap C) = 2$. Since O_i is not contained in the tangential variety of C, the line T_i is quasi-transversal to C and it intersects C at exactly two points. Take any R_1 as above and assume constructed $R_1, \dots, R_i$, $1 \leq i < b$, as above so that $E_i := L_1 \cup R_1 \cup \cdots \cup R_i$ is a comb with spine L_1 and $\sharp((R_1 \cup \cdots R_i) \cap C) = 2i$. We take as O_{i+1} a general point of L_1 so that $R_{i+1} \cap R_j = \emptyset$ for all $j < i$. In this way we construct a comb $E := L_1 \cup R_1 \cup \cdots \cup R_b$ such that $\sharp(R_i \cap C) = 2$ for all i, $\sharp(E \cap C) = 2b + 1$ and $O \notin E$. Since $C \cong \mathbb{P}^1$ and $\deg(\mathcal{O}_C(1, b)) = 2b$, we have $h^i(C, \mathcal{I}_{E \cap C}(1, b)) = 0$, $i = 0, 1$. Hence $h^i(Q, \mathcal{I}_{E \cap Q}(1, b)) = h^0(Q, \mathcal{I}_O) = 0$, $i = 1, 2$. Hence Theorem 1 is true in this case.

(iv) Now assume $a = 2$. By step (ii) we may assume $b > 0$. Fix two lines $T_i \in |\mathcal{O}_Q(1, 0)|$, $i = 1, 2$, such that $T_1 \neq T_2$ and $L_1 \cap (T_1 \cup T_2) = \emptyset$. Any 3 disjoint lines of $\mathbb{P}^3$ are contained in a smooth quadric. Hence for general lines $L_2, \dots, L_{b+1}$ meeting L_1, T_1 and T_2 the set $L_1 \cup R_1 \cup \cdots \cup R_{b+1}$ is a degree $b+2$ comb intersecting Q in O, O' and $b + 1$ points of each line T_i. For $b + 2 \leq j \leq d - 1$, let R_j be a general line intersecting L_1. Set $F := L_1 \cup R_1 \cup \cdots \cup R_{d-1}$, $F' := R_1 \cup \cdots \cup R_{b+1}$ and $F'' := L_1 \cup R_{b+2} \cup \cdots \cup R_{d-1}$. Since $F'' \cap T_i = \emptyset$ and $\deg(F \cap T_i) = b + 1$, we have $h^i(Q, \mathcal{I}_{F \cap Q}(a, b)) = h^i(Q, \mathcal{I}_{F''}(0, b))$, $i = 0, 1$. Apply step (ii) for the pair $(0, b)$ to F''.

(v) Now assume $a \geq 3$. By steps (ii) and (iii) we may assume $b \geq 2$. We use induction on a. We adapt step (iv) in the following way. By step (i) we may assume $d \ge b+2$. Fix two lines $T_i \in |\mathcal{O}_Q(1,0)|$, $i=1,2$, such that $T_1 \ne T_2$ and $L_1 \cap (T_1 \cup T_2) = \emptyset$. Any 3 disjoint lines of $\mathbb{P}^3$ are contained in a smooth quadric. Hence for general lines $L_2, \dots, L_{b+1}$ meeting L_1, T_1 and T_2 the set $L_1 \cup R_1 \cup \cdots \cup R_{b+1}$ is a degree $b+1$ comb intersecting Q in O, O' and $b+1$ points of each line T_i. Let $F'' \subset \mathbb{P}^3$ be a general comb of degree $d-b-2$ with L_1 as its spine. By the inductive assumption we may assume $h^0(Q, \mathcal{I}_{F''\cap Q}(a-2,b)) = \max\{0, (a-1)(b+1)-2d-2b-2\}$ and $h^1(Q, \mathcal{I}_{F''\cap Q}(a-2,b)) = \max\{0, 2d-2b-2-(a-1)(b+1)\}$. Set $F := F'' \cup R_1 \cup \cdots \cup R_{b+1}$. Since $F'' \cap T_i = \emptyset$ and $\deg(F \cap T_i) = b+1$, we have $h^i(Q, \mathcal{I}_{F\cap Q}(a,b)) = h^i(Q, \mathcal{I}_{F''}(0,b))$, $i=0,1$. $\square$

Lemma 1. *Fix integers $r \ge 3$, $d > 0$ and $t > 0$. Let $Q \subset \mathbb{P}^r$ be a quadric hypersurface with rank 0, i.e. $Q = 2H$ with H a hyperplane. Then there is a degree d tree $Y \subset \mathbb{P}^r$ such that Q contains no line of Y, $h^0(Q, \mathcal{I}_{Y\cap Q}(t)) = \max\{0, \binom{r+t}{r} - \binom{r+t-2}{r} - 2d\}$ and $h^1(Q, \mathcal{I}_{Y\cap Q}(t)) = \max\{0, 2d - \binom{r+t}{r} + \binom{r+t-2}{r}\}$.*

Proof. Since the case $t=1$ is trivial, we may assume $t \ge 2$. Set $u := \lfloor (\binom{r+t}{r} - \binom{r+t-2}{r} - 2d)/2 \rfloor$ and $w := \lceil (\binom{r+t}{r} - \binom{r+t-2}{r} - 2d)/2 \rceil$. For any two distinct points O, P of a projective space, let $\langle \{O, P\} \rangle$ denote the line spanned by O and P.

(i) First assume $d \le u$. In this case we need to find Y such that

$$h^1(Q, \mathcal{I}_{Q\cap Y}(t)) = 0.$$

We use induction on d, starting from the obvious case $d=1$. Fix a tree $A \subset \mathbb{P}^r$ such that Q contains no component of A, $\deg(A) = d-1$ and $h^1(Q, \mathcal{I}_{A\cap Q}(t)) = 0$, i.e. $h^0(Q, \mathcal{I}_{A\cap Q}(t)) = \binom{r+t}{r} - \binom{r+t-2}{r} - 2d - 2$. Let τ be the type of A. Let $\mathcal{B}$ be the base scheme of the $\mathcal{O}_Q$-sheaf $\mathcal{I}_{A\cap Q}(t)$. By definition $\mathcal{B} \subset Q$ and $H^0(Q, \mathcal{I}_{\mathcal{B}}(t)) = H^0(Q, \mathcal{I}_{A\cap Q}(t))$. Since $t \ge 2$ and the restriction map $\rho_t : H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(Q, \mathcal{O}_Q(t))$ is surjective, $\mathcal{B}$ is the base scheme of the $\mathcal{O}_{\mathbb{P}^r}$-sheaf $\mathcal{I}_{A\cap Q, \mathcal{O}_{\mathbb{P}^r}}(t)$. It is sufficient to find a line $L \subset \mathbb{P}^r$ such that $L \nsubseteq Q$, $A \cup L$ is a tree and $h^0(Q, \mathcal{I}_{(A\cup L)\cap Q}(t)) = \binom{r+t}{r} - \binom{r+t-2}{r} - 2d$. Since ρ_t is surjective and $(A \cup L) \cap Q \subset Q$, it is sufficient to prove $h^0(\mathbb{P}^r, \mathcal{I}_{(A\cup L)\cap Q, \mathcal{O}_{\mathbb{P}^r}}(t)) = h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q, \mathcal{O}_{\mathbb{P}^r}}(t)) - 2$.

(i1) First assume $H \subseteq \mathcal{B}$. Since no component of A is contained in H, we get $h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) = h^0(\mathcal{P}^r, \mathcal{I}_{A\cap H}(t-1))$. Deforming A to a general tree with type τ we may assume that $A \cap H$ is formed by $d-1$ general points of H. Hence $h^0(\mathbb{P}^r, \mathcal{I}_{A\cap H}(t-1)) = \binom{r+t-1}{r-1} - \min\{\binom{r+t-2}{r-1}, d-1\}$. If $d-1 \le \binom{r+t-2}{r-1}$, then we get $h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) - h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) = \binom{r+t-1}{d-1} + (d-1) > 2(d-1)$, absurd. Now assume $d-1 > \binom{r+t-2}{r-1}$. In this case we get $h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) - h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) =$

$h^0(Q, \mathcal{O}_Q(t))$, i.e. $h^0(Q, \mathcal{I}_{A\cap Q}(t)) = 0$, contradicting the assumption $d-1 < u$.

(i2) Now assume $H \nsubseteq \mathcal{B}$, but that the scheme $H \cap \mathcal{B}$ contains a degree t hypersurface of H. Equivalently, assume that the restriction map $\rho_H : H^0(Q, \mathcal{I}_{A\cap H}(t)) \to H^0(H, \mathcal{O}_H(t))$ has rank 1. The inequalities in step (i1) must be modified just by one. We get again a contradiction. Hence in step (i4) we are allowed to assume that ρ_H has rank ≥ 2.

(i3) Assume $d \leq r$. First assume $r = 3$. Let $2L \subset \mathbb{P}^2$ be a double line. Let $E = L_1 \cup L_2 \subset \mathbb{P}^2$ be a reducible conic such that $L \nsubseteq E$. The degree 4 scheme $2L \cap E$ satisfies $h^0(\mathbb{P}^2, \mathcal{I}_{E\cap 2L}(2)) = 2$. Hence $h^1(\mathbb{P}^2, \mathcal{I}_{E\cap 2L}(2)) = 0$. Hence $h^1(2L, \mathcal{I}_{E\cap 2L}(2)) = 0$. See $2L$ as a hyperplane section $N \cap Q$ of Q. Notice that $E \cap 2L$ is the base locus of the linear system $|\mathcal{I}_{E\cap Q}(2)|$. Hence for any $P \in Q \setminus 2L$ we have $h^0(\mathbb{P}^3, \mathcal{I}_{(E\cap 2L)\cup\{P\}}(2)) = 5$. Fix a general quadric surface Q_1 containing $(E \cap L) \cup \{P\}$. Since $h^0(\mathbb{P}^2, \mathcal{I}_{E\cap 2L}(2)) = 2$, Q_1 is not a cone with vertex containing P. Hence there is $i \in \{1, 2\}$ and $O \in L_i \setminus L_1 \cap L_2$ such that that there is a quadric surface Q_1 containing $\{P\} \cup (E \cap 2L)$, but not the tangent vector w of $\langle\{O, P\}\rangle$ at P. Hence $h^1(\mathbb{P}^3, \mathcal{I}_{(E\cap 2L)\cup w}(2)) = 0$. Since $(2L \cap E) \cup w \subset Q$, we get $h^1(Q, \mathcal{I}_{(2L\cap E)\cup w}(2)) = 0$. Since $t \geq 2$ we have $h^1(Q, \mathcal{I}_{(2L\cap E)\cup w}(2)) = 0$. Since $E \cup \langle\{O, P\}\rangle$ is a tree, we conclude the case $r = 3$ and $d \leq 3$. Now assume $r \geq 4$. We use induction on r. Let $M \subset \mathbb{P}^r$ be a general hyperplane. Set $Q' := Q \cap M$. Notice that Q' is the double of the hyperplane $H \cap M$ of M. Let $\rho' : H^0(Q, \mathcal{O}_Q(2)) \to H^0(Q, \mathcal{O}_Q(2))$ be the restriction map. If $d < r$ we take a degree d tree $B \subset M$ such that $h^1(Q', \mathcal{I}_{Q'\cap B}(2)) = 0$. Hence $h^1(Q, \mathcal{I}_{Q\cap B}(2)) = 0$. Since $t \geq 2$, we get $h^1(Q, \mathcal{I}_{Q\cap B}(t)) = 0$. Now assume $d = r$. The inductive assumption give the existence of a degree $d-1$ tree $A \subset M$ such that $h^1(Q', \mathcal{I}_{A\cap Q'}(2)) = 0$. Let $L \subset \mathbb{P}^r$ be a line such that $L \nsubseteq M$ and L contains a smooth point of A. Notice that $A \cup L$ is a tree. Since the restriction map is surjective and $t \geq 2$, it is sufficient to find L such that $h^0(\mathbb{P}^r, \mathcal{I}_{(A\cup Q')\cup L\cap Q}(2)) = h^0(\mathbb{P}^r, \mathcal{I}_{(A\cup Q')}(2)) - 2$. This is done as in the case $r = 3$, because A spans M.

(i4) Assume $d \geq r+1$. Fix a general $f \in H^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t))$ and set $U := \{f = 0\}$. Bertini's theorem says that U is smooth outside $\mathcal{B}$. By step (i1) U does not contains (i2). By step (i2) $U \cap H$ is not contained in $\mathcal{B} \cap H$. Hence there is $P \in U \cap H$ such that P is smooth at U. Since $d-1 \geq r$ and we may take as A a general tree of type τ, the tree A spans $\mathbb{P}^r$. Since A spans $\mathbb{P}^r$ there is $S \subset A_{reg}$ such that $\sharp(S) = r$, the tangent vectors at P of the lines $\langle\{O, P\}\rangle$, $O \in S$, are linearly independent and for each $O \in S$ $A \cup \langle\{O, P\}\rangle$ is a tree (notice that for some f we may take as P a general point of H). Since U is smooth at P, we may find

$O \in A_{reg}$ such that $A \cup \langle\{O,P\}\rangle$ is a tree and the tangent vector w of $\langle\{O,P\}\rangle$ at P is not contained in U. Hence $f \notin H^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup w}(t))$. Since $P \notin \mathcal{B}$, we have $h^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup\{P\}}(t)) = h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) - 1$. Since $f \in H^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup\{P\}}(t))$, while $f \notin H^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup w}(t))$ and $f \notin H^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup w}(t))$, we again obtain $h^0(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup w}(t)) = h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) - 2$. Hence $h^1(\mathbb{P}^r, \mathcal{I}_{(A\cap Q)\cup w}(t)) = 0$.

(ii) Now assume $d > u$. In this range we need to find Y with $h^0(Q, \mathcal{I}_{Y\cap Q}(t)) = 0$. As in step (i) of the proof of Theorem 1 to do all cases with $d > u$ it is sufficient to do the case $d = w$. If $w = u$, then we just proved this case in step (i). Now assume $w = u + 1$, i.e. assume that $\binom{r+t}{r} - \binom{r+t-2}{r}$ is an odd integer. Let $A \subset \mathbb{P}^r$ be a degree u tree such that $h^1(Q, \mathcal{I}_{A\cap Q}(t)) = 0$, i.e. $h^0(Q, \mathcal{I}_{A\cap Q}(t)) = 1$. First assume that H is not contained in the base locus $\mathcal{B}$ of $\mathcal{I}_{A\cap Q}(t)$. We may take as degree d tree any tree $A \cup L$ with L containing a general point of H. Now assume $H \subseteq \mathcal{B}$. For general A we have $h^0(\mathbb{P}^r, \mathcal{I}_{A\cap Q}(t)) = h^0(\mathbb{P}^r, \mathcal{I}_{A\cap H}(t-1) = \binom{r+t-1}{r} - u < h^0(\mathbb{P}^r, \mathcal{I}_Q t))$, a contradiction. □

Proof of Theorem 2. Any two quadric hypersurfaces of $\mathbb{P}^r$ with the same rank are projectively equivalent. Hence Theorem 2 is true for one rank e hyperquadric if and only if it is true for all rank e hyperquadrics. Lemma 1 proves the case $e = 0$. Now assume $e > 0$. There is a flat family of rank e hyperquadrics of $\mathbb{P}^r$ with as a flat limit a double hyperplane $2H$. Apply Lemma 1 to $2H$ and then use the semicontinuity theorem for cohomology. □

3. Intersection with a Hyperplane

Let $M \subseteq \mathbb{P}^r$ be a reduced variety. A *tangent vector* of M is a degree 2 connected zero-dimensional scheme $Z \subset M_{reg}$. Fix a type $\tau : \{2, \dots, d-1\} \to \{1, \dots, d-1\}$. For each $i \in \{1, \dots, d\}$ the function $\tau|\{1, \dots, i\}$ is a type for degree i trees.

Remark 1. Let $H \subset \mathbb{P}^r$ be a hyperplane and $Y \subset \mathbb{P}^r$ any tree. If no line of Y is contained in H, then the scheme $Y \cap H$ is the disjoint union of finitely many points and tangent vectors.

Any tree of degree d has $d-1$ singular points. Fix $Y = L_1 \cup \cdots \cup L_d \in Z(r, d, \tau)$. For each $P \in \mathrm{Sing}(Y)$ let $\tau(P)$ be the two indices $i, j \in \{1, \dots, d\}$ such that $\{P\} = L_i \cap L_j$. For each line $L \subseteq Y$ set $\sigma(L) := \sharp(S \cap L)$. Set $\sigma(i) = \sigma(L_i)$. The function σ is uniquely determined by τ. A *final line* of Y is a line L with $\sigma(L) \le 1$ (and hence $\sigma(L) = 1$ if $d \ge 2$). Any bamboo of degree $d \ge 2$ has exactly two ordering of its components associated to the type of any bamboo. Y is comb $\Longrightarrow$ there is a line $L \subseteq Y$ such that $\sigma(L) = d - 1 \Longrightarrow$

$\sigma(1) = d-1$. A comb has a unique type. Assume $d \ge 4$. The *wire* $w(Y)$ of Y is the maximal cardinality of a set $S' \subseteq \mathrm{Sing}(Y)$ such that the linear space $\langle S' \rangle$ spanned by S' contains no line of Y. Obviously $w(Y) \le \min\{d-1, r\}$. The wire $w(\tau)$ is the integer $w(Y)$, where Y is a general element of $Z(r,d,\tau)$. For many τ there are $Y_1, Y_2 \in Z(r,d,\tau)$ such that $w(Y_1) < w(\tau) < w(Y_2)$.

Remark 2. For any tree $Y \subset \mathbb{P}^r$ the wire $w(Y)$ is the only integer with the following properties:

(i) There is a hyperplane of $\mathbb{P}^r$ containing no line of Y and containing exactly $w(Y)$ singular points of Y.

(ii) Every hyperplane of $\mathbb{P}^r$ containing at least $w(Y)+1$ singular points of Y contains a component of Y.

Proposition 1. *Fix an integer $d > 0$, a hyperplane $H \subset \mathbb{P}^r$ and a set $S \subset H$ such that $\sharp(S) = d$ and no 3 of the points of S are collinear. Fix any type τ for trees of degree d. Then there is $Y \in Z(r,d,\tau)$ such that $Y \cap H = S$.*

Proof. Take an ordering $P_1, \dots, P_d$ of the points of S. Let L_1 be a general line through P_1. Hence we may assume $d \ge 2$ and use induction on d. Take $A \in Z(r, d-1, \tau|\{1,\dots,d-1\})$ such that $A \cap H = \{P_1, \dots, P_{d-1}\}$. Let E be the plane spanned by $L_{\tau(d)}$ and by P_d. We claim that $L_{\tau(d)}$ is the only line of A contained in E. Assume that this is not true and call B another line of $A \cap E$. The set $\{P_d\} \cup (L_{\tau(d)} \cup B) \cap H$ contains 3 collinear points, contradicting our assumption. Hence we may take $Y = A \cup L_d$ with L a general line through P_d contained in the plane E. □

Proposition 2. *Fix a hyperplane $H \subset \mathbb{P}^r$, $r \ge 3$, a finite set $S \subset H$ and $P \in S$. Then there is a comb $Y \subset \mathbb{P}^r$ such that $Y \cap H = S$ and P is the intersection of H with the spine of Y if and only if $\sharp(D \cap S) \le 2$ for each line $D \subset H$ containing P.*

Proof. First assume $\sharp(D \cap S) \le 2$ for each line $D \subset H$ containing P.Take an ordering $P_1, \dots, P_d$ of the points of S with $P_1 := P$. Let L_1 be a general line through P_1. Hence we may assume $d \ge 2$ and use induction on d. Take a degree $d-1$ comb A with L_1 as its such that $A \cap H = \{P_1, \dots, P_{d-1}\}$. The plane B spanned by L_1 and P_d contain no points of $A \setminus L$, because $T \cap L_1 \neq \emptyset$ the line D through P_1 and P_2 would contain the point $T \cap H$. □

Remark 3. Fix an integer $d \ge 2$. Let $Y \subset \mathbb{P}^r$ be a tree. The following conditions are equivalent:

1. Y is a bamboo;

2. Y has only two final lines;

3. no line of Y intersect at least two other components of Y.

Proposition 3. *Assume d even. Let $H \subset \mathbb{P}^r$ be a hyperplane. Let $Z \subset H$ be a disjoint union of $d/2$ tangent vectors. Assume $\deg(D \cap Z) \leq 2$ for all lines $D \subset H$. Then there is a tree $Y \subset \mathbb{P}^r$ such that $Y \cap H = Z$. Any such tree is a bamboo.*

Proof. We use induction on the integer $d/2$, the case $d = 2$ being obvious. Take any $P \in Z_{red}$. Call v the connected component of Z with P as its reduction and set $Z' := Z \setminus v$. Let $A = L_1 \cup \cdots \cup L_{d/2-1} \subset \mathbb{P}^r$ be a bamboo such that $Z' = Z \cap B$. Let B be the plane spanned by $L_{d/2-1}$ and P. We claim that there is no line $L \subset A$ such that $L \neq L_{d/2-1}$ and $L \subset B$. Assume the existence of L. The set $H \cap B$ is the line spanned by $L_{d/2-1} \cap H$ and P. Since $L \cap H \subset H \cap B$, we get $\deg(D \cap Z) \geq 3$, a contradiction. Fix a general plane $U \subset \mathbb{P}^r$ containing R. Since $v \subset R$, we have $R \cap A = \emptyset$. Hence $U \cap A$ is finite. Hence we may take as L_d a general line in U containing P.

Part 3 of Remark 3 implies that any such tree is a bamboo. □

Theorem 3. *Fix integers $d \geq 2a \geq 0$, $d > 0$. Let $H \subset \mathbb{P}^r$ be a hyperplane. Let $Z \subset H$ be a disjoint union of a tangent vectors and $d - 2a$ points. Assume $\deg(D \cap Z) \leq 2$ for all lines $D \subset H$. Then there is a degree d bamboo $Y \subset \mathbb{P}^r$ such that $Y \cap H = Z$.*

Proof. If $d = 2a$, then apply Proposition 3. Now assume $d > 2a$. Take $Z = S \sqcup W$ with S union of the reduced connect components of Z and W the union of the tangent vectors. First apply the proof of Proposition 1 to S (say obtaining a bamboo $L_1 \cup \cdots L_{d-2a}$ and then apply the proof of Proposition 3 to W, starting from any $P \in W_{red}$ and taking as L_{d-2a+1} a general line through P intersecting L_{d-2a+1}. □

Remark 4. Fix an odd integer $d \geq 3$ and a hyperplane $H \subset \mathbb{P}^r$. Let $Z \subset H$ be a disjoint union of $(d-1)/2$ tangent vector and one point P. Assume the existence of a degree d tree $Y \subset \mathbb{P}^r$ such that $Z = H \cap Y$ and call L the line of Y containing P. Then the projective curve $\overline{Y \setminus L} \subset \mathbb{P}^r$ is a disjoint union of bamboos of even degree.

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