A CONTINUED FRACTION OF RAMANUJAN AND THEIR EXPLICIT VALUES

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Abstract: In this paper, we study new continued fraction of Ramanujan, their modular identities and some explicit evaluation of their.

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1. Introduction

In Chapter 16 of his second notebook [2], Ramanujan develops the theory of theta-function and is defined by

$$f(a, b) := \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1,$$
$$= (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad (1)$$

where $(a; q)_0 = 1$ and $(a; q)_{\infty} = (1 - a)(1 - aq)(1 - aq^2) \cdots$.

Following Ramanujan, we defined

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$$\varphi(q) := f(q, q) = \sum_{n=-} q^{n^2} = \frac{(-q; -q)}{(q; -q)}, \quad (2)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)}{(q; q^2)}, \quad (3)$$

$$f(-q) := f(-q, -q^2) = \sum_{n=-} (-1)^n q^{\frac{n(3n-1)}{2}} = (q; q) \quad (4)$$

and

$$\chi(q) := (-q; q^2) \quad . \quad (5)$$

Ramanujan recorded many q -continued fractions and some of their explicit values in his second notebook [7] and in his lost notebook [8]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [1, p. 11, Entry 11]:

$$\begin{aligned} & \frac{(-a) \quad (b) \quad - (a) \quad (-b)}{(-a) \quad (b) \quad + (a) \quad (-b)} \\ &= \frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \dots \end{aligned} \quad (6)$$

where either q , a , and b are complex numbers with $\text{mod } q < 1$, or q , a , and b are complex numbers with $a = bq^m$ for some integer m . Several elegant q -continued fractions can be expressed in terms of Ramanujan's theta-functions. The most famous of them is the celebrated Rogers-Ramanujan continued fraction $R(q)$ is defined as

$$R(q) := \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \dots, \quad |q| < 1, \quad (7)$$

On page 365 of his Lost Notebook [8], Ramanujan recorded five identities showing the relationships between $R(q)$ and five continued fractions $R(-q)$, $R(q^2)$, $R(q^3)$, $R(q^4)$, and $R(q^5)$. He also recorded these identities at the scattered places of his Notebooks [7]. L. J. Rogers [9] established the modular equations relating $R(q)$ and $R(q^n)$ for $n=2,3,5$, and 11. The last of these equations cannot be found in Ramanujan's works.

The Ramanujan's cubic continued fraction $G(q)$ is defined as

$$G(q) := \frac{q^{1/3} f(-q, -q^5)}{f(-q^3, -q^3)} = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \dots, \quad |q| < 1, \quad (8)$$

The continued fraction (8) was first introduced by Ramanujan in his second letter to G. H. Hardy [5]. He also recorded the continued fraction (8) on page 365 of his Lost Notebook [8] and claimed that there are many results for $G(q)$ similar the results obtained for the famous Rogers-Ramanujan continued fraction (7).

Motivated by the above cited works on the continued fractions, changing q to q^2 and b to $-ab$ in (7), we obtain

$$\begin{aligned} & \frac{(-a; q^2)}{(-a; q^2)} \frac{(-ab; q^2)}{(-ab; q^2)} - \frac{(a; q^2)}{+ (a; q^2)} \frac{(ab; q^2)}{(ab; q^2)} \\ &= \frac{a+ab}{1-q^2} + \frac{(a+abq^2)(aq^2+ab)}{1-q^6} + \frac{q^2(a+abq^4)(aq^4+ab)}{1-q^{10}} + \dots \end{aligned} \quad (9)$$

If $a = q$ and $b = -q$ in the above equation (9), we get

$$\begin{aligned} & \frac{(-q; q^2)}{(-q; q^2)} \frac{(q^2; q^2)}{(q^2; q^2)} - \frac{(q; q^2)}{+ (q; q^2)} \frac{(-q^2; q^2)}{(-q^2; q^2)} \\ &= \frac{q}{1+q} + \frac{q^3(q-1)}{1+q^3} + \frac{q^5(q^3-1)}{1+q^5} + \dots \end{aligned} \quad (10)$$

In this paper, we study the Ramanujan's continued fraction $U(q)$ defined by

$$\kappa(q) = \frac{q}{1+q} + \frac{q^3(q-1)}{1+q^3} + \frac{q^5(q^3-1)}{1+q^5} + \dots \quad (11)$$

which is the right hand side of the equality (10).

In Section 3, we establish some modular relations connecting $U(q)$ and $U(q^n)$. In Section 4, we find the some explicit evaluations of $U(q)$.

2. Preliminary Results

In this section, we collect the necessary results required to prove our main results.

Lemma 1.

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4) \quad (12)$$

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2) \quad (13)$$

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2) \quad (14)$$

For the proofs of (12), (13) and (14), see [2, Entry 25(i), (iii) and (vi), p.40].

Lemma 2.

$$\varphi(q) = \sqrt{z} \quad (15)$$

and

$$\varphi(-q) = \sqrt{z}(1-t)^{1/4}. \quad (16)$$

For the proofs of (15) and (16), see [2, Entry 10(i),(ii), p.122].

Lemma 3. *If β is of degree 2 over α , then*

$$(1 - \sqrt{1 - \alpha})(1 - \sqrt{\beta}) = 2\sqrt{\beta(1 - \alpha)}. \quad (17)$$

For a proof of (17), see [1, Entry 17.3.1, p.385].

Lemma 4. *If β has degree 3 over α , then*

$$(\alpha\beta)^{1/4} + \{(1 - \alpha)(1 - \beta)\}^{1/4} = 1. \quad (18)$$

For a proof of (18), see [2, Entry 5(ii), p.230].

3. Relation between $\kappa(q)$ and $\kappa(q^n)$

Theorem 5. *We have*

$$\kappa(q) := \frac{\varphi(q) - 1}{\varphi(q) + 1} \quad (19)$$

Proof. From the equations (10) and (11), we get

$$\kappa(q) = \frac{\frac{(-q, q^2)}{(q, q^2)} \frac{(q^2, q^2)}{(-q^2, q^2)} - 1}{\frac{(-q, q^2)}{(q, q^2)} \frac{(q^2, q^2)}{(-q^2, q^2)} + 1} \quad (20)$$

Employing the equation (2) in the above equation (20), we obtain

$$\kappa(q) := \frac{\varphi(q) - 1}{\varphi(q) + 1} \quad (21)$$

the above equation can be written as (19). □

Corollary 6. *We have*

$$\varphi(q) := \frac{\kappa(q) + 1}{1 - \kappa(q)} \quad (22)$$

Proof. Easily from the equations (21). $\square$

Theorem 7. If $x := \kappa(q)$, $y := \kappa(-q)$, and $z := \kappa(q^4)$, then

$$2xy - x - zx - y + 2z - zy = 0. \quad (23)$$

Proof. Employing the equation (12) and the equations (22), we obtain (23). $\square$

Theorem 8. If $x := \kappa(q)$, $y := \kappa(-q)$, and $z := \kappa(-q^2)$, then

$$2xyz - x - xz^2 - y - yz^2 + 2z = 0. \quad (24)$$

Proof. Squaring the equation (13) and using equation (22), we obtain (24). $\square$

Theorem 9. If $x := \kappa(q)$, $y := \kappa(-q)$, and $z := \kappa(-q^2)$, then

$$x + y - 2z - zy - zx + 2yx = 0. \quad (25)$$

Proof. Using the equations (14) and (22), we obtain (25). $\square$

Theorem 10. If $x := \kappa(q)$, $y := \kappa(q^2)$, and $z := \kappa(q^4)$, then

$$\begin{aligned} & (zy^2 + 1 + y^2 + z - 3y - z^2y)x^2 \\ & + (2y - 4zy^2 + 2z^2y - 4z + 4zy)x \\ & - y + zy^2 - 3z^2y + z^2y^2 + z^2 + z = 0. \end{aligned} \quad (26)$$

Proof. Using the equations (12) and (14) eliminating $\varphi(-q)$, we obtain

$$2\varphi(q)\varphi(q^4) = \varphi^2(q) + 2\varphi^2(q^4) - \varphi^2(q^2). \quad (27)$$

Squaring the above equation (27), we obtain

$$2\varphi^2(q)\varphi^2(q^2) + 4\varphi^2(q^2)\varphi^2(q^4) - \varphi^4(q) - \varphi^4(q^2) - 4\varphi^4(q^4) = 0. \quad (28)$$

Employing the equations (22) in the above equation (28), we get (26). $\square$

Theorem 11. If $x := \kappa(q)$, $y := \kappa(-q)$, $z := \kappa(q^2)$, and $w := \kappa(-q^2)$, then

$$\begin{aligned} & (2zy^2 - w^2z^2 - w^2 - 1 - 2wy^2 + 4wz + 2w^2zy^2 - z^2 \\ & - 2wz^2y^2)x^2 + (2y + 2w^2y + 2z^2y + 4wz^2y - 8wzy \\ & - 4zy - 4w^2zy + 4wy + 2w^2z^2y)x - w^2z^2y^2 - w^2y^2 \\ & - y^2 - z^2y^2 - 2wz^2 + 2z + 4wzy^2 - 2w + 2w^2z = 0. \end{aligned} \quad (29)$$

Proof. From the equations (15) and (16), we get

$$\frac{\varphi(-q)}{\varphi(q)} = (1-t)^{1/4}, \quad 0 < t < 1. \quad (30)$$

The equation (17) can be written as

$$\beta = \left[\frac{1 - \sqrt{1 - \alpha}}{1 + \sqrt{1 - \alpha}} \right]^2. \quad (31)$$

Employing the equations (22) and (30) in the above equation (31), we obtain

$$\begin{aligned} & (-w^2 z^2 y^2 - w^2 z^2 x^2 + 2yx + 4wyz + 2w^2 yx + 2zy^2 x^2 + 4wzx^2 \\ & + 4wzy^2 - 4zyx - 2wy^2 x^2 + 2z^2 yx + 2w^2 z - 2wz^2 - 2w - w^2 y^2 \\ & - w^2 x^2 - z^2 y^2 - z^2 x^2 + 2z - y^2 - x^2 + 4wz^2 yx - 2wz^2 y^2 x^2 \\ & - 4w^2 zyx + 2w^2 zy^2 x^2 + 2w^2 z^2 yx - 8wzyx)(1 - 2w^2 y^2 \\ & + 2wz^2 x^2 - 2yx - 4wyz - 2w^2 yx + 2wz^2 y^2 - 2w^2 zx^2 + 4zyx \\ & + z^2 y^2 x^2 - 2z^2 yx + w^2 y^2 x^2 + w^2 + z^2 - 4wz^2 yx + 4w^2 zyx \\ & - 2w^2 z^2 yx + 8wzyx + w^2 z^2 y^2 x^2 - 4wzy^2 x^2 + w^2 z^2 - 2zy^2 \\ & - 4wz + 2wy^2 + y^2 x^2 + 2wx^2 - 2zx^2) = 0 \end{aligned} \quad (32)$$

By examining the behavior of the above factors near $q = 0$, we can find a neighborhood about the origin, where the first factor is zero; whereas other factors are not zero in this neighborhood. By the Identity Theorem first factor vanishes identically. This completes the proof. $\square$

Theorem 12. If $x := \kappa(q)$, $y := \kappa(-q)$, $z := \kappa(q^3)$, and $w := \kappa(-q^3)$, then

$$\begin{aligned} & 6y^2 w^2 - 4w^3 y^3 - 12w^3 z^2 y - 12wz^3 y^2 - 12wz^2 y - 12w^3 zy^2 - 4wz^4 y^3 \\ & - 12w^3 z^3 y^2 - 12wz^2 y^3 + 12w^2 z^3 y + 24w^2 z^2 y^2 + 12w^2 z^3 y^3 - 4w^3 z^4 y \\ & + w^4 z^4 y^4 + 4w^4 zy^3 - 4w^3 z^3 y^4 - 12w^3 z^2 y^3 + 12w^2 zy + 12w^2 zy^3 + w^4 \\ & - 12wzy^2 + 4w^4 z^3 y - 4wzy^4 + 6w^2 z^2 y^4 + 6w^2 z^4 y^2 + 6w^4 z^2 y^2 + 4zy \\ & + 4z^3 y^3 - 4wy + 6y^2 z^2 - 4wz^3 + 6w^2 z^2 - 4w^3 z + z^4 + y^4 + (24wz^3 y \\ & - 4zy^4 - 4y - 4w^4 z^3 y^4 - 12w^4 z^3 y^2 - 4w^4 z^4 y - 72w^2 z^2 y^3 + 24w^3 zy \\ & + 40wzy + 72wz^2 y^2 + 40w^3 y^3 z - 12w^2 zy^4 - 72w^2 z^3 y^2 + 12w^3 z^4 y^2 \\ & - 12w^2 z^3 y^4 + 12wz^4 y^2 - 72w^2 zy^2 - 12w^4 z^2 y + 40wz^3 y^3 - 72w^2 z^2 y \end{aligned}$$

$$\begin{aligned}
& + 4w^3z^4y^4 + 72w^3z^2y^2 + 24wzy^3 + 12w^3z^2y^4 - 12w^4zy^2 + 12wz^2y^4 \\
& + 40w^3z^3y - 12w^4y^3z^2 - 12w^2z^4y + 24w^3z^3y^3 - 12w^2z^4y^3 - 12z^2y \\
& - 4z^3 - 12w^2z^3 - 4z^4y^3 - 4w^4y^3 + 12w^3y^2 + 12wz^2 - 12z^2y^3 - 4w^4z \\
& - 12zy^2 - 12w^2z - 12z^3y^2 + 12w^3z^2 - 12w^2y^3 + 4wz^4 + 4wy^4 + 4w^3 \\
& - 12w^2y + 12wy^2)x^3 + (-12w^3y^3 + 12w^4zy + 24y^2w^2 + 12w^4z^3y^3 \\
& - 12wy - 72w^3z^2y - 72wz^3y^2 - 72wz^2y - 72w^3zy^2 - 72w^3z^3y^2 + 6z^2 \\
& - 12wz^3y^4 - 12wz^4y^3 - 72wz^2y^3 + 6w^4y^4z^2 + 6w^4z^4y^2 - 12wz^4y \\
& - 12w^3z^4y^3 + 72w^2z^3y + 168w^2z^2y^2 - 12w^3z^4y + 12zy + 72w^2zy^3 \\
& + 72w^2z^3y^3 + 12w^4zy^3 + 72w^2zy - 12w^3z^3y^4 - 72w^3z^2y^3 + 12zy^3 \\
& + 6w^2z^4y^4 - 72wzy^2 + 12w^4z^3y - 12wzy^4 + 24w^2z^2y^4 + 24w^2z^4y^2 \\
& - 12w^3zy^4 + 24w^4z^2y^2 + 12z^3y^3 + 24y^2z^2 + 24w^2z^2 - 12w^3z + 6w^2 \\
& + 6y^2 - 12w^3z^3 - 12wz - 12wy^3 - 12wz^3 + 12z^3y + 6z^2y^4 + 6w^2z^4 \\
& - 12w^3y + 6w^2y^4 + 6w^4z^2 + 6w^4y^2 + 6z^4y^2)x^2 + (-4w^4zy^4 - 4z \\
& + 4w + 40wz^3y - 12w^4z^3y^2 - 72w^2z^2y^3 + 40w^3zy + 24wzy - 72w^2z^3y^2 \\
& + 72wz^2y^2 + 24w^3y^3z + 4wz^4y^4 - 12w^2zy^4 + 12w^3z^4y^2 + 40wzy^3 \\
& - 12w^2z^3y^4 + 12wz^4y^2 - 72w^2zy^2 - 12w^4z^2y + 24wz^3y^3 - 72w^2z^2y \\
& - 4w^4z^4y^3 + 72w^3z^2y^2 + 12w^3z^2y^4 - 12w^4zy^2 + 12wz^2y^4 + 12wz^2 \\
& - 12w^4y^3z^2 - 12w^2z^4y + 40w^3z^3y^3 - 12w^2z^4y^3 + 4w^3y^4 - 4w^4y \\
& + 24w^3z^3y - 12w^2z^3 + 12w^3y^2 - 12z^2y^3 - 4w^4z^3 - 4z^4y - 12zy^2 \\
& - 4y^3 - 4z^3y^4 + 4w^3z^4 - 12w^2z + 12w^3z^2 - 12w^2y^3 + 12wy^2 - 12z^2y \\
& - 12z^3y^2 - 4wz - 12w^2y)x + (4w^4zy + 4w^4z^3y^3 - 12w^3z^2y - 12wz^2y \\
& 1 - 12wz^3y^2 + 6y^2w^2 - 12w^3zy^2 - 12w^3z^3y^2 - 12wz^2y^3 - 4wz^4y \\
& - 4wz^3y^4 + w^4z^4 + 12w^2z^3y + 24w^2z^2y^2 - 12wzy^2 + 12w^2z^3y^3 \\
& - 4w^3z^4y^3 - 12w^3z^2y^3 + 12w^2zy - 4w^3z^3 + 6w^2z^2y^4 + 6w^2z^4y^2 \\
& - 4w^3zy^4 + 6w^4z^2y^2 - 4w^3y + 6y^2z^2 + 6w^2z^2 - 4wy^3 + 12w^2zy^3 \\
& + 4z^3y + z^4y^4 + 4zy^3 + w^4y^4)x^4 = 0.
\end{aligned} \tag{33}$$

Proof. The equation (17) can be written as

$$\alpha\beta = [1 - \{(1 - \alpha)(1 - \beta)\}^{1/4}]^4. \tag{34}$$

Employing the equations (22) and (30) in the above equation (36), we obtain (35). $\square$

Theorem 13. If $x := \kappa(q)$, $y := \kappa(-q)$, $z := \kappa(q^4)$, and $w := \kappa(-q^4)$, then

$$\begin{aligned}
& 16wz^3y^4 - 4w^4zy^4 - 4w^4z^3y^4 - 6w^4y^4z^2 + 4wz^4y^4 - w^4z^4y^4 + 16wzy^4 \\
& - 24w^2zy^4 - 24w^2z^3y^4 + 4w^3z^4y^4 + 16w^3z^3y^4 + 24w^3z^2y^4 + 24wz^2y^4 \\
& - 6w^2z^4y^4 - 36w^2z^2y^4 + 16w^3zy^4 + (32w^3y^3 + 16w^4zy - 32w^4z^3y^3 \\
& + 4y - 96w^3z^2y - 96wz^2y + 32wz^4y^3 + 4w^4z^4y + 192wz^2y^3 + 16zy \\
& - 64w^3zy - 64wz^3y - 64wzy - 16wz^4y + 24z^2y - 192w^2z^3y^3 - 16wy \\
& + 96w^2z^3y - 16w^3z^4y + 24w^4z^2y + 144w^2z^2y + 32w^3z^4y^3 - 32w^4zy^3 \\
& + 192w^3z^2y^3 + 96w^2zy - 192w^2zy^3 + 16w^4z^3y - 64w^3z^3y + 24w^2z^4y \\
& - 32z^3y^3 + 32wy^3 + 16z^3y + 4w^4y - 32zy^3 - 16w^3y + 4z^4y + 24w^2y)x^3 \\
& + 4w^3y^4 + 8z - 8w - 8w^3 + 8z^3 - y^4 - 4zy^4 + 48w^2z^3 - z^4y^4 - 6z^2y^4 \\
& - 48wz^2 + 8w^4z^3 - 6w^2y^4 + 8w^4z - 4z^3y^4 - 8w^3z^4 + 48w^2z - 48w^3z^2 \\
& + 4wy^4 - w^4y^4 - 8wz^4 + (-1 + 8w^4zy^4 + 8w^4z^3y^4 - 8wz^4y^4 + 48w^2zy^4 \\
& + 48w^2z^3y^4 - 8w^3z^4y^4 - 48w^3z^2y^4 - 48wz^2y^4 - w^4 - 8w^3y^4 - w^4z^4 \\
& - 4z + 16wz^3 - 36w^2z^2 + 16w^3z + 4w - 6w^2 - 6z^2 + 4w^3 - 4z^3 - z^4 \quad (35) \\
& + 16w^3z^3 + 16wz + 8zy^4 - 24w^2z^3 + 24wz^2 - 4w^4z^3 - 6w^2z^4 - 4w^4z \\
& + 8z^3y^4 + 4w^3z^4 - 24w^2z - 6w^4z^2 + 24w^3z^2 - 8wy^4 + 4wz^4)x^4 \\
& + (24w^4z^3y^2 - 36y^2w^2 + 96wz^3y^2 + 96w^3zy^2 + 96w^3z^3y^2 - 6w^4z^4y^2 \\
& - 144wz^2y^2 + 96wzy^2 + 144w^2z^3y^2 - 24w^3z^4y^2 - 24wz^4y^2 + 24zy^2 \\
& - 216w^2z^2y^2 + 144w^2zy^2 - 144w^3z^2y^2 + 24w^4zy^2 - 36w^2z^4y^2 \\
& - 36w^4z^2y^2 - 36y^2z^2 - 6y^2 - 24w^3y^2 + 24z^3y^2 - 24wy^2 - 6w^4y^2 \\
& - 6z^4y^2)x^2 + (-16w^3y^3 - 32w^4zy + 16w^4z^3y^3 + 192w^3z^2y + 192wz^2y \\
& - 16wz^4y^3 + 144w^2z^2y^3 - 96wz^2y^3 - 64w^3y^3z + 32wz^4y - 192w^2z^3y \\
& + 96w^2z^3y^3 + 32w^3z^4y - 64wz^3y^3 - 16w^3z^4y^3 + 4w^4z^4y^3 + 16w^4zy^3 \\
& - 96w^3z^2y^3 - 64wzy^3 - 192w^2zy + 96w^2zy^3 - 32w^4z^3y + 24w^4y^3z^2 \\
& - 64w^3z^3y^3 + 24w^2z^4y^3 + 16z^3y^3 - 32zy + 32wy + 4z^4y^3 - 16wy^3 \\
& + 4y^3 - 32z^3y + 4w^4y^3 + 24z^2y^3 + 16zy^3 + 32w^3y + 24w^2y^3)x = 0.
\end{aligned}$$

Proof. The equation (17) can be written as

$$(1 - \sqrt[4]{1 - \alpha})(1 - \sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1 - \alpha)}. \quad (36)$$

Employing the equations (22) and (30) in the above equation (36), we obtain (35). $\square$

4. Explicit Evaluation of $U(q)$

Theorem 14. *We have*

$$(i) \quad \kappa(e^{-n\pi}) = \frac{\varphi(-e^{-n\pi}) - 1}{\varphi(-e^{-n\pi}) + 1} \quad \text{and} \quad (ii) \quad \kappa(-e^{-n\pi}) = \frac{\varphi(e^{-n\pi}) - 1}{1 + \varphi(e^{-n\pi}) + 1} \quad (37)$$

Proof. Put $q := e^{-n\pi}$ and $q := -e^{-n\pi}$ in the equation (19), we obtain (37). $\square$

In his first notebook, Ramanujan's second notebook [7] recorded many elementary values of $\varphi(q)$. In particular, he recorded $\varphi(e^{-n\pi})$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4, 8, 1/2, \text{ and } 1/4$. Ramanujan also recorded values of $\varphi(e^{-n\pi})$ for $n = 3, 5, 7, 9, \text{ and } 45$. Proof of these can found in [3]. Yi [10] also evaluated $\varphi(e^{-n\pi})$ for $n = 1, 2, 3, 4, 5, \text{ and } 6$ and $\varphi(-e^{-n\pi})$ for $n = 1, 2, 4, 6, 8, 10, \text{ and } 12$. Noting from [3, Entry 1(i) and (ii), p.325], we have

$$\varphi(e^{-\pi}) = \frac{\pi^{1/4}}{\Gamma(3/4)} \quad \text{and} \quad \varphi(-e^{-\pi}) = 2^{-1/4} \frac{\pi^{1/4}}{\Gamma(3/4)}. \quad (38)$$

Employing (38) in the equation (37), we get

$$\kappa(e^{-\pi}) = \frac{\frac{\pi^{1/4}}{\Gamma(3/4)} - 2^{1/4}}{\frac{\pi^{1/4}}{\Gamma(3/4)} + 2^{1/4}} \quad \text{and} \quad \kappa(-e^{-\pi}) = \frac{\frac{\pi^{1/4}}{\Gamma(3/4)} - 1}{\frac{\pi^{1/4}}{\Gamma(3/4)} + 1}.$$

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