CONVOLUTIONAL CODES DERIVED FROM MELAS CODES

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Abstract: A family of convolutional codes is constructed algebraically in this note. These new convolutional codes are derived from Melas codes. The proposed codes have basic generator matrices, consequently, they are non catastrophic.

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1. Introduction

Several papers addressing constructions of convolutional codes as well as their corresponding properties have been presented in the literature [2, 12, 16, 14, 15, 17, 18, 7, 19, 3, 5, 4, 1, 6, 11]. In [2], the author constructed an algebraic structure for convolutional codes; in [12], the author construct unit memory convolutional codes with good parameters. Constructions of maximum-distance-separable (MDS) convolutional codes (in the sense that the codes attain the generalized Singleton bound introduced in [17, Theorem 2.2]), were presented in [17, 19, 4]. Concerning the optimality with respect to other bounds we have [16, 15]. Strongly MDS convolutional codes were constructed in [3]. In [18, 7, 1], the authors constructed convolutional BCH codes. In [5], doubly-cyclic convolutional codes were constructed and in [6], the authors described cyclic convolutional codes by means of the matrix ring. Finally, in [11], the author constructed families of convolutional codes derived from group character

codes.

In this note we construct algebraically convolutional codes derived from Melas codes by applying the well-known method introduced by Piret [14] and after generalized by Aly *et al.* [1, Theorem 3], which consists in the construction of convolutional codes derived from block codes.

The new convolutional codes have parameters

$$\bullet (q^m - 1, q^m - m - 1, m)_q,$$

where $q \neq 2$ is a prime power, q even, $n = q^m - 1$ is the code length, $m \geq 2$, $k = q^m - m - 1$ is the code dimension, $\delta = m$ is the degree of the code, and $d_f \geq 3$, where d_f is the free distance of the code;

$$\bullet (2^m - 1, 2^m - m - 1, m)_2,$$

where m is odd and $d_f \geq 5$.

The work is organized as follows. Section 2 presents a brief review on Melas codes and basic definitions and known results on convolutional codes. In Section 3, we present the contributions of the paper by constructing new convolutional codes derived from Melas codes and, in Section 4, a summary of the work is given.

2. Preliminaries

In this section we give necessary background for the proposed code construction.

Notation. We assume throughout that p denotes a prime number, q is a prime power and $\mathbb{F}_q$ is a finite field with q elements. As usual, the parameters of a linear code are given by $[n, k, d]_q$, where n is the code length, k is the code dimension and d is the minimum (Hamming) distance of the code.

2.1. Melas Codes

We assume that the reader is familiar with basics on cyclic codes. For details we refer to [13, 8]. Recall that the Melas code $\mathbb{M}_p(m)$ is a cyclic code with parameters $[n, n - 2m, d \geq 3]_p$, where p is a prime number and $n = p^m - 1$ (see [21, 20]). If α is a primitive element of $\mathbb{F}_{p^m}$ then a parity check matrix for $\mathbb{M}_p(m)$ is given by

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \end{bmatrix}$$

considered as a matrix over F_p . This class of codes was generalized in [10] to q -ary alphabets, where q is a prime power.

2.2. Convolutional Codes

The following background on convolutional codes can be found in [9, 8]. We say that a polynomial encoder matrix $G(D) \in \mathbb{F}_q[D]^{k \times n}$ is *basic* if $G(D)$ has a polynomial right inverse. A basic generator matrix of a convolutional code C is called *reduced* (or *minimal* [19, 8, 6]) if the overall constraint length $\delta = \sum_{i=1}^k \delta_i$ has the smallest value among all basic generator matrices of C ; in this case δ is called the *degree* of the code. We define the *weight* of $\mathbf{v}(D) \in \mathbb{F}_q[D]^n$ as $\text{wt}(\mathbf{v}(D)) = \sum_{i=1}^n \text{wt}(v_i(D))$, where $\text{wt}(v_i(D))$ is the number of nonzero coefficients of $v_i(D)$.

Definition 1. [9] A rate k/n convolutional code C with parameters $(n, k, \delta)_q$ is a submodule of $\mathbb{F}_q[D]^n$ generated by a reduced basic matrix $G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n}$, that is, $C = \{\mathbf{u}(D)G(D) \mid \mathbf{u}(D) \in \mathbb{F}_q[D]^k\}$, where $\delta = \sum_{i=1}^k \delta_i$ is the *degree*, $\delta_i = \max_{1 \leq j \leq n} \{\deg g_{ij}\}$, $\mu = \max_{1 \leq i \leq k} \{\delta_i\}$ is the *memory* and $d_f = \text{wt}(C) = \min\{\text{wt}(\mathbf{v}(D)) \mid \mathbf{v}(D) \in C, \mathbf{v}(D) \neq 0\}$ is the *free distance* of C .

The Euclidean inner product of two n -tuples $\mathbf{u}(D) = \sum_i \mathbf{u}_i D^i$, $\mathbf{v}(D) = \sum_j \mathbf{v}_j D^j$ in $\mathbb{F}_q[D]^n$ is defined by $\langle \mathbf{u}(D) \mid \mathbf{v}(D) \rangle = \sum_i \mathbf{u}_i \cdot \mathbf{v}_i$. If C is a convolutional code then its (Euclidean) dual code is defined by $C^\perp = \{\mathbf{u}(D) \in \mathbb{F}_q[D]^n \mid \langle \mathbf{u}(D) \mid \mathbf{v}(D) \rangle = 0, \text{ for all } \mathbf{v}(D) \in C\}$.

Let $[n, k, d]_q$ be a block code whose parity check matrix H is partitioned into disjoint submatrices H_0 and H_1 , each of which has n columns and $\text{rk} H_0 \geq \text{rk} H_1$, such that $H = \begin{bmatrix} H_0 & H_1 \end{bmatrix}^T$, obtaining the polynomial matrix $G(D) = H_0 + \tilde{H}_1 D$, where $\tilde{H}_1$ is derived from H_1 by adding zero-rows to complete $\text{rk} H_0$ rows in total. The matrix $G(D)$ generates a convolutional code V with $\text{rk} H_0$ rows in total.

Theorem 2. [1, Theorem 3] Suppose that $C \subseteq \mathbb{F}_q^n$ is an $[n, k, d]_q$ code with parity check matrix $H \in \mathbb{F}_q^{(n-k) \times n}$ partitioned into submatrices H_0, H_1 as above such that $\text{rk} H_0 \geq \text{rk} H_1$. Then $G(D)$ is a reduced basic generator matrix. Moreover, if d_f and $d_f^\perp$ denote the free distances of V and $V^\perp$, respectively, d_i denote the minimum distance of the code $C_i = \{\mathbf{v} \in \mathbb{F}_q^n \mid \mathbf{v} \tilde{H}_i^t = 0\}$ and $d^\perp$

is the minimum distance of $C^\perp$, then one has $\min\{d_0 + d_1, d\} \leq d_f^\perp \leq d$ and $d_f \geq d^\perp$.

3. The New Codes

Most of known convolutional codes are constructed case by case or even by exhaustive computational search. In this light, we propose the construction of a family of convolutional codes derived from Melas codes by means of an algebraic method. The resulting codes are non catastrophic since they have basic generator matrices. We are now ready to show the main result:

Theorem 3. *Let $q \neq 2$ be a prime power, q even, and $n = q^m - 1$, where $m \geq 2$. Then there exists an $(q^m - 1, q^m - m - 1, m)_q$ convolutional code with free distance $d_f \geq 3$.*

Proof. We assume the same notation adopted in Subsection 2.2. Let $\mathbb{M}_q(m)$ be the Melas code of length $n = q^m - 1$. Suppose that α is a primitive element of $\mathbb{F}_{q^m}$; then a parity check matrix for $\mathbb{M}_q(m)$ is given by

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \end{bmatrix},$$

considered as a matrix over F_q . More precisely, let β be any basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. We expand all entries of H with respect to β , resulting in a new matrix $H_{\mathbb{F}_q}$ over $\mathbb{F}_q$ having $2m$ rows. Since $\mathbb{M}_q(m)$ has dimension $n - 2m$ it follows that $H_{\mathbb{F}_q}$ has rank $2m$, so all rows of $H_{\mathbb{F}_q}$ are linearly independent. Further, we split $H_{\mathbb{F}_q}$ into disjoint submatrices H_0 and H_1 as follows:

$$H_{\mathbb{F}_q} = \begin{bmatrix} H_0 \\ \text{---} \\ H_1 \end{bmatrix},$$

where $\text{rk}H_0 = m = \text{rk}H_1$. In other words, H_0 is derived from the matrix

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix}$$

by expanding each entry with respect to the basis β , and H_1 is derived from the matrix

$$\begin{bmatrix} 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \end{bmatrix}$$

by expanding each entry with respect to β . From Theorem 2 and by construction, we can form the non catastrophic unit memory convolutional code V of

length $n = q^m - 1$, generated by the matrix $G(D) = H_0 + H_1D$. Since both matrices H_0 and H_1 have rank m , then V has dimension m and degree m .

Let us consider the (Euclidean) dual $V^\perp$ of the convolutional code V . We know that $V^\perp$ has length $n = q^m - 1$, dimension $n - m$ and degree m . To compute the minimum distance $d_f^\perp$ of $V^\perp$, consider the cyclic code C_0 with parity check matrix H_0 . Then it follows from the BCH bound that C_0 has minimum distance $d_0 \geq 2$. The same logic applies to the cyclic code C_1 with parity check matrix H_1 , so $d_1 \geq 2$. It is easy to see that for $q \neq 2$, q even, the minimum distance of the Melas code is $d \geq 3$. Applying again Theorem 2, it follows that $d_f^\perp \geq 3$. Therefore, one can get an $(q^m - 1, q^m - m - 1, m)_q$ convolutional code with free distance $d_f \geq 3$, as required. $\square$

In the case of binary convolutional codes of length $n = 2^m - 1$, with m odd, one can derive convolutional codes having free distance $d_f \geq 5$, as shown in the following theorem.

Theorem 4. *Consider that $n = 2^m - 1$, where $m \geq 3$ is an odd integer. Then there exists an $(2^m - 1, 2^m - m - 1, m)_2$ convolutional code with free distance $d_f \geq 5$.*

Proof. The proof follows the same line to that of Theorem 3. It is known that binary Melas codes of length $n = 2^m - 1$ for m odd has parameters $[n, n - 2m, d \geq 5]$ (see [13, p. 206]). Assume that

$$H = \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \end{bmatrix},$$

is a parity check matrix for the code (considered as a matrix over F_2) and let β be any basis of $\mathbb{F}_{2^m}$ over $\mathbb{F}_2$. Expanding all entries of H with respect to β one has a new matrix $H_{\mathbb{F}_2}$ over $\mathbb{F}_2$ with $2m$ rows in total, each of which are linearly independent. We next split $H_{\mathbb{F}_2}$ into disjoint submatrices H_0 and H_1 as follows:

$$H_{\mathbb{F}_2} = \begin{bmatrix} H_0 \\ \text{---} \\ H_1 \end{bmatrix}, \text{ where } \text{rk} H_0 = m = \text{rk} H_1, \text{ where } H_0, H_1 \text{ are derived from}$$

$$\begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & \alpha^{-1} & \alpha^{-2} & \cdots & \alpha^{-(n-1)} \end{bmatrix},$$

respectively. By construction and based on Theorem 2, we can form the non catastrophic unit memory convolutional code V of length $n = 2^m - 1$, dimension

m and degree m , generated by the matrix $G(D) = H_0 + H_1D$. The (Euclidean) dual $V^\perp$ of V has length $n = 2^m - 1$, dimension $n - m$ and degree m . Consider the code C_0 with parity check matrix H_0 . Then one concludes from the BCH bound that C_0 has minimum distance $d_0 \geq 3$ since the defining set of C_0 contains the sequence $\{1, 2\}$ of two consecutive integers. The same justification applies to the cyclic code C_1 with parity check matrix H_1 , since the defining set of C_1 contains the sequence $\{n-2, n-1\}$ of two consecutive integers; so $d_1 \geq 3$. As was said previously, the minimum distance of $\mathbb{M}_2(m)$ satisfies $d \geq 5$. From Theorem 2 we have $d_f^\perp \geq 5$. Thus an $(2^m - 1, 2^m - m - 1, m)_2$ convolutional code with free distance $d_f \geq 5$ can be constructed, and the proof is complete. $\square$

Example 5. If we consider $q = 4$ and $m = 2$ in Theorem 3, one obtains an $(15, 13, 2)_4$ convolutional code with free distance $d_f \geq 3$. Analogously, when $q = 2$ and $m = 3$, one obtains from Theorem 4 an $(7, 4, 3)_2$ convolutional code with free distance $d_f \geq 5$.

4. Summary

We have constructed a family of non catastrophic convolutional codes derived from Melas codes. These codes are constructed algebraically and not by computational search.

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