MODULATION WAVE EQUATION FOR SPRING-BLOCK ORBITS WITH NONLINEAR RATE AND STATE FRICTION

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Abstract: A four-dimensional parameter model that captures the essence of spring-block propagation with rate- and state- friction laws was stated. In order to establish the existence of waves phenomena for all velocity-dependent critical value of stiffness and stick-slip amplitude for stiffness variations, we derived the modulation equation for our model. Conditions under which the amplitude of the modulation wave equation produces many small solutions near the instability threshold for finitely large enough time scales were established.

Key Words: spring-block, modulation, rate-state-friction, instability threshold, existence

1. Introduction

Non-linear frictional spring-block model systems have been extensively investigated in search of a better understanding and accurate prediction of earthquake dynamics [1-7]. However, a spring-block system cannot replicate an actual earthquake fault without a wealth of laboratory-scale experimental friction data to describing fully these stick-slip processes. Since the introduction of the laboratory determined friction laws (rate- and state- friction law) by Dieterich and Ruina in the late 70s [8, 9], it's been established that both velocity dependence of friction and time dependent of static friction were important in understanding and predicting earthquake dynamics.

The basic idea underlying spring-block models with rate- and state- friction (RSF) laws is that the system considers the effects of inertia and deformations through the introduction of masses and springs. The rate implied instantaneous rate of deformation which is dependent on the friction laws while the state determined the system's internal state which depended on the friction laws. The aforementioned details of the system include such effects as, e.g., dependence of friction on environmental conditions (temperature, humidity, presence of lubricant), history of friction contact (memory effect), wear of materials and surface roughness [10-16]. However, while earthquakes are recurrent and aperiodic on a continuum time scale, the stick-slip of spring-block oscillations has mostly been periodic on a short time scale

Even though few models of RSF capture the essence of stick-slip propagations from difference in values between the static and dynamic coefficients of friction [6, 10, 17-21], they cannot account for the existence of a velocity-dependent critical value of stiffness [17] and the increase of stick-slip amplitude for decreasing stiffness [6, 18, 20-21]. Annan (2012), for example, proved the existence of periodic traveling wave for non coupled, $\ell = 0$, and for small perturbation, $\ell \approx 0$ cases [6]. However, to establish the existence of waves phenomena for all ℓ we need to examine the modulated waves instead of the basic periodic traveling waves.

In this paper, we derive the modulated wave orbits for spring-block model with non-linear RSF law that allows the amplitude of the wave to vary slowly with time and space. Conditions under which the amplitude of the modulated wave equation produces many small solutions near the instability threshold for finitely large enough time scales are established.

The rest of the paper is arranged as follows: The 4-dimensional parametric model of spring-block propagation with rate and state dependent friction law was stated in Section 2. The derivation of the modulation wave equation is established in Section 3. Conditions under which the modulation equation generated many small solutions near the instability threshold for finitely large enough time scales were established in Section 4. We then give the conclusion in Section 5.

2. Model Formulation

We adapt the dimensionless spring-block model derived in [6] for the Burridge and Knopoff model [2] where the friction force is replaced by rate- and state-

dependent friction laws and $x_j(t)$, $\theta_j(t) \in \mathbb{R}$ as

$$\ddot{x}_j + x_j - \ell^2(x_{j+1} - 2x_j + x_{j-1}) = F(\dot{x}_j, \theta_j); \quad j \in \mathbb{Z}. \quad (1)$$

Since the state variable, θ_j , is time-dependent, we must have a time evolution law for θ_j together with the friction force. Here, we assume the dimensionless form of the Dieterich's law (also known as slowness law) [9] to describe the time-dependent properties of the friction law as

$$\dot{\theta}_j = 1 - \theta_j(V + \dot{x}_j). \quad (2)$$

Thus, (2) describes the time-dependent increase of the state variable even at $V = 0$ where V is the dimensionless loading rate. Following the model derivation of spring-block model with RSF law presented in [3, 22], the system equation is presented as

$$\begin{cases} \ddot{X}_j + X_j = \ell^2(X_{j+1} - 2X_j + X_{j-1}) + a_1\dot{X}_j + b_1\psi_j + \Gamma(\dot{X}_j, \psi_j); \\ \dot{\psi}_j = \alpha_1\dot{X}_j + \alpha_2\psi_j + \alpha_3\psi_j\dot{X}_j, \quad \text{for } j \in \mathbb{Z} \end{cases}, \quad (3)$$

where $\Gamma(\dot{X}_j, \psi_j) = a_2X^2 + a_3X^3 + o(X^4) + b_2\psi^2 + b_3\psi^3 + o(\psi^4)$, $a_1 = -a(1+V)^{-1}$, $b_1 = -bV$ and a_i, b_i, α_i for $i = 1, 2, 3$ are dimensionless parameters. Since a_1, b_1, α_1 and α_2 are negative, we make a hypothesis that

$$a_1, b_1, \alpha_1 \text{ and } \alpha_2 \text{ are non-positive.} \quad (4)$$

Thus, the system (3) depends on a four-dimension parameter $P = (a_1, b_1, \alpha_1, \alpha_2)$.

In his paper, Annan (2013) developed the general setup of equation (3) using spectral method approach to describe the model in space and time of a spring-block periodic pattern waves close to the critical variety of parameter space for small ℓ . The method will not work well in a case where we are interested in a large class of solutions for all ℓ . Therefore, instead of using a periodic propagation of waves approach, we develop a modulated wave equation for our model. The general idea of modulation is that if we consider the physical domain $\xi = \mathbb{R}^d \times \Sigma$, where Σ is a bounded cross-section, with $\varepsilon \in \mathbb{R}^d$ and $r \in \Sigma$, then the basic pattern of (3) takes the form

$$p(t, \varepsilon, r) = \varphi_{0, k_0}(r) \lambda(t, \varepsilon) \quad \text{where } \lambda(t, \varepsilon) = e^{i(\omega t + \langle k_0, \varepsilon \rangle)}.$$

Here, ω and k_0 belonging to $\mathbb{R}^d$ denote the temporal frequency and the spatial wave vector of the basic pattern, respectively. Therefore, the modulation pattern equation is of the form

$$\hbar^A(t) = A(\tau, \varsigma)p(t, x) + C_c,$$

where the amplitude function $A(\tau, \varsigma) \in \mathbb{C}$ depends on a slow time variable τ and a large spatial variable ς . The amplitude $|A(\tau, \varsigma)|$ gives the intensity of the basic pattern while phase changes of A give rise to changes of the pattern in the temporal frequency and in the spatial wave vector.

3. Modulation Equation Derivation

Let $\lambda(\varepsilon, r)$ and $\varphi_{\varepsilon, r}$ be the eigenvalue and eigenvector of the linear operator $\mathcal{L}(\varepsilon, ir)$ derived in [3], then $X = \varphi_{\varepsilon, r}[e^{\lambda(\varepsilon, r)t + irj}]_j$ is a solution to the linear equation. If the solution has a single pair of eigenvalues that crosses the complex plane at $\varepsilon = 0$, then for a certain r_0 we have $\lambda(0, r_0) = i\omega$. Thus, we can write the expression for the largest real eigenvalue for $\mathcal{L}(\varepsilon, ir)$ as [23] by

$$\lambda(\varepsilon, r) = \lambda_0(\varepsilon)(r - r_0) + \lambda_1(\varepsilon)(r - r_0)^2 + \lambda_2(\varepsilon)(r - r_0)^2 + o(|r - r_0|^3). \quad (5)$$

If we cross the imaginary axis from left to right at $\varepsilon = 0$ and consider $Re(\lambda_0(0)) = 0$, $\frac{dRe(\lambda_0(0))}{d\varepsilon} > 0$ and $Re(\lambda_1(0)) = 0$, then for $(\varepsilon, r) \rightarrow (0, r_0)$ we have

$$\varepsilon = \rho\delta^2 \text{ for } \rho = \pm 1 \text{ and } r = r_0 + \delta\kappa.$$

Thus, (5) reduces to

$$\lambda(\varepsilon, r) = i\omega + \delta^2\rho\lambda_{0,1} + \delta\kappa\lambda_1(0) - \delta^2\Lambda^2(0) + O(\delta^3), \quad (6)$$

with the solution of the linear problem in the form

$$e^{\lambda(\varepsilon, r) + irj} = e^{(\rho\lambda_{0,1} - \Lambda_0\kappa^2)\delta^2 t + i\kappa\delta(j + i\lambda_1(0)) + O(\delta^2)} p(t, j). \quad (7)$$

Here, we have denoted $p(t, j)$ by $e^{i(\omega t + r_0 j)\varphi_{\delta^2\rho, r_0 + \delta\kappa}} \approx e^{i(\omega t + r_0 j)\varphi_{0, r_0}}$ as the only mode that is not dumped at $\varepsilon = 0$. We see that the amplitude of (7) depends on the slow time variable $\delta^2 t$ and on the slow space variable $\varsigma = \delta(j - C_g t)$ where $C_g = -i\lambda_1(0)$. So, we expect that there exist solutions to the original non-linear system that are close to $\hbar^A(t)$ for small $\varepsilon > 0$ if the solution is of the form $\hbar^A(t) = A(\tau, \varsigma)p(t, x) + C_c$, where C_c is the complex conjugates.

In addition, as $\delta^2 \rightarrow 0$, we expect the modulated waves to be of small amplitude. Therefore, we make a scaling and assume that the amplitude is of order $O(\delta)$. Hence the modulated wave orbits form of (3) is

$$\hbar^A(t) = \begin{pmatrix} X^A(t) \\ \dot{X}^A(t) \\ \psi^B(t) \end{pmatrix} \text{ for } \begin{cases} X^A(t) = \delta A(\delta^2 t, \delta(j - C_g t))e^{i\omega t + ir_0 j} + C_c, \\ \psi_j^B(t) = \delta B(\delta^2 t, \delta(j - C_g t))e^{i\omega t + ir_0 j} + C_c. \end{cases} \quad (8)$$

We now claim that if the solution close to (8) exists, then substituting the solution into the system (3) and determining the power of δ will lead to the amplitude equation

$$\partial_\tau A = (c + id)A + (\vartheta + i\alpha)\partial_{\varsigma} A + (a + ib)|A|^2 A, \quad A(\tau, \varsigma) \in \mathbb{C}.$$

Proposition 3.1. *The group velocity of the modulated wave (8) is zero when $C_g = 0$.*

Proof. By definition $C_g = -i\frac{\partial\lambda(0,0)}{\partial r}$. Obtaining the derivative again with the implicit equation

$$p(\lambda(\varepsilon, r), \varepsilon, r) = 0 \quad \Rightarrow \quad \frac{\partial\lambda(0,0)}{\partial r} = \frac{\partial_r p(i\omega, 0, 0)}{\partial_X p(i\omega, 0, 0)}.$$

Since $i\omega$ is a simple root for $\varepsilon = 0$ and $r = 0$, it implies $\partial_X p(i\omega, 0, 0) \neq 0$ and so $\partial_r p(i\omega, 0, 0) = 0$. $\square$

Also, since we are looking for a solution of the form

$$\begin{cases} X_j(t) = \delta A(\delta^2 t, \delta j)e^{i\omega t} + C_c + O(\delta^2), \\ \psi_j(t) = \delta B(\delta^2 t, \delta j)e^{i\omega t} + C_c + O(\delta^2). \end{cases} \quad (9)$$

We notice from (3) that the non-linear terms contain $e^{2i\omega t}$, $e^{3i\omega t}$, ... terms with amplitude δ^2 , δ^3 , Therefore, we introduce the following series approach to account for the terms

$$X_j(t) = \sum_{k=0}^{\infty} \sum_{n \in \mathbb{Z}} \delta^k A_{kn}(\tau, \varsigma) e^{i\omega t}; \quad \psi_j(t) = \sum_{k=0}^{\infty} \sum_{n \in \mathbb{Z}} \delta^k B_{kn}(\tau, \varsigma) e^{i\omega t}, \quad (10)$$

where the amplitudes A_{kn} and B_{kn} are complex terms $A_{k-n} = \bar{A}_{kn}$ and $B_{k-n} = \bar{B}_{kn}$ for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Also,

$$\tau = \delta^2 t \text{ (slow time variable) and } \varsigma = \delta j \text{ (slow space variable).} \quad (11)$$

Annan (2013) has shown the existence of a bifurcation in model system (3) at some critical value of the hypothesis parameters (4) at $\Delta^c = [a_1^c, b_1^c, \alpha_1^c, \alpha_2^c]$. The researcher proved that at $\varepsilon = 0$, the spectrum of the linear operator $\mathcal{L}$ has a complex pair of element crossing the imaginary axis at $\pm i\omega$ with $\omega = \sqrt{a_1^c \alpha_2^c - \alpha_1^c b_1^c + 1}$. Specifically, the model system was uniformly stable at the origin but when crossing the threshold of instability at Δ^c , there existed a wide range of unstable modes, r [3]. Therefore, the origin is unstable for $\varepsilon > 0$.

Since the origin is unstable for any $\varepsilon > 0$, we consider $\varepsilon = +\delta^2$ and substitute (10) into model system (3) to have

$$\begin{cases} \ddot{X}_j + X_j = (\Delta_X)_j + (a_1^c + a_{11}\delta^2 + O(\delta^2))\dot{X}_j + a_2\dot{X}_j^2 + a_3\dot{X}_j^3 \\ \quad + (b_1^c + b_{11}\delta^2 + O(\delta^2))\psi_j + b_2\psi_j^2 + b_3\psi_j^3 + N_1(\dot{X}_j) + N_2(\psi_j), \\ \dot{\psi}_j = (\alpha_1^c + \alpha_{11}\delta^2 + O(\delta^2))\dot{X}_j + (\alpha_2^c + \alpha_{11}\delta^2 + O(\delta^2))\psi_j + \alpha_3\psi_j\dot{X}_j, \end{cases} \quad (12)$$

where $N_1(x)$, $N_2(x) = O(x^4)$, $\Delta_X = X_{j+1} - 2X_j + X_{j-1}$ and $j \in \mathbb{Z}$.

Theorem 3.2. Suppose that (3) has solutions of the form

$$\begin{cases} X_j(t) = X_j^A(t) + O(\delta^2), \\ \psi_j(t) = \psi_j^B + O(\delta^2), \end{cases} \quad \text{with} \quad \begin{cases} X_j^A(t) = \delta A(\tau, \varsigma)e^{i\omega t} + C_c \\ \psi_j^B = \delta B(\tau, \varsigma)e^{i\omega t} + C_c \end{cases}$$

for all $j \in \mathbb{Z}$ and $\delta \in (0, \delta_0)$, where A is a smooth function $[0, T] \times \mathbb{R} \rightarrow \mathbb{C}$. Then A is a solution to the complex scalar Ginzburg-Landau equation

$$\partial_\tau A = (c + id)A + (\vartheta + i\alpha)\partial_{\varsigma\varsigma} A + (a + ib)|A|^2 A, \quad A(\tau, \varsigma) \in \mathbb{C}. \quad (13)$$

with the following coefficients

$$\begin{aligned} c + id &= \frac{(i\omega - \alpha_2^c)b_{11}\Omega_{11} + b_1^c(\beta_{11}\Omega_{11} + i\omega\alpha_{11})}{(2i\omega - a_1^c)(i\omega - \alpha_2^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \\ \vartheta + i\alpha &= \frac{(i\omega - \alpha_2^c)\ell^2}{(2i\omega - a_1^c)(i\omega - \alpha_2^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \\ a + ib &= \frac{(i\omega - \alpha_2^c)(4a_2\omega^2\Omega_{22} + 2b_2\Omega_{22}'\bar{\Omega}_{11} + 3b_3\Omega_{11}|\Omega_{11}|^2 + 3i\omega^3a_3)}{(a_1^c - 2i\omega)(i\omega - \alpha_2^c) + b_1^c(\alpha_1^c - \Omega_{11})} \\ &\quad + \frac{i\omega b_1^c\alpha_3(2\Omega_{22}\Omega_{11} + \Omega_{20} - \Omega_{22}')}{(a_1^c - 2i\omega)(i\omega - \alpha_2^c) + b_1^c(\alpha_1^c - \Omega_{11})}. \end{aligned} \quad (14)$$

and B given by the expression $B = \frac{i\omega\alpha_1^c}{i\omega - \alpha_2^c}A$.

Proof. To prove that given a solution to the Ginzburg-Landau equation, we can determine all the amplitudes A_{kn} for $1 \leq k \leq 3$ and $-k \leq n \leq k$, let's consider X and ψ defined in (10) and express the series at order δ^3 where $\lambda(t) = e^{i\omega t}$. Then

$$\begin{aligned} \dot{X}_j(t) &= i\omega A_{11}\delta\lambda + i\omega A_{21}\delta^2\lambda + 2i\omega A_{22}\delta^2\lambda^2 + \partial_\tau A_{10}\delta^3\lambda^0 + 2i\omega A_{32}\delta^3\lambda^2 \\ &\quad + (\partial_\tau A_{11} + 3i\omega A_{31})\delta^3\lambda + 3i\omega A_{33}\delta^3\lambda^3 + C_c + O(\delta^3), \end{aligned}$$

$$\begin{aligned} \dot{X}_j^2(t) &= 2\omega^2|A_{11}|^2\delta^2\lambda^0 - \omega^2 A_{11}^2\delta^2\lambda^2 + 2\omega^2(\bar{A}_{11}A_{21} + A_{11}\bar{A}_{21})\delta^3\lambda^0 \\ &\quad + 4\omega^2\bar{A}_{11}A_{22}\delta^3\lambda - 2\omega^2 A_{11}A_{21}\delta^3\lambda^2 - 4\omega^2 A_{11}A_{22}\delta^3\lambda^3 + C_c + O(\delta^3), \end{aligned}$$

$$\dot{X}_j^3(t) = 3i\omega^2|A_{11}|^2 A_{11}\delta^3\lambda - i\omega^3 A_{11}^3\delta^3\lambda^3 + C_c + O(\delta^3),$$

$$\ddot{X}_j(t) = -\omega^2 A_{11} \delta \lambda - \omega^2 A_{21} \delta^2 \lambda - 4\omega^2 A_{22} \delta^2 \lambda^2 - 4\omega^2 A_{32} \delta^3 \lambda^2 \\ + (2i\omega \partial_\tau A_{11} - \omega^2 A_{32}) \delta^3 \lambda - 9\omega^2 A_{33} \delta^3 \lambda^3 + C_c + O(\delta^3),$$

$$\Delta_X(t) = \partial_{\zeta\zeta} A_{10} \delta^3 \lambda^0 + \partial_{\zeta\zeta} A_{11} \delta^3 \lambda + C_c + O(\delta^3),$$

$$\psi_j^2(t) = (B_{10}^2 + 2|B_{11}|^2) \delta^2 \lambda^0 + 2B_{10} B_{11} \delta^2 \lambda + B_{11}^2 \delta^2 \lambda^2 + 2B_{11} \delta^3 \lambda^3 \\ + 2(B_{10} B_{20} + B_{11} \bar{B}_{21} + \bar{B}_{11} B_{21}) \delta^3 \lambda^0 + 2(B_{10} B_{21} + B_{10} \bar{B}_{20} + \bar{B}_{11} B_{22}) \delta^3 \lambda \\ + 2(B_{10} B_{22} + B_{11} B_{21}) \delta^3 \lambda^2 + C_c + O(\delta^3),$$

$$\psi_j^3(t) = B_{10}^3 \delta^3 \lambda^0 + 3(B_{10}^2 B_{11} + B_{11}^2 \bar{B}_{11}) \delta^3 \lambda + 3B_{11}^2 B_{10} \delta^3 \lambda^2 \\ + B_{11}^3 \delta^3 \lambda^3 + C_c + O(\delta^3),$$

$$\psi_j(t) \dot{X}_j(t) = i\omega(A_{11} \bar{B}_{11} - B_{11} \bar{A}_{11}) \delta^2 \lambda^0 + i\omega B_{10} A_{11} \delta^2 \lambda + i\omega B_{11} A_{11} \delta^2 \lambda^2 \\ + i\omega(\bar{B}_{11} A_{21} - B_{11} \bar{A}_{21} + A_{11} \bar{B}_{21} - \bar{A}_{11} B_{21}) \delta^3 \lambda^0 \\ + i\omega(B_{10} A_{21} + 2A_{22} \bar{B}_{11} + A_{11} B_{20} - \bar{A}_{11} B_{22}) \delta^3 \lambda^1 \\ + i\omega(2B_{10} A_{22} + B_{11} A_{21} + A_{11} B_{21}) \delta^3 \lambda^2 \\ + i\omega(2A_{22} B_{11} + A_{11} B_{22}) \delta^3 \lambda^3 + C_c + O(\delta^3).$$

Substituting the above expressions into (3) and comparing each order $\delta^k \lambda^n$ for $1 \leq k \leq 3$ and $-k \leq n \leq k$ gives the following results

$$\text{The } \delta \lambda^0 \text{ term gives : } \begin{cases} A_{10} = b_1^c B_{10} \\ 0 = \alpha_2^c B_{10} \end{cases} \Rightarrow A_{10} = B_{10} = 0.$$

$$\text{The } \delta \lambda \text{ term gives : } \begin{cases} -\omega A_{11} + A_{11} = a_1^c i\omega A_{11} + b_1^c B_{11} \\ i\omega B_{11} = \alpha_1^c i\omega A_{11} + \alpha_2^c B_{11} \end{cases},$$

$$\Rightarrow B_{11} = \Omega_{11} A_{11} \quad \text{where} \quad \Omega_{11} = \frac{i\omega \alpha_1^c}{i\omega - \alpha_2^c} \in \mathbb{C}. \quad (15)$$

The first equation is written as $p(i\omega, 0, 0)A_{11} = 0$, which is satisfied for any A_{11} since $p(i\omega, 0, 0) = 0$.

$$\text{The } \delta^2 \lambda^0 \text{ term gives : } \begin{cases} A_{20} = b_1^c B_{20} + A_2 a_2 \omega^2 |A_{11}|^2 + b_2 |B_{11}|^2 \\ 0 = \alpha_2^c B_{20} + \alpha_3^c i\omega(A_{11} \bar{B}_{11} - B_{11} \bar{A}_{11}) \end{cases},$$

$$\Rightarrow \begin{cases} B_{20} = \Omega_{20} |A_{11}|^2, & \Omega_{20} = \frac{2\omega \alpha_3}{\alpha_3^c} \text{Im} \Omega_{11} \in \mathbb{R}, \\ A_{20} = \Omega'_{20} |A_{11}|^2, & \Omega'_{20} = b_1^c \Omega_{20} + 2a_2 \omega^2 + 2b_2 |\Omega_{11}|^2 \in \mathbb{R}. \end{cases}, \quad (16)$$

$$\text{The } \delta^2 \lambda^1 \text{ term gives : } \begin{cases} -\omega^2 A_{21} + A_{21} = i\omega a_1^c A_{21} + b_1^c B_{21} \\ i\omega B_{21} = i\omega \alpha_1^c A_{21} + \alpha_2^c B_{21} \end{cases},$$

From the second equation we get $B_{21} = \Omega_{11}A_{21}$ and it follows that the first equation can be written as $p(i\omega, 0, 0)A_{21} = 0$ which is satisfied for any A_{21} . The $\delta^2\lambda^2$ term gives:

$$\begin{cases} -4\omega^2 A_{22} + A_{22} = 2i\omega a_1^c A_{22} + b_1^c B_{22} - \omega^2 a_2 A_{11}^2 + b_2 B_{11}^2 \\ 2i\omega B_{22} = 2i\omega \alpha_1^c A_{22} + \alpha_2^c B_{22} + i\omega a_3 B_{11} A_{11}. \end{cases}$$

$$\Rightarrow \begin{cases} A_{22} = \Omega'_{22} A_{11}^2, & \Omega'_{22} = \frac{\eta(i\omega)}{p(2i\omega, 0, 0)} \in \mathbb{C}, \\ B_{22} = \Omega_{22} A_{11}^2, & \Omega_{22} = \frac{2i\omega \Omega'_{22} \alpha_1^c + i\omega \alpha_3 \Omega_{11}}{2i\omega - \alpha_2^c} \in \mathbb{C} \end{cases}, \quad (17)$$

where $\eta(i\omega) = -2i\omega^3 a_2 + a_2 \alpha_2^c + i\omega \Omega_{11}(\alpha_3 b_1^c + 2b_2 \Omega_{11}) - \alpha_2^c b_2 \Omega_{11}^2$. To obtain $\delta^3\lambda^0$, we have

$$\begin{cases} A_{30} = b_1^c B_{30} + 2a_2 \omega^2 (\bar{A}_{11} A_{21} + A_{11} \bar{A}_{21}) + 2b_2 (B_{11} \bar{B}_{21} + \bar{B}_{11} B_{21}), \\ 0 = \alpha_2^c B_{30} + i\omega \alpha_3 (\bar{B}_{11} A_{21} - B_{11} \bar{A}_{21} + A_{11} \bar{B}_{21} - \bar{A}_{11} B_{21}). \end{cases}$$

$$\Rightarrow \begin{cases} B_{30} = 4\Omega_{30} \operatorname{Re}(A_{11} \bar{A}_{21}), & \Omega_{30} = \frac{\omega \alpha_3}{\alpha_2^c} \operatorname{Im} \bar{\Omega}_{11} \in \mathbb{R}, \\ A_{30} = \Omega'_{30} \operatorname{Re}(A_{11} \bar{A}_{21}), & \Omega'_{30} = \Omega_{30} + a_2 \omega^2 + b_2 |\Omega_{11}|^2 \in \mathbb{R}. \end{cases}, \quad (18)$$

The $\delta^3\lambda^1$ term gives:

$$\begin{cases} -\omega^2 A_{31} + 2i\omega \partial_\tau A_{11} + A_{31} = \ell^2 \partial_{\zeta\zeta} A_{11} + a_1^c (A_{22} \partial_\tau A_{11} + i\omega A_{31}) + b_1^c B_{31} \\ \quad 4a_2 \omega^2 \bar{A}_{11} A_{22} + 2b_2 \bar{B}_{11} B_{22} + 3b_3 B_{11}^2 \bar{B}_{11} + 3i\omega^3 a_3 A_{11} |A_{11}|^2 \\ \quad + b_{11} \Omega_{11} A_{11} + i\omega a_{11} A_{11} \\ i\omega B_{31} + \partial_\tau B_{11} = a_1^c (\partial_\tau A_{11} + i\omega A_{31}) + \alpha_2^c B_{31} + \beta_{11} B_{11} \\ \quad + i\omega \alpha_3 (2A_{22} \bar{B}_{11} + A_{11} B_{20} - \bar{A}_{11} B_{22}) + i\omega \alpha_{11} A_{11}. \end{cases}$$

Using (15) and (17) and the fact that $p(i\omega, 0, 0) = 0$, the second equation is reduced to

$$B_{31} = \lambda_1 A_{31} + \lambda_2 \partial_\tau A_{11} + \lambda_3 A_{11} |A_{11}|^2 + \lambda_4 A_{11}, \quad (19)$$

where

$$\begin{aligned} \lambda_1 &= \frac{i\omega \alpha_1^c}{i\omega - \alpha_2^c}, & \lambda_2 &= \frac{\alpha_1^c - \Omega_{11}}{i\omega - \alpha_2^c}, \\ \lambda_3 &= \frac{i\omega \alpha_3 (2\Omega_{22} \bar{\Omega}_{11} + \Omega_{20} - \Omega'_{22})}{i\omega - \alpha_2^c}, & \lambda_4 &= \frac{\beta_{11} \Omega_{11} + i\omega \alpha_{11}}{i\omega - \alpha_2^c}. \end{aligned}$$

Substituting (19) into the first part of the equation gives

$$\partial_\tau A_{11} = (c + id)A_{11} + (\vartheta + i\alpha)\partial_{\zeta\zeta} A_{11} + (a + ib)|A_{11}|^2 A_{11}, \quad (20)$$

With

$$\begin{aligned} c + id &= \frac{(i\omega - \alpha_2^c)b_{11}\Omega_{11} + b_1^c(\beta_{11}\Omega_{11} + i\omega\alpha_{11})}{(2i\omega - a_1^c)(i\omega - \alpha_2^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \\ \vartheta + i\alpha &= \frac{(i\omega - \alpha_2^c)\ell^2}{(2i\omega - a_1^c)(i\omega - \alpha_2^c) - b_1^c(\alpha_1^c - \Omega_{11})}, \end{aligned} \quad (21)$$

$$a + ib = \frac{(i\omega - \alpha_2^c)(4a_2\omega^2\Omega_{22} + 2b_2\Omega_{22}'\bar{\Omega}_{11} + 3b_3\Omega_{11}|\Omega_{11}|^2 + 3i\omega^3a_3)}{(a_1^c - 2i\omega)(i\omega - \alpha_2^c) + b_1^c(\alpha_1^c - \Omega_{11})} + \frac{i\omega b_1^c\alpha_3(2\Omega_{22}\Omega_{11} + \Omega_{20} - \Omega_{22}')}{(a_1^c - 2i\omega)(i\omega - \alpha_2^c) + b_1^c(\alpha_1^c - \Omega_{11})}. \quad (22)$$

Finally, $\delta^3\lambda^2$ and $\delta^3\lambda^3$ are obtained respectively as

$$\begin{cases} -4\omega^2 A_{32} + A_{32} = 2i\omega a_1^c A_{32} + b_1^c B_{32} - 2\omega^2 a_2 A_{11} A_{21} + 2b_2 B_{11} B_{21}, \\ 2i\omega B_{32} = 2i\omega \alpha_1^c A_{32} + \alpha_2^c B_{32} + i\omega a_3 (B_{11} A_{21} + A_{11} B_{21}). \end{cases}$$

$$\Rightarrow \begin{cases} A_{32} = \Omega_{32}' A_{11} A_{21}, \quad \Omega_{32}' = \frac{2(b_2\Omega_{11}^2 - a_2\omega^2)(2i\omega - \alpha_2^c) + 2i\omega b_1^c\alpha_3 - \alpha_2^c\Omega_{11}}{(1 - 4\omega^2 - 2i\omega a_1^c)(2i\omega - \alpha_2^c) - i\omega\alpha_2^c\alpha_1^c} \in \mathbb{C}, \\ B_{32} = \Omega_{32} A_{11} A_{21}, \quad \Omega_{32} = \frac{2i\omega(\Omega_{32}'\alpha_1^c + \alpha_3\Omega_{11})}{2i\omega - \alpha_2^c} \in \mathbb{C} \end{cases}, \quad (23)$$

and

$$\begin{cases} (1 - 9\omega^2)A_{33} = 3i\omega a_1^c A_{33} + b_1^c B_{33} - 4\omega^2 a_2 A_{11} A_{22} \\ \quad + 2b_2 B_{11} B_{22} - i\omega^3 a_3 A_{11}^3 + b_3 B_{11}^3, \\ 3i\omega B_{33} = 3i\omega \alpha_1^c A_{33} + \alpha_2^c B_{33} + i\omega a_3 (2B_{11} A_{22} + A_{11} B_{22}). \end{cases}$$

When equations (15) and (17) are applied

$$\Rightarrow A_{33} = \Omega_{33}' A_{11}^3 \text{ and } B_{33} = \Omega_{33} A_{11}^3; \text{ with} \quad (24)$$

$$\begin{aligned} \Omega_{33}' &= \frac{i\omega b_1^c\alpha_3(2\Omega_{22}'\Omega_{11} + \Omega_{22})}{(3i\omega - \alpha_2^c)(1 - 9\omega^2 - 3i\omega a_1^c) - 3i\omega\alpha_1^c b_1^c} \\ &\quad + \frac{(3i\omega - \alpha_2^c)(2b_2\Omega_{11}\Omega_{22} - i\omega^3 a_3 + b_3\Omega_{11}^3 - 4a_2\omega^2\Omega_{22}')}{(3i\omega - \alpha_2^c)(1 - 9\omega^2 - 3i\omega a_1^c) - 3i\omega\alpha_1^c b_1^c} \in \mathbb{C}. \\ \Omega_{33} &= \frac{3i\omega\alpha_1^c\Omega_{33}' + i\omega\alpha_3(2\Omega_{22}'\Omega_{11} + \Omega_{22})}{(3i\omega - \alpha_2^c)} \in \mathbb{C}. \end{aligned}$$

To compare the δ^3 order terms we have

$$\begin{aligned} A_{10} = B_{10} = 0, \quad B_{11} = \Omega_{11} A_{11}, \quad B_{20} = \Omega_{20} |A_{11}|^2, \quad A_{20} = \Omega_{20}' |A_{11}|^2, \\ A_{22} = \Omega_{22}' A_{11}^2, \quad B_{22} = \Omega_{22} A_{11}^2, \quad A_{33} = \Omega_{33}' A_{11}^3, \quad B_{33} = \Omega_{33} A_{11}^3. \end{aligned} \quad (25)$$

Also, we know that B_{31} is a function of A_{11} while ∂_τ and A_{11} satisfies (20). In addition

$$\begin{aligned} B_{21} = \Omega_{11} B_{21}, \quad B_{30} = 4\Omega_{30} \text{Re}(A_{11} \bar{A}_{21}), \quad A_{30} = 4\Omega_{30}' \text{Re}(A_{11} \bar{A}_{21}), \\ A_{32} = \Omega_{32}' A_{11} A_{21}, \quad B_{32} = \Omega_{32} A_{11} A_{21} \text{ and } A_{31} = A_{21} = 0. \end{aligned} \quad (26)$$

Thus, from equations (19), (25) and (26) A_{11} is a solution of (20) with B_{31} being a function of A_{11} and $\partial_\tau A_{11}$ and $A_{10} = B_{10} = A_{21} = B_{21} = A_{30} = B_{30} = A_{31} = A_{32} = B_{32} = 0$. Therefore, given a solution of (20), we can evaluate all

the A_{kn} and B_{kn} solutions for $1 \leq k \leq 3$ and $-k \leq n \leq k$ using the series equations (10) to obtain

$$\begin{cases} X_j^A(t) = \delta A_{11} \lambda + \delta^2 \Omega'_{20} |A_{11}|^2 + \delta^2 \Omega'_{22} A_{11}^2 \lambda^2 + \delta^3 \Omega'_{33} A_{11}^3 \lambda^3 + C_c + O(\delta^3), \\ \psi_j^B(t) = \delta \Omega_{11} A_{11} \lambda + \delta^2 \Omega_{20} |A_{11}|^2 + \delta^2 \Omega_{22} A_{11}^2 \lambda^2 + \delta^3 B_{31} \lambda \\ \quad + \delta^3 \Omega_{33} A_{11}^3 \lambda^3 + C_c + O(\delta^3). \end{cases} \quad (27)$$

□

4. Conclusion

The parameters affecting the existence of periodic solutions generated with rate and state friction in spring-block model were stated. Derivation of the modulation equation for the model with non-linear rate-and state- friction law that allows the amplitude of the wave to vary slowly with time and space was presented. Conditions under which the modulation wave equation produced many small solutions near the instability threshold for finitely large enough time scales were established.

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