VECTOR BUNDLES ON A QUADRIC SURFACE
WHICH ARE POSITIVE ON EACH LINE

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY

Abstract: Let $Q$ be a smooth quadric surface and $E$ a vector bundle on $Q$. We say that $E$ is weakly line positive if for each line $T \subset Q$ the bundle $E|T$ is a direct sum of line bundles of degree $\geq 0$. Here we classify the quadruples $(a, b, u, v) \in \mathbb{Z}^4$ such that there is a weakly line positive extension of $\mathcal{O}_Q(a, b)$ by $\mathcal{O}_Q(u, v)$.

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1. Introduction

Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. We have $\text{Pic}(Q) \cong \mathbb{Z}^2$ and we will write $\mathcal{O}_Q(a, b)$ for the line bundle of bidegree $(a, b)$ on $Q$. Let $\pi_i : Q \to \mathbb{P}^1$, $i = 1, 2$, be the two projections with $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathcal{O}_Q(1, 0)$ and $\pi_2^*(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathcal{O}_Q(0, 1)$. Let $E$ be a rank two vector bundle on $Q$. We say that $E$ is $\pi_1$-uniform (resp. $\pi_2$-uniform) if there are integers $c \geq d$ such that for all $T \in |\mathcal{O}_Q(1, 0)|$ (resp all $T \in |\mathcal{O}_Q(0, 1)|$) the vector bundle $E|T$ is a direct sum of a line bundle of degree $c$ and a line bundle of degree $d$. A vector bundle $F$ is said to be weakly line positive if for each $T \in (|\mathcal{O}_Q(1, 0)| \cup |\mathcal{O}_T(0, 1)|)$ (i.e. for each line $T \subset Q$) the vector bundle bundle $F|T$ is a direct sum of line bundles of degree $\geq 0$. Notice
that every vector bundle spanned outside finitely many points is weakly line positive. In this note we prove the following result.

**Theorem 1.** There is a weakly line positive extension of \( O_Q(a, b) \) by \( O_Q(u, v) \) if and only if \( a \geq 0, \ b \geq 0 \) and one of the following conditions is satisfied:

1. \( u \geq 0 \) and \( v \geq 0 \).
2. \( v = b \) and \( -a \leq u < 0 \).
3. \( v > b \) and \( -a + 1 \leq u < 0 \).
4. \( u = a \) and \( -b \leq v < 0 \).
5. \( u > a \) and \( -b + 1 \leq v < 0 \).

2. The Proof

We need the following remarks ([1]).

**Remark 1.** Let \( E \) be a \( \pi_2 \)-uniform vector bundle of type \((t, t)\). Then \( \pi_2^*(E(-t, -0)) \) is a rank two vector bundle and \( E \cong \pi_2^*(\pi_2^*(E(-t, -0)))(t, 0) \). Hence \( E \cong O_Q(t, e) \oplus O_Q(t, f) \) for some integers \( e, f \).

**Remark 2.** Let \( E \) be a \( \pi_2 \)-uniform vector bundle of type \((t, c)\) for some \( t > c \). Then \( \pi_2^*(E(-t, -0)) \) is a rank one bundle and \( \pi_2^*(\pi_2^*(E(-t, -0)))(t, 0) \) is a rank one saturated subbundle of \( E \). Hence there are integers \( e, f \) such that \( E \) is an extension of \( O_Q(c, e) \) by \( O_Q(t, f) \).

**Remark 3.** Assume \( v > b \) and take any extension \( E \) of \( O_Q(a, b) \) by \( O_Q(-a, v) \). Then \( E|T \) has splitting type \((v, b)\) for each \( T \in |O_Q(0, 1)| \). Hence \( E \) is \( \pi_1 \)-uniform.

**Proposition 1.** Fix integers \( a > 0, \) and \( v > b \). Then there is no vector bundle \( E \) on \( Q \) spanned outside finitely many points and which is an extension of \( O_Q(a, b) \) by \( O_Q(-a, v) \).

**Proof.** Notice that \( c_2(E) = a(v - b) \). Assume the existence of such a vector bundle \( E \). Since \( O_Q(a, b) \) is a quotient of \( E \), it is spanned outside a finite set. Hence \( b \geq 0 \). Since \( E \) is spanned outside finitely many points, for each \( D \in |O_Q(0, 1)| \) the bundle \( E|D \) is spanned outside finitely many points. Hence it has splitting type \( a_1 \geq a_2 \geq 0 \). Since \( a_1 + a_2 = 0 \), we get \( a_1 = a_2 = 0 \). Remark
1 gives \( E \cong \mathcal{O}_Q(0,e) \oplus \mathcal{O}_Q(0,f) \) for some \( e,f \). Hence \( c_2(E) = 0 \neq a(v-b) \), a contradiction. \( \square \)

**Lemma 1.** Take integers \( a \geq 0 \), \( b \geq 0 \) and \( u \geq -a \). Then there is a spanned vector bundle \( E \) which is an extension of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,b) \).

**Proof.** If \( u \geq 0 \), then every extension of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,b) \), because \( \mathcal{O}_Q(a,b) \) is spanned and \( h^1(\mathcal{O}_Q(u,b)) = 0 \). Now assume \( 0 > u \geq -a \). There is an extension \( F \) of \( \mathcal{O}_{\mathbb{P}^1}(a) \) by \( \mathcal{O}_{\mathbb{P}^1}(u) \) which is spanned. Hence \( G := \pi_1^*(F) \) is spanned. Since \( \mathcal{O}_Q(0,b) \) is spanned, \( E := G(0,b) \) is spanned. \( \square \)

**Lemma 2.** Fix integers \( a, u \) such that \( -a < u < 0 \). Let \( \mathcal{B} \) be the set of all \( \epsilon \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-a)) \) such that the extension \( E \) of \( \mathcal{O}_{\mathbb{P}^1}(a) \) by \( \mathcal{O}_{\mathbb{P}^1}(u) \) induced by \( \epsilon \) is not spanned. Then \( \mathcal{B} \) has codimension \( \geq 2 \) in \( H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-a)) \).

**Proof.** Fix an integer \( t > 0 \) and set \( F := \mathcal{O}_{\mathbb{P}^1}(a+u-t) \oplus \mathcal{O}_{\mathbb{P}^1}(-t) \). We have \( h^1(F \otimes F^\vee) \geq 2t - a - u - 1 \geq 2 \). Use the deformation theory of vector bundles on \( \mathbb{P}^1 \). \( \square \)

**Proof of Theorem 1.** Every extension \( E \) of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,v) \) has \( \mathcal{O}_Q(a+u,b+v) \) as its determinant. Hence if there is a weakly line positive extension \( E \) of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,v) \), then \( a + u \geq 0 \) and \( b + v \geq 0 \). Hence from now on we assume \( u \geq -a \) and \( v \geq b \). Assume \( a < 0 \). Since \( u \geq -a > 0 > a \), for each \( T \in |\mathcal{O}_Q(0,1)| \) the vector bundle \( E|T \) is the direct sum of a line bundle of degree \( u \) and a line bundle of degree \( a < 0 \), a contradiction. In the same way we check that \( b \geq 0 \) is a necessary condition. From now on we assume \( a \geq 0 \) and \( b \geq 0 \). If \( u \geq 0 \) and \( v \geq 0 \), then every extension of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,v) \) is spanned and in particular it is weakly line positive. Now assume \( u < 0 \) and \( v < 0 \). Since \( a \geq 0 \) and \( b \geq 0 \), we have \( h^1(\mathcal{O}_Q(u-a,u-b)) = 0 \) (K"unneth formula). Hence any extension of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,v) \) splits. Hence no such an extension is weakly line positive. Hence it is sufficient to look at all cases with \( uv < 0 \). Interchanging the role of the two rulings of \( Q \) we see that it is sufficient to analyze all cases with \( -a \leq u < 0 \) and \( v \geq 0 \). If \( v < b \), then \( h^1(\mathcal{O}_Q(u-a,u-b)) = 0 \) (K"unneth formula). Hence to get a weakly line positive extension of \( \mathcal{O}_Q(a,b) \) by \( \mathcal{O}_Q(u,v) \) we need to assume \( v \geq b \). If \( v = b \), then there is a spanned extension (Lemma 1). From now on we assume \( v > b \). First assume \( u = -a \). Since \( a \geq 0 \) and \( b \geq 0 \), we may repeat the proof of Lemma 1 and get a contradiction. Now assume \( u \geq -a + 1 \) (and hence \( a \geq 2 \)). Lemma 2 gives the existence of \( \epsilon_i \in H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-a)) \), \( i = 1,2 \), such that for all \( (\lambda,\mu) \in \mathbb{K}^2 \setminus \{(0,0)\} \) the extension \( F_{\lambda\epsilon_1+\mu\epsilon_2} \) of \( \mathcal{O}_{\mathbb{P}^1}(a) \) by \( \mathcal{O}_{\mathbb{P}^1}(u) \) is spanned. K"unneth formula gives \( H^1(\mathcal{O}_Q(u-a,b-v)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(u-a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(v-b)) \). Since \( v-b > 0 \),
there are two linearly independent $\alpha_i \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(v-b))$ with no common zero. Set $\epsilon := \alpha_1 \epsilon_1 + \alpha_2 \epsilon_2 \in H^1(\mathcal{O}_Q(u-a,b-v))$ and let $E$ be the extension of $\mathcal{O}_Q(a,b)$ by $\mathcal{O}_Q(u,v)$ induced by $\epsilon$. Since $v > b$, for all $T \in |\mathcal{O}_Q(0,1)|$ the vector bundle $E|T$ has splitting type $(v,b)$. Fix $D \in |\mathcal{O}_Q(0,1)|$, say $D = \pi^1(o)$ with $o \in \mathbb{P}^1$. We have $E|D \cong F_{\alpha_1(o)\epsilon_1+\alpha_2(o)\epsilon_2}$. Hence $E|D$ is spanned. Hence $E$ is weakly line positive.\qed

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References