FURTHER RESEARCH ON THE M/M/1 RETRIAL QUEUEING MODEL WITH SPECIAL RETRIAL TIMES

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Abstract: We study the spectrum of the underlying operator which corresponds to the M/M/1 retrial queueing model with special retrial times and obtain that all $\eta \in \left(-\lambda, \frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{2}\right)$ are eigenvalues of the underlying operator with geometric multiplicity one.

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1. Introduction

According to Gomez-Corral [1], the M/M/1 retrial queueing model with special retrial times can be described by the following system of equations:

$$\frac{dp_0(t)}{dt} = -\lambda p_0(t) + \beta \int_0^\infty Q_0(x, t) dx, \quad (1)$$

$$\frac{\partial p_n(x, t)}{\partial t} + \frac{\partial p_n(x, t)}{\partial x} = -(\lambda + \alpha) p_n(x, t), \quad \forall n \geq 1, \quad (2)$$

$$\frac{\partial Q_0(x, t)}{\partial t} + \frac{\partial Q_0(x, t)}{\partial x} = -(\lambda + \beta) Q_0(x, t), \quad (3)$$

$$\frac{\partial Q_n(x, t)}{\partial t} + \frac{\partial Q_n(x, t)}{\partial x} = -(\lambda + \beta)Q_n(x, t) + \lambda Q_{n-1}(x, t), \quad \forall n \geq 1, \quad (4)$$

$$p_n(0, t) = \beta \int_0^\infty Q_n(x, t) dx \quad \forall n \geq 1, \quad (5)$$

$$Q_0(0, t) = \lambda p_0(t) + \alpha \int_0^\infty p_1(x, t) dx, \quad (6)$$

$$Q_n(0, t) = \lambda \int_0^\infty p_n(x, t) dx + \alpha \int_0^\infty p_{n+1}(x, t) dx, \quad \forall n \geq 1, \quad (7)$$

$$p_0(0) = 1, \quad p_n(x, 0) = 0, \quad \forall n \geq 1, \quad Q_j(x, 0) = 0, \quad \forall j \geq 0. \quad (8)$$

Where $p_0(t)$ is the probability that at time t there is no customer and the server is idle. $p_n(x, t)$ ($n \geq 1$) represents the probability that at time t the server is idle and there are n customers in the system with elapsed retrial time x . $Q_n(x, t)dx$ ($n \geq 0$) is the probability that at time t the server is busy, there are n customers in the system with elapsed service time of the customer undergoing service lying in $[x, x + dx)$. λ is the arrival rate of customers. α is the repeated attempt rate of customers. β is the service rate of the server.

In 1999, Gomez-Corral [1] established the M/M/1 retrial queueing model with special retrial times by using the supplementary variable technique and obtained the existence of its steady-state solution by using the probability generating function. In 2005, by using C_0 -semigroup theory of linear operators Gupur [2] proved that the model has a unique positive time-dependent solution which satisfies the probability condition. In 2005, Zhang and Gupur [9] studied the resolvent set of the underlying operator which corresponds to the M/M/1 retrial queueing model with special retrial times and obtained that all points on the imaginary axis except 0 belong to the resolvent set of the underlying operator. In 2006, Jiang and Gupur [6] proved that 0 is an eigenvalue of the underlying operator with geometric and algebraic multiplicity one and 0 is an eigenvalue of its adjoint operator. These results together with theorem 14 in Gupur et al [4] imply that the time-dependent solution of this model strongly converges to its steady-state solution. In 2009, Lv and Gupur [7] researched eigenvalues of the underlying operator on the left half complex plane and obtained that the underlying operator has one eigenvalue on the left half complex plane. In 2010, Ismayil and Gupur [5] studied eigenvalues of the underlying operator on the left half complex plane and obtained that if $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4}$, then the operator has uncountably infinitely many eigenvalues on the left half complex plane. In 2011, Gupur [3] has proved that the results in Ismayil and Gupur [5] also hold for $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} = \frac{1}{4}$. So far, any other results have not been found about this model in the literature. On the basis of the work by Ismayil and Gupur [5],

we study eigenvalues of the underlying operator on the left half complex plane and found that the underlying operator also has uncountably infinitely many eigenvalues on the left side of the eigenvalues which was given by Ismayil and Gupur [5].

In this paper, we use the notations in Gupur [2]. Choose the state space as follows.

$$X = \{g = (p, Q) \mid \|g\| = \|p\| + \|Q\| < \infty\},$$

here

$$\begin{aligned} p &\in \mathbb{R} \times L^1[0, \infty) \times L^1[0, \infty) \times L^1[0, \infty) \times \cdots, \\ Q &\in L^1[0, \infty) \times L^1[0, \infty) \times L^1[0, \infty) \times \cdots, \\ \|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0, \infty)}, \quad \|Q\| = \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0, \infty)}. \end{aligned}$$

It is obvious that X is a Banach space. We denote

$$\begin{aligned} \Gamma_0 &= \begin{pmatrix} e^{-x} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \beta & 0 & 0 & \cdots \\ 0 & 0 & \beta & 0 & \cdots \\ 0 & 0 & 0 & \beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\ \Gamma_2 &= \begin{pmatrix} \lambda e^{-x} & \alpha & 0 & 0 & 0 & \cdots \\ 0 & \lambda & \alpha & 0 & 0 & \cdots \\ 0 & 0 & \lambda & \alpha & 0 & \cdots \\ 0 & 0 & 0 & \lambda & \alpha & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \end{aligned}$$

In the following we define operators and their domains.

$$\begin{aligned} A(p, Q)(x) &= \left(\begin{pmatrix} -\lambda & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_0 \\ p_1(x) \\ p_2(x) \\ \vdots \end{pmatrix}, \right. \\ &\quad \left. \begin{pmatrix} -\frac{d}{dx} & 0 & 0 & \cdots \\ 0 & -\frac{d}{dx} & 0 & \cdots \\ 0 & 0 & -\frac{d}{dx} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ \vdots \end{pmatrix} \right), \end{aligned}$$

$$\begin{aligned}
D(A) = & \left\{ (p, Q) \in X \left| \begin{array}{l} \frac{dp_i(x)}{dx} \in L^1[0, \infty), (i \geq 1), \frac{dQ_n(x)}{dx} \in L^1[0, \infty) (n \geq 1), \\ p_i(x) \text{ and } Q_n(x) \text{ are absolutely continuous and satisfy} \\ p(0) = \int_0^\infty \Gamma_0 p(x) dx + \int_0^\infty \Gamma_1 Q(x) dx, Q(0) = \int_0^\infty \Gamma_2 p(x) dx \end{array} \right. \right\}, \\
U(p, Q)(x) = & \left(\left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & \cdots \\ 0 & -(\lambda + \alpha) & 0 & 0 & \cdots \\ 0 & 0 & -(\lambda + \alpha) & 0 & \cdots \\ 0 & 0 & 0 & -(\lambda + \alpha) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) \left(\begin{array}{c} p_0 \\ p_1(x) \\ p_2(x) \\ p_3(x) \\ \vdots \end{array} \right) \right. \\
& \left. \left(\begin{array}{ccccc} -(\lambda + \beta) & 0 & 0 & 0 & \cdots \\ \lambda & -(\lambda + \beta) & 0 & 0 & \cdots \\ 0 & \lambda & -(\lambda + \beta) & 0 & \cdots \\ 0 & 0 & \lambda & -(\lambda + \beta) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \right) \left(\begin{array}{c} Q_0(x) \\ Q_1(x) \\ Q_2(x) \\ Q_3(x) \\ \vdots \end{array} \right) \right) \Bigg|, \\
E(p, Q)(x) = & \left(\begin{array}{c} \int_0^\infty \beta Q_0(x) dx \\ 0 \\ 0 \\ \vdots \end{array} \right), \quad D(U) = X, \quad D(E) = X.
\end{aligned}$$

Then the above system of equations (1)–(8) can be written as an abstract Cauchy problem in the Banach space X :

$$\begin{cases} \frac{d(p, Q)(t)}{dt} = (A + U + E)(p, Q)(t), & \forall t \in (0, \infty) \\ (p, Q)(0) = (p(0), Q(0)), \\ p(0) = (1, 0, 0, \dots), Q(0) = (0, 0, 0, \dots) \end{cases} \quad (9)$$

Gupur [2] has obtained the following results.

Theorem 1. $A + U + E$ generates a positive contraction C_0 –semigroup $T(t)$. $T(t)$ is isometric for the initial value. So, the system (9) has a unique positive time-dependent solution which satisfies the probability condition.

Zhang and Gupur [9], Jiang and Gupur [6] have gotten the following results.

Theorem 2. If $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < 1$, then 0 is an eigenvalue of $(A + U + E)$ and $(A + U + E)^*$ with geometric multiplicity one.

$$\left\{ \gamma \in \mathbb{C} \left| \sup \left\{ \frac{\lambda}{|\gamma + \lambda|}, \frac{\lambda\alpha|\gamma + \lambda + \beta|(Re\gamma + \lambda + \alpha + \beta)}{(Re\gamma + \lambda + \alpha)(Re\gamma + \lambda + \beta)(|\gamma + \lambda + \alpha||\gamma + \lambda + \beta| - \alpha\beta)} \right. \right. \right. \\
\left. \left. \left. + \frac{\lambda}{Re\gamma + \lambda + \alpha}, \frac{\lambda|\gamma + \lambda + \alpha||\gamma + \lambda + \beta|(Re\gamma + \lambda + \alpha + \beta)}{(Re\gamma + \lambda + \alpha)(Re\gamma + \lambda + \beta)(|\gamma + \lambda + \alpha||\gamma + \lambda + \beta| - \alpha\beta)} \right\} < 1 \right\}$$

belongs to the resolvent set of $(A + U + E)^*$. In particular, all points on the imaginary axis except 0 belong to the resolvent set of $A + U + E$. Therefore, the time-dependent solution of the system (9) strongly converges to its steady-state solution.

Lv and Gupur [7], Ismayil and Gupur [5] deduced the following results.

Theorem 3. If $\frac{\lambda(\alpha+\lambda)}{\alpha\beta} < \frac{1}{4}$, then for all $\theta \in (0, 1]$, $\frac{-(2\lambda+\alpha+\beta)+\sqrt{(\alpha+\beta)^2+4\lambda\beta}}{4}\theta$ are eigenvalues of $(A + U + E)$ with geometric multiplicity one.

2. Main Results

Theorem 4. If $\frac{1}{4} < \frac{\lambda}{\beta} < \frac{\lambda(\alpha+\lambda)}{\alpha\beta} < 1$, then all

$$\eta \in \left(-\lambda, \frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2} \right)$$

are eigenvalues of $(A + U + E)$ with geometric multiplicity one.

Proof. Consider the equation $[(A + U + E) - \eta I](p, Q) = 0$, which is equivalent to

$$-(\lambda + \eta)p_0 + \beta \int_0^\infty Q_0(x)dx = 0, \quad (10)$$

$$-\frac{dp_n(x)}{dx} - (\lambda + \eta + \alpha)p_n(x) = 0, \quad \forall n \geq 1, \quad (11)$$

$$-\frac{dQ_0(x)}{dx} - (\lambda + \eta + \beta)Q_0(x) = 0, \quad (12)$$

$$-\frac{dQ_n(x)}{dx} - (\lambda + \eta + \beta)Q_n(x) + \lambda Q_{n-1}(x) = 0, \quad \forall n \geq 1, \quad (13)$$

$$p_n(0) = \beta \int_0^\infty Q_n(x)dx, \quad \forall n \geq 1, \quad (14)$$

$$Q_0(0) = \lambda p_0 + \alpha \int_0^\infty p_1(x)dx, \quad (15)$$

$$Q_n(0) = \lambda \int_0^\infty p_n(x)dx + \alpha \int_0^\infty p_{n+1}(x)dx, \quad \forall n \geq 1. \quad (16)$$

If we take $m = \lambda + \eta$, then (10)–(16) become

$$mp_0 = \beta \int_0^\infty Q_0(x)dx, \quad (17)$$

$$\frac{dp_n(x)}{dx} = -(m + \alpha)p_n(x), \quad \forall n \geq 1, \quad (18)$$

$$\frac{dQ_0(x)}{dx} = -(m + \beta)Q_0(x), \quad (19)$$

$$\frac{dQ_n(x)}{dx} = -(m + \beta)Q_n(x) + \lambda Q_{n-1}(x), \quad \forall n \geq 1, \quad (20)$$

$$p_n(0) = \beta \int_0^\infty Q_n(x) dx, \quad \forall n \geq 1, \quad (21)$$

$$Q_0(0) = \lambda p_0 + \alpha \int_0^\infty p_1(x) dx, \quad (22)$$

$$Q_n(0) = \lambda \int_0^\infty p_n(x) dx + \alpha \int_0^\infty p_{n+1}(x) dx, \quad \forall n \geq 1. \quad (23)$$

Solving (18), (19) and (20) we have

$$p_n(x) = a_n e^{-(m+\alpha)x}, \quad \forall n \geq 1, \quad (24)$$

$$Q_0(x) = b_0 e^{-(m+\beta)x}, \quad (25)$$

$$Q_n(x) = b_n e^{-(m+\beta)x} + \lambda e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} Q_{n-1}(\tau) d\tau, \quad \forall n \geq 1. \quad (26)$$

By inserting (25) into (17) and using $m + \beta > 0$ we obtain

$$\begin{aligned} mp_0 &= \beta \int_0^\infty Q_0(x) dx = \beta \int_0^\infty b_0 e^{-(m+\beta)x} dx = \frac{\beta}{m + \beta} b_0 \\ &\implies \\ b_0 &= \frac{m + \beta}{\beta} mp_0. \end{aligned} \quad (27)$$

From (25), (22), (24), (27) and $m + \alpha > 0$ it follows that

$$\begin{aligned} b_0 &= Q_0(0) = \lambda p_0 + \alpha \int_0^\infty p_1(x) dx = \lambda p_0 + \alpha \int_0^\infty a_1 e^{-(m+\alpha)x} dx \\ &= \lambda p_0 + \frac{\alpha}{m + \alpha} a_1 \\ &\implies \\ \frac{\alpha}{m + \alpha} a_1 &= b_0 - \lambda p_0 = \frac{m + \beta}{\beta} mp_0 - \lambda p_0 \\ &\implies \\ a_1 &= \frac{(m + \alpha)(m^2 + m\beta - \lambda\beta)}{\alpha\beta} p_0. \end{aligned} \quad (28)$$

By combining (24), (21), (26) and (25) with (27) and (28), and using $m + \beta > 0$, $\lim_{x \rightarrow \infty} \frac{x}{e^{(m+\beta)x}} = 0$ we deduce

$$\begin{aligned}
 a_1 &= p_1(0) = \beta \int_0^\infty Q_1(x) dx \\
 &= \beta \int_0^\infty \left[b_1 e^{-(m+\beta)x} + \lambda e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} Q_0(\tau) d\tau \right] dx \\
 &= \frac{\beta}{m+\beta} b_1 + \lambda \beta \int_0^\infty e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} Q_0(\tau) d\tau dx \\
 &= \frac{\beta}{m+\beta} b_1 + \lambda \beta \int_0^\infty e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} b_0 e^{-(m+\beta)\tau} d\tau dx \\
 &= \frac{\beta}{m+\beta} b_1 + \lambda \beta b_0 \int_0^\infty x e^{-(m+\beta)x} dx \\
 &= \frac{\beta}{m+\beta} b_1 \\
 &\quad + \lambda \beta b_0 \left\{ -\frac{1}{m+\beta} x e^{-(m+\beta)x} \Big|_0^\infty + \frac{1}{m+\beta} \int_0^\infty e^{-(m+\beta)x} dx \right\} \\
 &= \frac{\beta}{m+\beta} b_1 + \frac{\lambda \beta}{(m+\beta)^2} b_0 \\
 &= \frac{\beta}{m+\beta} b_1 + \frac{\lambda \beta}{(m+\beta)^2} \times \frac{m+\beta}{\beta} m p_0 \\
 &= \frac{\beta}{m+\beta} b_1 + \frac{\lambda m}{m+\beta} p_0 \\
 &\implies \\
 \frac{\beta}{m+\beta} b_1 &= a_1 - \frac{\lambda m}{m+\beta} p_0 \\
 &= \frac{(m+\alpha)(m^2+m\beta-\lambda\beta)}{\alpha\beta} p_0 - \frac{\lambda m}{m+\beta} p_0 \\
 &= \frac{(m+\alpha)(m+\beta)(m^2+m\beta-\lambda\beta) - m\lambda\alpha\beta}{\alpha\beta(m+\beta)} p_0 \\
 &\implies \\
 b_1 &= \frac{(m+\alpha)(m+\beta)(m^2+m\beta-\lambda\beta) - m\lambda\alpha\beta}{\alpha\beta^2} p_0. \tag{29}
 \end{aligned}$$

Similarly, (26), (23), (24), (28), (29) and $m + \alpha > 0$ give

$$b_1 = Q_1(0) = \lambda \int_0^\infty p_1(x) dx + \alpha \int_0^\infty p_2(x) dx$$

$$\begin{aligned}
&= \lambda \int_0^\infty a_1 e^{-(m+\alpha)x} dx + \alpha \int_0^\infty a_2 e^{-(m+\alpha)x} dx \\
&= \frac{\lambda}{m+\alpha} a_1 + \frac{\alpha}{m+\alpha} a_2 \\
&\implies \\
\frac{\alpha}{m+\alpha} a_2 &= b_1 - \frac{\lambda}{m+\alpha} a_1 \\
&= \frac{(m+\alpha)(m+\beta)(m^2+m\beta-\lambda\beta) - m\lambda\alpha\beta}{\alpha\beta^2} p_0 \\
&\quad - \frac{\lambda}{m+\alpha} \times \frac{(m+\alpha)(m^2+m\beta-\lambda\beta)}{\alpha\beta} p_0 \\
&= \left[\frac{(m+\alpha)^2(m+\beta)(m^2+m\beta-\lambda\beta) - (m+\alpha)m\lambda\alpha\beta}{\alpha\beta^2(m+\alpha)} \right. \\
&\quad \left. - \frac{\lambda\beta(m+\alpha)(m^2+m\beta-\lambda\beta)}{\alpha\beta^2(m+\alpha)} \right] p_0 \\
&\implies \\
a_2 &= \frac{1}{(\alpha\beta)^2} \left[(m+\alpha)^2(m+\beta)(m^2+m\beta-\lambda\beta) \right. \\
&\quad \left. - \lambda\beta(m+\alpha)(m^2+m\beta-\lambda\beta+m\alpha) \right] p_0. \tag{30}
\end{aligned}$$

By using (26), (23), (24), (21) and the Fubini theorem we calculate

$$\begin{aligned}
b_{n+1} &= Q_{n+1}(0) = \lambda \int_0^\infty p_{n+1}(x) dx + \alpha \int_0^\infty p_{n+2}(x) dx \\
&= \lambda \int_0^\infty a_{n+1} e^{-(m+\alpha)x} dx + \alpha \int_0^\infty a_{n+2} e^{-(m+\alpha)x} dx \\
&= \frac{\lambda}{m+\alpha} a_{n+1} + \frac{\alpha}{m+\alpha} a_{n+2}, \quad \forall n \geq 1. \tag{31} \\
a_{n+1} &= p_{n+1}(0) = \beta \int_0^\infty Q_{n+1}(x) dx \\
&= \beta \int_0^\infty \left[b_{n+1} e^{-(m+\beta)x} + \lambda e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} Q_n(\tau) d\tau \right] dx \\
&= \frac{\beta}{m+\beta} b_{n+1} + \lambda\beta \int_0^\infty e^{(m+\beta)\tau} Q_n(\tau) \int_\tau^\infty e^{-(m+\beta)x} dx d\tau \\
&= \frac{\beta}{m+\beta} b_{n+1} + \frac{\lambda\beta}{m+\beta} \int_0^\infty Q_n(\tau) d\tau
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\beta}{m+\beta} b_{n+1} + \frac{\lambda}{m+\beta} p_n(0) \\
 &= \frac{\beta}{m+\beta} b_{n+1} + \frac{\lambda}{m+\beta} a_n, \quad \forall n \geq 1.
 \end{aligned} \tag{32}$$

By inserting (31) into (32) and rearranging, it is immediately got

$$a_{n+2} = \frac{(m+\alpha)(m+\beta) - \lambda\beta}{\alpha\beta} a_{n+1} - \frac{\lambda(m+\alpha)}{\alpha\beta} a_n, \quad \forall n \geq 1. \tag{33}$$

If we set

$$\begin{aligned}
 s+t &= \frac{(m+\alpha)(m+\beta) - \lambda\beta}{\alpha\beta}, \quad s \cdot t = \frac{\lambda(m+\alpha)}{\alpha\beta}, \quad (s \neq t), \\
 f(x) &= x^2 - (s+t)x + st
 \end{aligned} \tag{34}$$

then $f(s) = f(t) = 0$. The definition of η implies

$$\begin{aligned}
 -\lambda < \eta &< \frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2} < 0 \\
 \implies \\
 0 < m = \eta + \lambda &< \frac{-(\alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2} < \lambda,
 \end{aligned} \tag{35}$$

$$\begin{aligned}
 \implies \\
 \frac{(m+\alpha)(m+\beta) - \lambda\beta}{\alpha\beta} &< 1.
 \end{aligned} \tag{36}$$

Since $\frac{\lambda}{\beta} > \frac{1}{4}$,

$$\frac{\lambda(m+\alpha)}{\alpha\beta} > \frac{1}{4}. \tag{37}$$

(36) and (37) give

$$\Delta = \left[\frac{(m+\alpha)(m+\beta) - \lambda\beta}{\alpha\beta} \right]^2 - 4 \times \frac{\lambda(m+\alpha)}{\alpha\beta} < 0.$$

Which means that s, t are complex conjugate roots of $f(x)$ and therefore

$$|s| = |t| = \sqrt{s \cdot t} = \sqrt{\frac{\lambda(m+\alpha)}{\alpha\beta}}. \tag{38}$$

The condition $\frac{\lambda(\lambda+\alpha)}{\alpha\beta} < 1$ and (35) imply $\frac{\lambda(m+\alpha)}{\alpha\beta} < 1$. From this together with (38) we know

$$0 < |s| = |t| < 1. \quad (39)$$

(33) and (34) give

$$\begin{aligned} a_{n+2} &= \frac{(m+\alpha)(m+\beta) - \lambda\beta}{\alpha\beta} a_{n+1} - \frac{\lambda(m+\alpha)}{\alpha\beta} a_n \\ &= (s+t)a_{n+1} - sta_n \\ &\implies \\ a_{n+2} - sa_{n+1} &= t(a_{n+1} - sa_n) = t^2(a_n - sa_{n-1}) = t^3(a_{n-1} - sa_{n-2}) \\ &= \dots\dots\dots = t^n(a_2 - sa_1), \quad n \geq 1 \\ &\implies \\ a_{n+2} - sa_{n+1} &= t^n(a_2 - sa_1), \\ sa_{n+1} - s^2a_n &= st^{n-1}(a_2 - sa_1), \\ s^2a_n - s^3a_{n-1} &= s^2t^{n-2}(a_2 - sa_1), \\ &\dots\dots\dots \\ s^{n-3}a_5 - s^{n-2}a_4 &= s^{n-3}t^3(a_2 - sa_1), \\ s^{n-2}a_4 - s^{n-1}a_3 &= s^{n-2}t^2(a_2 - sa_1), \\ s^{n-1}a_3 - s^na_2 &= s^{n-1}t(a_2 - sa_1). \end{aligned}$$

By adding all the above equations we have

$$\begin{aligned} &(a_{n+2} - sa_{n+1}) + (sa_{n+1} - s^2a_n) + (s^2a_n - s^3a_{n-1}) + \dots\dots\dots \\ &\quad + (s^{n-3}a_5 - s^{n-2}a_4) + (s^{n-2}a_4 - s^{n-1}a_3) + (s^{n-1}a_3 - s^na_2) \\ &= t^n(a_2 - sa_1) + st^{n-1}(a_2 - sa_1) + s^2t^{n-2}(a_2 - sa_1) + \dots \\ &\quad + s^{n-3}t^3(a_2 - sa_1) + s^{n-2}t^2(a_2 - sa_1) + s^{n-1}t(a_2 - sa_1) \\ &\implies \\ a_{n+2} - s^na_2 &= (t^n + st^{n-1} + s^2t^{n-2} + \dots + s^{n-3}t^3 + s^{n-2}t^2 + s^{n-1}t)(a_2 - sa_1) \\ &= \left(\frac{s^{n+1} - t^{n+1}}{s - t} - s^n \right) (a_2 - sa_1) \\ &\implies \\ a_{n+2} &= s^{n+1}a_1 + \frac{s^{n+1} - t^{n+1}}{s - t}(a_2 - sa_1), \quad \forall n \geq 1. \end{aligned} \quad (40)$$

Which implies

$$a_{n+1} = s^n a_1 + \frac{s^n - t^n}{s - t}(a_2 - sa_1), \quad (41)$$

$$a_n = s^{n-1} a_1 + \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1). \quad (42)$$

Combining (31) with (41) and (42) we determine

$$\begin{aligned} b_n &= \frac{\lambda}{m + \alpha} a_n + \frac{\alpha}{m + \alpha} a_{n+1} \\ &= \frac{\lambda}{m + \alpha} \left[s^{n-1} a_1 + \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1) \right] \\ &\quad + \frac{\alpha}{m + \alpha} \left[s^n a_1 + \frac{s^n - t^n}{s - t}(a_2 - sa_1) \right] \\ &= \frac{\lambda s^{n-1}}{m + \alpha} a_1 + \frac{\alpha s^n}{m + \alpha} a_1 \\ &\quad + \frac{\lambda}{m + \alpha} \times \frac{s^{n-1} - t^{n-1}}{s - t}(a_2 - sa_1) + \frac{\alpha}{m + \alpha} \times \frac{s^n - t^n}{s - t}(a_2 - sa_1) \\ &= \frac{s^{n-1}(\lambda + \alpha s)}{m + \alpha} a_1 + \frac{s^{n-1}(\lambda + \alpha s) - t^{n-1}(\lambda + \alpha t)}{(m + \alpha)(s - t)}(a_2 - sa_1), \quad \forall n \geq 1. \end{aligned} \quad (43)$$

From (24), (41), $m + \alpha > 0$, (39), (28) and (30) we estimate

$$\begin{aligned} \|p\| &= |p_0| + \sum_{n=1}^{\infty} \|p_n\|_{L^1[0, \infty)} = |p_0| + \sum_{n=1}^{\infty} \int_0^{\infty} |p_n(x)| dx \\ &= |p_0| + \sum_{n=1}^{\infty} \int_0^{\infty} \left| a_n e^{-(m+\alpha)x} \right| dx = |p_0| + \frac{1}{m + \alpha} \sum_{n=1}^{\infty} |a_n| \\ &= |p_0| + \frac{1}{m + \alpha} \left(|a_1| + \sum_{n=1}^{\infty} |a_{n+1}| \right) \\ &= |p_0| + \frac{1}{m + \alpha} \left[|a_1| + \sum_{n=1}^{\infty} \left| s^n a_1 + \frac{s^n - t^n}{s - t}(a_2 - sa_1) \right| \right] \\ &\leq |p_0| + \frac{1}{m + \alpha} \left(|a_1| + \sum_{n=1}^{\infty} |s^n a_1| + |a_2 - sa_1| \sum_{n=1}^{\infty} \left| \frac{s^n - t^n}{s - t} \right| \right) \\ &\leq |p_0| + \frac{1}{m + \alpha} \left(|a_1| + |a_1| \sum_{n=1}^{\infty} |s|^n + \frac{|a_2 - sa_1|}{|s - t|} \sum_{n=1}^{\infty} |s|^n + \frac{|a_2 - sa_1|}{|s - t|} \sum_{n=1}^{\infty} |t|^n \right) \end{aligned}$$

$$\begin{aligned}
&= |p_0| + \frac{1}{m+\alpha} \left(\frac{|a_1|}{1-|s|} + \frac{|a_2-sa_1|}{|s-t|} \times \frac{|s|}{1-|s|} + \frac{|a_2-sa_1|}{|s-t|} \times \frac{|t|}{1-|t|} \right) \\
&< \infty.
\end{aligned} \tag{44}$$

By (26), $m + \beta > 0$, the Fubini theorem, (25) and (43) we know

$$\begin{aligned}
\|Q_n\|_{L^1[0,\infty)} &= \int_0^\infty |Q_n(x)| dx \\
&\leq \int_0^\infty |b_n| e^{-(m+\beta)x} dx + \lambda \int_0^\infty e^{-(m+\beta)x} \int_0^x e^{(m+\beta)\tau} |Q_{n-1}(\tau)| d\tau dx \\
&= \frac{1}{m+\beta} |b_n| + \lambda \int_0^\infty |Q_{n-1}(\tau)| e^{(m+\beta)\tau} \int_\tau^\infty e^{-(m+\beta)x} dx d\tau \\
&= \frac{1}{m+\beta} |b_n| + \frac{\lambda}{m+\beta} \int_0^\infty |Q_{n-1}(\tau)| d\tau \\
&\leq \frac{1}{m+\beta} |b_n| + \frac{\lambda}{m+\beta} \int_0^\infty |b_{n-1}| e^{-(m+\beta)\tau} d\tau \\
&\quad + \frac{\lambda^2}{m+\beta} \int_0^\infty e^{-(m+\beta)\tau} \int_0^\tau |Q_{n-2}(y)| e^{(m+\beta)y} dy d\tau \\
&= \frac{1}{m+\beta} |b_n| + \frac{\lambda}{(m+\beta)^2} |b_{n-1}| \\
&\quad + \frac{\lambda^2}{m+\beta} \int_0^\infty |Q_{n-2}(y)| e^{(m+\beta)y} \int_y^\infty e^{-(m+\beta)\tau} d\tau dy \\
&= \frac{1}{m+\beta} |b_n| + \frac{\lambda}{(m+\beta)^2} |b_{n-1}| + \frac{\lambda^2}{(m+\beta)^2} \|Q_{n-2}\|_{L^1[0,\infty)} \\
&\leq \dots\dots \\
&= \frac{1}{m+\beta} |b_n| + \frac{\lambda}{(m+\beta)^2} |b_{n-1}| + \frac{\lambda^2}{(m+\beta)^3} |b_{n-2}| + \dots + \frac{\lambda^{n-2}}{(m+\beta)^{n-1}} |b_2| \\
&\quad + \frac{\lambda^{n-1}}{(m+\beta)^n} |b_1| + \frac{\lambda^n}{(m+\beta)^n} \|Q_0\|_{L^1[0,\infty)} \\
&= \frac{1}{m+\beta} |b_n| + \frac{\lambda}{(m+\beta)^2} |b_{n-1}| + \dots + \frac{\lambda^{n-1}}{(m+\beta)^n} |b_1| + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
&= \sum_{k=1}^n \frac{\lambda^{k-1}}{(m+\beta)^k} |b_{n-k+1}| + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
&= \sum_{k=1}^n \frac{\lambda^{k-1}}{(m+\beta)^k} \left| \frac{s^{n-k}(\lambda + \alpha s)}{m + \alpha} a_1 + \frac{s^{n-k}(\lambda + \alpha s) - t^{n-k}(\lambda + \alpha t)}{(m + \alpha)(s - t)} (a_2 - sa_1) \right|
\end{aligned}$$

$$\begin{aligned}
 & + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
 \leq & \sum_{k=1}^n \frac{\lambda^{k-1} |s|^{n-k}}{(m+\beta)^k} \left| \frac{\lambda + \alpha s}{m + \alpha} \right| |a_1| + \sum_{k=1}^n \frac{\lambda^{k-1} |s|^{n-k}}{(m+\beta)^k} \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \\
 & + \sum_{k=1}^n \frac{\lambda^{k-1} |t|^{n-k}}{(m+\beta)^k} \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
 = & |s|^n \left| \frac{\lambda + \alpha s}{m + \alpha} \right| |a_1| \sum_{k=1}^n \frac{\lambda^{k-1}}{[|s|(m+\beta)]^k} + |s|^n \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \sum_{k=1}^n \frac{\lambda^{k-1}}{[|s|(m+\beta)]^k} \\
 & + |t|^n \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \sum_{k=1}^n \frac{\lambda^{k-1}}{[|t|(m+\beta)]^k} + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
 = & |s|^n \left| \frac{\lambda + \alpha s}{m + \alpha} \right| |a_1| \frac{\frac{1}{|s|(m+\beta)} \left[1 - \left(\frac{\lambda}{|s|(m+\beta)} \right)^n \right]}{1 - \frac{\lambda}{|s|(m+\beta)}} \\
 & + |s|^n \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{\frac{1}{|s|(m+\beta)} \left[1 - \left(\frac{\lambda}{|s|(m+\beta)} \right)^n \right]}{1 - \frac{\lambda}{|s|(m+\beta)}} \\
 & + |t|^n \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{\frac{1}{|t|(m+\beta)} \left[1 - \left(\frac{\lambda}{|t|(m+\beta)} \right)^n \right]}{1 - \frac{\lambda}{|t|(m+\beta)}} + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0| \\
 = & \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m+\beta) - \lambda} \left[|s|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right] \\
 & + \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m+\beta) - \lambda} \left[|s|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right] \\
 & + \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m+\beta) - \lambda} \left[|t|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right] \\
 & + \frac{\lambda^n}{(m+\beta)^{n+1}} |b_0|. \tag{45}
 \end{aligned}$$

If we denote

$$\begin{aligned}
 A_n &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m+\beta) - \lambda} \left[|s|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right], \\
 B_n &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m+\beta) - \lambda} \left[|s|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right], \\
 C_n &= \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m+\beta) - \lambda} \left[|t|^n - \left(\frac{\lambda}{m+\beta} \right)^n \right],
 \end{aligned}$$

then (45) is equivalent to

$$\|Q_n\|_{L^1[0,\infty)} \leq A_n + B_n + C_n + \frac{\lambda^n}{(m+\beta)^{n+1}}|b_0|, \quad \forall n \geq 1 \quad (46)$$

$\Rightarrow$

$$\begin{aligned} \|Q\| &= \sum_{n=0}^{\infty} \|Q_n\|_{L^1[0,\infty)} = \|Q_0\|_{L^1[0,\infty)} + \sum_{n=1}^{\infty} \|Q_n\|_{L^1[0,\infty)} \\ &\leq \frac{1}{m+\beta}|b_0| + \sum_{n=1}^{\infty} \left(A_n + B_n + C_n + \frac{\lambda^n}{(m+\beta)^{n+1}}|b_0| \right). \end{aligned} \quad (47)$$

The condition $\frac{1}{4} < \frac{\lambda}{\beta} < \frac{\lambda(\alpha+\lambda)}{\alpha\beta} < 1$ and $m > 0$ implies $0 < \frac{\lambda}{m+\beta} < 1$ and therefore (39) gives

$$\begin{aligned} \sum_{n=1}^{\infty} A_n &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left[|s|^n - \left(\frac{\lambda}{m + \beta} \right)^n \right] \\ &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} |s|^n \\ &\quad - \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left(\frac{\lambda}{m + \beta} \right)^n \\ &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m + \beta) - \lambda} \left(\frac{|s|}{1 - |s|} \right) \\ &\quad - \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \frac{|a_1|}{|s|(m + \beta) - \lambda} \left(\frac{\lambda}{\eta + \beta} \right) \\ &< \infty. \end{aligned} \quad (48)$$

$$\begin{aligned} \sum_{n=1}^{\infty} B_n &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left[|s|^n - \left(\frac{\lambda}{m + \beta} \right)^n \right] \\ &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} |s|^n \\ &\quad - \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left(\frac{\lambda}{m + \beta} \right)^n \\ &= \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m + \beta) - \lambda} \left(\frac{|s|}{1 - |s|} \right) \\ &\quad - \left| \frac{\lambda + \alpha s}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|s|(m + \beta) - \lambda} \left(\frac{\lambda}{\eta + \beta} \right) \end{aligned}$$

$$< \infty. \quad (49)$$

$$\begin{aligned} \sum_{n=1}^{\infty} C_n &= \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left[|t|^n - \left(\frac{\lambda}{m + \beta} \right)^n \right] \\ &= \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m + \beta) - \lambda} \sum_{n=1}^{\infty} |t|^n \\ &\quad - \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m + \beta) - \lambda} \sum_{n=1}^{\infty} \left(\frac{\lambda}{m + \beta} \right)^n \\ &= \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m + \beta) - \lambda} \left(\frac{|t|}{1 - |t|} \right) \\ &\quad - \left| \frac{\lambda + \alpha t}{m + \alpha} \right| \left| \frac{a_2 - sa_1}{s - t} \right| \frac{1}{|t|(m + \beta) - \lambda} \left(\frac{\lambda}{\eta + \beta} \right) \\ &< \infty. \end{aligned} \quad (50)$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\lambda^n}{(m + \beta)^{n+1}} |b_0| &= \frac{|b_0|}{m + \beta} \times \left(\frac{\frac{\lambda}{m + \beta}}{1 - \frac{\lambda}{m + \beta}} \right) = \frac{(\eta + \lambda)\lambda}{\beta(\eta + \beta)} |p_0| \\ &< \infty. \end{aligned} \quad (51)$$

By combining (47) with (48),(49),(50) and (51) we obtain

$$\begin{aligned} \|Q\| &= \frac{1}{m + \beta} |b_0| + \sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} B_n + \sum_{n=1}^{\infty} C_n + \sum_{n=1}^{\infty} \frac{\lambda^n}{(m + \beta)^{n+1}} |b_0| \\ &< \infty. \end{aligned} \quad (52)$$

From this together with (44) we deduce the desired result:

$$\|(p, Q)\| = \|p\| + \|Q\| < \infty. \quad (53)$$

Which shows that all $\eta \in \left(-\lambda, \frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2} \right)$ are eigenvalues of $A + U + E$. Moreover, from (24), (25), (26), (27), (28), (29), (30), (42) and (43) we know that geometric multiplicity of each η is one. $\square$

Theorem 3 and Theorem 4 show that all points in

$$\begin{aligned} &\left(\frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{4}, 0 \right] \\ &\cup \left(-\lambda, \frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2} \right) \end{aligned}$$

are eigenvalues of $(A + U + E)$. Naturally, we ask that the points in

$$\left(\frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{2}, \frac{-(2\lambda + \alpha + \beta) + \sqrt{(\alpha + \beta)^2 + 4\lambda\beta}}{4} \right)$$

are eigenvalues of $A + U + E$ or not? This is our next research work.

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