REGULAR ELEMENTS OF THE COMPLETE SEMIGROUPS
OF BINARY RELATIONS OF THE CLASS $\Sigma_7(X, 8)$

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Abstract: In this paper let $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ be a subsemilattice of $X$—semilattice of unions $D$ where $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8$, $T_1 \neq \emptyset$, $T_4 \setminus T_3 \neq \emptyset$, $T_3 \setminus T_4 \neq \emptyset$, $T_6 \setminus T_7 \neq \emptyset$, $T_7 \setminus T_6 \neq \emptyset$, $T_3 \cup T_4 = T_5$, $T_6 \cup T_7 = T_8$, then we characterize the class each element of which is isomorphic to $Q$ by means of the characteristic family of sets, the characteristic mapping and the generate set of $Q$. Moreover, we calculate the number of regular elements of $B_X(D)$ for a finite set $X$.

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1. Introduction

Let $X$ be an arbitrary nonempty set. Recall that a binary relation on $X$ is a subset of the cartesian product $X \times X$. The binary operation $\circ$ on $B_X$ (the set
of all binary relations on $X$) defined by for $\alpha, \beta \in B_X$

$$(x, z) \in \alpha \circ \beta \iff (x, y) \in \alpha \text{ and } (y, z) \in \beta,$$

for some $y \in X$ is associative. Therefore $B_X$ is a semigroup with respect to the operation $\circ$. This semigroup is called the \textit{semigroup of all binary relations} on the set $X$.

Let $D$ be a nonempty set of subsets of $X$ which is closed under the union i.e., $\cup D' \in D$ for any nonempty subset $D'$ of $D$. In that case, $D$ is called a \textit{complete $X$–semilattice of unions}. The union of all elements of $D$ is denoted by the symbol $\tilde{D}$. Clearly, $\tilde{D}$ is the largest element of $D$.

Let $X$ be an arbitrary nonempty set and $m$ be an arbitrary cardinal number. $\Sigma (X, m)$ is the class of all complete $X$–semilattices of unions of power $m$.

Let $\bar{D}$ and $D'$ be some nonempty subsets of the complete $X$–semilattices of unions. We say that a subset $\bar{D}$ generates a set $D'$ if any element from $D'$ is a set-theoretic union of the elements from $\bar{D}$.

Note that the semilattice $D$ is partially ordered with respect to the set-theoretic inclusion. Let $\emptyset \neq D' \subseteq D$ and

$$N(D, D') = \{Z \in D \mid Z \subseteq Z' \text{ for any } Z' \in D'\}.$$  

It is clear that $N(D, D')$ is the set of all lower bounds of $D'$. If $N(D, D') \neq \emptyset$ then $\Lambda(D, D') = \cup N(D, D')$ belongs to $D$ and it is the \textit{greatest lower bound} of $D'$.

Further, let $x, y \in X$, $Y \subseteq X$, $\alpha \in B_X$, $T \in D$, $\emptyset \neq D' \subseteq D$ and $t \in \bar{D}$. Then we have the following notations,

$$y\alpha = \{x \in X \mid (y, x) \in \alpha\}, Y\alpha = \bigcup_{y \in Y} y\alpha,$$

$$V(D, \alpha) = \{Y\alpha \mid Y \in D\}, D_t = \{Z' \in D \mid t \in Z'\},$$

$$D'_T = \{Z' \in D' \mid T \subseteq Z'\}, \bar{D}_T = \{Z' \in D' \mid Z' \subseteq T\}.$$  

Let $f$ be an arbitrary mapping from $X$ into $D$. Then one can construct a binary relation $\alpha_f$ on $X$ by $\alpha_f = \bigcup_{x \in X} (\{x\} \times f(x))$. The set of all such binary relations is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation $\circ$. In this case $B_X(D)$, is called a \textit{complete semigroup of binary relations} defined by an $X$–semilattice of unions $D$. This structure was comprehensively investigated in Diasemidze [6].

If $\alpha \circ \beta \circ \alpha = \alpha$ for some $\beta \in B_X(D)$ then a binary relation $\alpha$ is called a \textit{regular element} of $B_X(D)$. 

Let $\alpha \in B_X$, $Y^\alpha_T = \{y \in X \mid y\alpha = T\}$ and

$$V[\alpha] = \begin{cases} V(X^*, \alpha), & \text{if } \emptyset \notin D, \\ V(X^*, \alpha), & \text{if } \emptyset \in V(X^*, \alpha), \\ V(X^*, \alpha) \cup \{\emptyset\}, & \text{if } \emptyset \notin V(X^*, \alpha) \text{ and } \emptyset \in D. \end{cases}$$

Then a representation of a binary relation $\alpha$ of the form $\alpha = \bigcup_{T \in V[\alpha]} (Y^\alpha_T \times T)$ is called quasinormal. Note that, if $\alpha = \bigcup_{T \in V[\alpha]} (Y^\alpha_T \times T)$ is a quasinormal representation of the binary relation $\alpha$, then $X = \bigcup_{T \in V(X^*, \alpha)} Y^\alpha_T$ and $Y^\alpha_T \cap Y^\alpha_{T'} \neq \emptyset$ for $T, T' \in V(X^*, \alpha)$ which $T \neq T'$. In [7] they show that, if $\beta$ is regular element of $B_X(D)$, then $V[\beta] = V(D, \beta)$ and a complete $X-$semilattice of unions $D$ is an $XI-$semilattice of unions if $\Lambda(D, D_t) \in D$ for any $t \in \bar{D}$ and $\bar{Z} = \bigcup_{T \in V(X^*, \alpha)} \Lambda(D, D_t)$ for any nonempty element $Z$ of $D$.

Let $D'$ be an arbitrary nonempty subset of the complete $X-$semilattice of unions $D$. A nonempty element $T \in D'$ is a nonlimiting element of $D'$ if $T \setminus l(D', T) = T \setminus \cup (D' \setminus D'_T) \neq \emptyset$. A nonempty element $T \in D'$ is limiting element of $D'$ if $T \setminus l(D', T) = \emptyset$.

The family $C(D)$ of pairwise disjoint subsets of the set $\bar{D} = \cup D$ is the characteristic family of sets of $D$ if the following hold

a) $\cap D \in C(D)$

b) $\cup C(D) = \bar{D}$

c) There exists a subset $C_Z(D)$ of the set $C(D)$ such that $Z = \cup C_Z(D)$ for all $Z \in D$.

A mapping $\theta : D \to C(D)$ is called characteristic mapping if $Z = (\cap D) \cup \bigcup_{Z' \in \bar{D}} \theta (Z')$ for all $Z \in D$.

The existence and the uniqueness of characteristic family and characteristic mapping is given in Diasemidze [8]. Moreover, it is shown that every $Z \in D$ can be written as $Z = \theta(\bar{Q}) \cup \bigcup_{T \in \bar{Q}(Z)} \theta (T)$, where $\bar{Q}(Z) = Q \setminus \{T \in Q \mid Z \subseteq T\}$.

A one-to-one mapping $\varphi$ between two complete $X-$semilattices of unions $D'$ and $D''$ is called a complete isomorphism if $\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$ for each
nonempty subset $D_1$ of the semilattice $D'$. Also, let $\alpha \in B_X(D)$. A complete isomorphism $\varphi$ between $XI$—semilattice of unions $Q$ and $D$ is called a complete $\alpha$— isomorphism if $Q = V(D, \alpha)$ and $\varphi(\emptyset) = \emptyset$ for $\emptyset \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$.

Let $Q$ and $D'$ are respectively some $XI$ and $X$—subsemilattices of the complete $X$—semilattice of unions $D$. Then

$$R_\varphi(Q, D') = \{\alpha \in B_X(D) \mid \alpha \text{ regular element, } \varphi \text{ complete } \alpha \text{—isomorphism}\}$$

where $\varphi : Q \to D'$ complete isomorphism and $V(D, \alpha) = Q$. Besides, let us denote

$$R(Q, D') = \bigcup_{\varphi \in \Phi(Q, D')} R_\varphi(Q, D') \text{ and } R(D') = \bigcup_{Q' \in \Omega(Q)} R(Q', D').$$

where

$$\Phi (Q, D') = \{\varphi \mid \varphi : Q \to D' \text{ is a complete } \alpha \text{—isomorphism } \exists \alpha \in B_X(D)\},$$

$$\Omega(Q) = \{Q' \mid Q' \text{ is } XI \text{—subsemilattices of } D \text{ which is complete isomorphic to } Q\}.$$ E. Schröder described the theory of binary relations in detail in the 1890s ([1]). The basic concepts and the properties of the theory were introduced in ”Principia mathematica” Whitehead and Russell([2]). The theory of binary relations has been improved by Riguet ([3] — [4]). Many researcher studied this theory using partial transformations as Vagner did ([5]). Regular elements of semigroup play an important role in semigroup theory. Therefore Diasamidze generate systematic rules for understanding structure of a semigroup of binary relations and characterization of regular elements of these semigroup in ([6] — [9]). In general he studied semigroups but, in particular, he investigates complete semigroups of the binary relations.

In this paper, we take in particular, $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ subsemilattice of $X$—semilattice of unions $D$ where the elements $T_i$, $i = 1, 2, \ldots, 8$ are satisfying the following properties, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8$, $T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8$, $T_1 \not\subset \emptyset$, $T_1 \backslash T_2 \not\subset \emptyset$, $T_1 \backslash T_3 \not\subset \emptyset$, $T_1 \backslash T_4 \not\subset \emptyset$, $T_1 \backslash T_5 \not\subset \emptyset$, $T_1 \backslash T_6 \not\subset \emptyset$, $T_1 \backslash T_7 \not\subset \emptyset$, $T_1 \backslash T_8 \not\subset \emptyset$, $T_1 \backslash T_1 = T_5$, $T_1 \backslash T_2 = T_7$, $T_1 \backslash T_3 = T_8$. We will investigate the properties of regular element $\alpha \in B_X(D)$ satisfying $V(D, \alpha) = Q$. Moreover, we will calculate the number of regular elements of $B_X(D)$ for a finite set $X$. 

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As general, we study the properties and calculate the number of regular elements of $B_X(D)$ satisfying $V(D, \alpha) = Q'$ where $Q'$ is a semilattice isomorphic to $Q$. So, we characterize the class for each element of which is isomorphic to $Q$ by means of the characteristic family of sets, the characteristic mapping and the generate set of $D$.

2. Preliminaries

**Theorem 2.1.** [9, Theorem 10] Let $\alpha$ and $\sigma$ be binary relations of the semigroup $B_X(D)$ such that $\alpha \circ \sigma \circ \alpha = \alpha$. If $D(\alpha)$ is some generating set of the semilattice $V(D, \alpha) \setminus \{\emptyset\}$ and $\alpha = \bigcup_{T \in V(D, \alpha)} (Y_\alpha^T \times T)$ is a quasinormal representation of the relation $\alpha$, then $V(D, \alpha)$ is a complete $XI-$ semilattice of unions. Moreover, there exists a complete $\alpha-$isomorphism $\varphi$ between the semilattice $V(D, \alpha)$ and $D' = \{T \sigma \mid T \in V(D, \alpha)\}$, that satisfies the following conditions:

a) $\varphi(T) = T\sigma$ and $\varphi(T)\alpha = T$ for all $T \in V(D, \alpha)$

b) $\bigcup_{T' \in \tilde{D}(\alpha)_T} Y_{T'}^\alpha \supseteq \varphi(T)$ for any $T \in D(\alpha)$,

c) $Y_T^\alpha \cap \varphi(T) \neq \emptyset$ for all nonlimiting element $T$ of the set $\tilde{D}(\alpha)_T$,

d) If $T$ is a limiting element of the set $\tilde{D}(\alpha)_T$, then the equality $\cup B(T) = T$

is always holds for the set $B(T) = \{Z \in \tilde{D}(\alpha)_T \mid Y_Z^\alpha \cap \varphi(T) \neq \emptyset\}$.

On the other hand, if $\alpha \in B_X(D)$ such that $V(D, \alpha)$ is a complete $XI-$semilattice of unions. If for a complete $\alpha-$isomorphism $\varphi$ from $V(D, \alpha)$ to a subsemilattice $D'$ of $D$ satisfies the conditions b) – d) of the theorem, then $\alpha$ is a regular element of $B_X(D)$.

**Theorem 2.2.** [7, Theorem 1.18.2] Let $D_j = \{T_1, \ldots, T_j\}$, $X$ be finite set and $\emptyset \neq Y \subseteq X$. If $f$ is a mapping of the set $X$, on the $D_j$, for which $f(y) = T_j$ for some $y \in Y$, then the numbers of those mappings $f$ of the sets $X$ on the set $D_j$ can be calculated by the formula $s = j^{\left|X \setminus Y\right|} \cdot \left(j^{|Y|} - (j - 1)^{|Y|}\right)$.

**Theorem 2.3.** [7, Theorem 6.3.5] Let $X$ is a finite set. If $\varphi$ is a fixed element of the set $\Phi(D, D')$ and $\left|\Omega(D)\right| = m_0$ and $q$ is a number of all automorphisms of the semilattice $D$ then $|R(D')| = m_0 \cdot q \cdot |R_{\varphi}(D, D')|$.
3. Results

Let $X$ be a finite set, $D$ be a complete $X$–semilattice of unions and $Q = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8\}$ be $X$–subsemilattice of unions of $D$ satisfies the following conditions

\[
T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_6 \subset T_8, \quad T_1 \subset T_2 \subset T_3 \subset T_5 \subset T_7 \subset T_8,
\]

\[
T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_6 \subset T_8, \quad T_1 \subset T_2 \subset T_4 \subset T_5 \subset T_7 \subset T_8,
\]

\[
T_1 \setminus T_3 \neq \emptyset, \quad T_3 \setminus T_4 \neq \emptyset, \quad T_6 \setminus T_7 \neq \emptyset, \quad T_7 \setminus T_6 \neq \emptyset,
\]

\[
T_3 \cup T_4 = T_5, T_6 \cup T_7 = T_8, \quad T_1 \neq \emptyset.
\]

The diagram of the $Q$ is shown in Figure 3.1. Let $C(Q) = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ is characteristic family of sets of $Q$ and $\theta : Q \rightarrow C(Q)$, $\theta(T_i) = P_i$ ($i = 1, 2, \ldots, 8$) is characteristic mapping.

Then, by using properties of characteristic family and characteristic mapping for each element $T_i \in Q$ we can write

\[
T_i = \theta(\tilde{Q}) \cup \bigcup_{T \in \tilde{Q}(T_i)} \theta(T), (i = 1, 2, \ldots, 8)
\]

where $\tilde{Q}(T_i) = Q \setminus \{Z \in Q \mid T_i \subseteq Z\}$, $\tilde{Q} = \cup Q = T_8$ and $\theta(\tilde{Q}) = \theta(T_8) = P_8$.

Hence,

\[
T_8 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_8)} \theta(T) = P_8 \cup P_7 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1,
\]

\[
T_7 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_7)} \theta(T) = P_8 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1,
\]

\[
T_6 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_6)} \theta(T) = P_8 \cup P_7 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1,
\]

\[
T_5 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_5)} \theta(T) = P_8 \cup P_4 \cup P_3 \cup P_2 \cup P_1,
\]

\[
T_4 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_4)} \theta(T) = P_8 \cup P_3 \cup P_2 \cup P_1,
\]

\[
T_3 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_3)} \theta(T) = P_8 \cup P_4 \cup P_2 \cup P_1,
\]

\[
T_2 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_2)} \theta(T) = P_8 \cup P_1,
\]

\[
T_1 = P_8 \cup \bigcup_{T \in \tilde{Q}(T_1)} \theta(T) = P_8 \cup \emptyset = P_8
\]
are obtained.

**Lemma 3.1.** $Q$ is $XI-$ semilattice of unions.

**Proof.** Let us show that the conditions of definition of $XI-$ semilattice of unions hold. First, let determine the greatest lower bounds of the each semilattice $Q_t$ in $Q$ for $t \in T_8$. Since $T_8 = P_8 \cup P_7 \cup P_6 \cup P_5 \cup P_4 \cup P_3 \cup P_2 \cup P_1$ and $P_i$ ($i = 1, 2, \ldots, 8$) are pairwise disjoint sets, by Equation (3.1) and the definition of $Q_t$, we get

$$Q_t = \begin{cases} Q, & t \in P_8 \\ \{T_8, T_6\}, & t \in P_7 \\ \{T_8, T_7\}, & t \in P_6 \\ \{T_8, T_7, T_6\}, & t \in P_5 \\ \{T_8, T_7, T_6, T_5, T_3\}, & t \in P_4 \\ \{T_8, T_7, T_6, T_5, T_4\}, & t \in P_3 \\ \{T_8, T_7, T_6, T_5, T_4, T_3\}, & t \in P_2 \\ \{T_8, T_7, T_6, T_5, T_4, T_3, T_2\}, & t \in P_1 \end{cases} \quad (3.2)$$

By using Equation (3.2) and the definition of $N(Q, Q_t)$, we get

$$N(Q, Q_t) = \begin{cases} \{T_1\}, & t \in P_8 \\ \{T_1, T_2, T_3, T_4, T_5, T_6\}, & t \in P_7 \\ \{T_1, T_2, T_3, T_4, T_5, T_7\}, & t \in P_6 \\ \{T_1, T_2, T_3, T_4, T_5\}, & t \in P_5 \\ \{T_1, T_2, T_3\}, & t \in P_4 \\ \{T_1, T_2, T_4\}, & t \in P_3 \\ \{T_1, T_2\}, & t \in P_2 \\ \{T_1, T_2\}, & t \in P_1 \end{cases} \quad (3.3)$$

From the Equation (3.3) the greatest lower bounds for each semilattice $Q_t$

$$\cup N(Q, Q_t) = \Lambda(Q, Q_t) = \begin{cases} T_1, & t \in P_8 \\ T_6, & t \in P_7 \\ T_7, & t \in P_6 \\ T_5, & t \in P_5 \\ T_3, & t \in P_4 \\ T_4, & t \in P_3 \\ T_2, & t \in P_2 \\ T_2, & t \in P_1 \end{cases} \quad (3.4)$$
are obtained. So, we get \( \Lambda(D, D_t) \in D \) for any \( t \in T_8 \). Now using the Equation (3.4), we have

\[
\begin{align*}
t \in T_1 &= P_8 \Rightarrow T_1 = \Lambda(Q, Q_t), \\
t \in T_2 &= P_8 \cup P_1 \Rightarrow t \in P_8 \text{ or } t \in P_1 \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2\} \\
  &\Rightarrow T_2 = T_1 \cup T_2 = \bigcup_{t \in T_2} \Lambda(Q, Q_t), \\
t \in T_3 &= P_8 \cup P_4 \cup P_2 \cup P_1 \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2, T_3\} \\
  &\Rightarrow T_3 = T_1 \cup T_2 \cup T_3 = \bigcup_{t \in T_3} \Lambda(Q, Q_t), \\
t \in T_4 &= P_8 \cup P_3 \cup P_2 \cup P_1 \Rightarrow \Lambda(Q, Q_t) \in \{T_1, T_2, T_4\} \\
  &\Rightarrow T_4 = T_1 \cup T_2 \cup T_4 = \bigcup_{t \in T_4} \Lambda(Q, Q_t), \\
t \in T_5 &= P_8 \cup P_4 \cup P_3 \cup P_2 \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4\} \\
  &\Rightarrow T_5 = T_1 \cup T_2 \cup T_3 \cup T_4 = \bigcup_{t \in T_5} \Lambda(Q, Q_t), \\
t \in T_6 &= P_8 \cup P_7 \cup P_5 \cup \ldots \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_6\} \\
  &\Rightarrow T_6 = T_1 \cup \ldots \cup T_6 = \bigcup_{t \in T_6} \Lambda(Q, Q_t), \\
t \in T_7 &= P_8 \cup P_6 \cup P_5 \cup \ldots \cup P_1 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\} \\
  &\Rightarrow T_7 = T_1 \cup \ldots \cup T_5 \cup T_7 = \bigcup_{t \in T_7} \Lambda(Q, Q_t), \\
t \in T_8 &= T_7 \cup T_6 \Rightarrow \Lambda(Q, Q_t) = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\} \\
  &\Rightarrow T_8 = T_6 \cup T_7 = \bigcup_{t \in T_8} \Lambda(Q, Q_t).
\end{align*}
\]

Then \( Q \) is a \( XI \)-semilattice of unions. \( \square \)

**Lemma 3.2.** Following equalities are true for \( Q \) where \( P_i \)'s are pairwise disjoint sets and union of these sets equals \( Q \).

\[
\begin{align*}
P_1 &= T_2 \setminus T_1, \quad P_2 = (T_4 \cap T_3) \setminus T_2, \quad P_3 = T_4 \setminus T_3, \quad P_4 = T_3 \setminus T_4, \\
P_5 &= (T_7 \cap T_6) \setminus T_5, \quad P_6 = T_7 \setminus T_6, \quad P_7 = T_6 \setminus T_7, \quad P_8 = T_1.
\end{align*}
\]

**Proof.** Considering the (3.1), it is easy to see that equalities are true. \( \square \)

**Lemma 3.3.** Let \( G = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\} \) be a generating set of \( Q \). Then the elements \( T_1, T_2, T_3, T_4, T_6, T_7 \) are nonlimiting elements of the set \( \tilde{G}_{T_1}, \tilde{G}_{T_2}, \tilde{G}_{T_3}, \tilde{G}_{T_4}, \tilde{G}_{T_6}, \tilde{G}_{T_7} \) respectively and \( T_5 \) is limiting element of the set \( \tilde{G}_{T_5} \).
Proof. Definition of $D'_T$, following equations

$$\begin{align*}
\tilde{G}_{T_1} &= \{T_1\}, \\
\tilde{G}_{T_2} &= \{T_1, T_2\}, \\
\tilde{G}_{T_3} &= \{T_1, T_2, T_3\}, \\
\tilde{G}_{T_4} &= \{T_1, T_2, T_4\}, \\
\tilde{G}_{T_5} &= \{T_1, T_2, T_3, T_4, T_5\}, \\
\tilde{G}_{T_6} &= \{T_1, T_2, T_3, T_4, T_5, T_6\}, \\
\tilde{G}_{T_7} &= \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}.
\end{align*}$$

(3.5)

are obtained. Now we get the sets $l(\tilde{G}_{T_i}, T_i), i \in \{1, 2, \ldots, 7\}$,

$$\begin{align*}
l(\tilde{G}_{T_1}, T_1) &= \bigcup(\tilde{G}_{T_1} \setminus \{T_1\}) = \emptyset, \\
l(\tilde{G}_{T_2}, T_2) &= \bigcup(\tilde{G}_{T_2} \setminus \{T_2\}) = T_1, \\
l(\tilde{G}_{T_3}, T_3) &= \bigcup(\tilde{G}_{T_3} \setminus \{T_3\}) = T_2, \\
l(\tilde{G}_{T_4}, T_4) &= \bigcup(\tilde{G}_{T_4} \setminus \{T_4\}) = T_2, \\
l(\tilde{G}_{T_5}, T_5) &= \bigcup(\tilde{G}_{T_5} \setminus \{T_5\}) = T_5, \\
l(\tilde{G}_{T_6}, T_6) &= \bigcup(\tilde{G}_{T_6} \setminus \{T_6\}) = T_5, \\
l(\tilde{G}_{T_7}, T_7) &= \bigcup(\tilde{G}_{T_7} \setminus \{T_7\}) = T_5.
\end{align*}$$

Then we find nonlimiting and limiting elements of $\tilde{G}_{T_i}, i \in \{1, 2, \ldots, 7\}$.

$$\begin{align*}
T_1 \setminus l(\tilde{G}_{T_1}, T_1) &= T_1 \setminus \emptyset = T_1 \neq \emptyset, \quad T_1 \text{ nonlimiting element} \\
T_2 \setminus l(\tilde{G}_{T_2}, T_2) &= T_2 \setminus T_1 \neq \emptyset, \quad T_2 \text{ nonlimiting element} \\
T_3 \setminus l(\tilde{G}_{T_3}, T_3) &= T_3 \setminus T_2 \neq \emptyset, \quad T_3 \text{ nonlimiting element} \\
T_4 \setminus l(\tilde{G}_{T_4}, T_4) &= T_4 \setminus T_2 \neq \emptyset, \quad T_4 \text{ nonlimiting element} \\
T_5 \setminus l(\tilde{G}_{T_5}, T_5) &= T_5 \setminus T_5 = \emptyset, \quad T_5 \text{ limiting element} \\
T_6 \setminus l(\tilde{G}_{T_6}, T_6) &= T_6 \setminus T_5 \neq \emptyset, \quad T_6 \text{ nonlimiting element} \\
T_7 \setminus l(\tilde{G}_{T_7}, T_7) &= T_7 \setminus T_5 \neq \emptyset, \quad T_7 \text{ nonlimiting element}
\end{align*}$$

Therefore, the elements $T_1, T_2, T_3, T_4, T_6, T_7$ are nonlimiting elements of the sets $\tilde{G}_{T_1}, \tilde{G}_{T_2}, \tilde{G}_{T_3}, \tilde{G}_{T_4}, \tilde{G}_{T_6}, \tilde{G}_{T_7}$, respectively and $T_5$ is limiting element of the set $\tilde{G}_{T_5}$. 

Now, we determine properties of a regular element $\alpha$ of $B_X(Q)$ where $V(D, \alpha) = Q$ and $\alpha = \bigcup_{i=1}^{8} (Y_i^\alpha \times T_i)$.

**Theorem 3.4.** Let $\alpha \in B_X(Q)$ be a quasinormal representation of the form $\alpha = \bigcup_{i=1}^{8} (Y_i^\alpha \times T_i)$ such that $V(D, \alpha) = Q$. $\alpha \in B_X(D)$ is a regular iff for
some complete $\alpha$-isomorphism $\varphi : Q \to D' \subseteq D$, the following conditions are satisfied:

\begin{align*}
Y_1^\alpha &\supseteq \varphi(T_1), \\
Y_1^\alpha \cup Y_2^\alpha &\supseteq \varphi(T_2), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha &\supseteq \varphi(T_3), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha &\supseteq \varphi(T_4), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha &\supseteq \varphi(T_5), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha &\supseteq \varphi(T_6), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha &\supseteq \varphi(T_7), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha &\supseteq \varphi(T_7).
\end{align*}

(3.6)

Proof. Let $G = \{T_1, T_2, T_3, T_4, T_5, T_6, T_7\}$ be a generating set of $Q$.

$\Rightarrow$: Since $\alpha \in B_X(D)$ is regular and $V(D, \alpha) = Q X I$–semi-lattice of unions, by Theorem 2.1, there exists a complete isomorphism $\varphi : Q \to D'$. By Theorem 2.1 (a), satisfying $\varphi(T) \alpha = T$ for all $T \in V(D, \alpha)$. So, $\varphi$ is complete $\alpha$-isomorphism. Applying the Theorem 2.1 (b) we have

\begin{align*}
Y_1^\alpha &\supseteq \varphi(T_1), \\
Y_1^\alpha \cup Y_2^\alpha &\supseteq \varphi(T_2), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha &\supseteq \varphi(T_3), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha &\supseteq \varphi(T_4), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha &\supseteq \varphi(T_5), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha &\supseteq \varphi(T_6), \\
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha &\supseteq \varphi(T_7).
\end{align*}

(3.7)

Moreover, considering that the elements $T_1, T_2, T_3, T_4, T_5, T_6, T_7$ are nonlimiting and using the Theorem 2.1 (c), following properties

\begin{align*}
Y_1^\alpha \cap \varphi(T_1) &\neq \emptyset, \\
Y_2^\alpha \cap \varphi(T_2) &\neq \emptyset, \\
Y_3^\alpha \cap \varphi(T_3) &\neq \emptyset, \\
Y_4^\alpha \cap \varphi(T_4) &\neq \emptyset, \\
Y_5^\alpha \cap \varphi(T_5) &\neq \emptyset, \\
Y_6^\alpha \cap \varphi(T_6) &\neq \emptyset, \\
Y_7^\alpha \cap \varphi(T_7) &\neq \emptyset.
\end{align*}

(3.8)

are obtained. From $Y_1^\alpha \supseteq \varphi(T_1)$, $Y_2^\alpha \cap \varphi(T_1) \neq \emptyset$ always ensured. Also by using $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \supseteq \varphi(T_3)$ and $Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \supseteq \varphi(T_4)$, we get

\begin{align*}
Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha &\supseteq (Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha) \cup (Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha) \\
&\supseteq \varphi(T_3) \cup \varphi(T_4) \cup Y_5^\alpha \\
&= \varphi(T_5) \cup Y_5^\alpha \\
&\supseteq \varphi(T_5).
\end{align*}

Thus there is no need the condition $Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \supseteq \varphi(T_5)$. Therefore there exist an $\alpha$–isomorphism $\varphi$ which holds given conditions.
\[\Leftrightarrow: \text{Since } V(D, \alpha) = Q, V(D, \alpha) \text{ is } XI-\text{semilattice of unions. Let } \varphi : Q \rightarrow D' \subseteq D \text{ be complete } \alpha-\text{isomorphism which holds given conditions. So, considering Equation (3.6), satisfying Theorem 2.1 (a) -- (c). Remembering that } T_5 \text{ is a limiting element of the set } \hat{G}_{T_5}, \text{ we constitute the set } B(T_5) = \left\{ Z \in \hat{G}_{T_5} \mid Y^\alpha_Z \cap \varphi(T_5) \neq \emptyset \right\}. \text{ If } Y^\alpha_4 \cap \varphi(T_5) = \emptyset \text{ we have}
\]
\[
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \cup Y^\alpha_4 = (Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3) \cup (Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_4) \\
\geq \varphi(T_3) \cup \varphi(T_4) = \varphi(T_5)
\]
So we get \[Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \supseteq \varphi(T_5) \supseteq \varphi(T_4)\] which is a contradiction with \[Y^\alpha_4 \cap \varphi(T_4) \neq \emptyset.\] Therefore \(T_4 \in B(T_5)\). If \(Y^\alpha_3 \cap \varphi(T_5) = \emptyset\) we have
\[
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \cup Y^\alpha_4 = (Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3) \cup (Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_4) \\
\geq \varphi(T_3) \cup \varphi(T_4) = \varphi(T_5)
\]
So we get \[Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \supseteq \varphi(T_5) \supseteq \varphi(T_4)\] which is a contradiction with \[Y^\alpha_3 \cap \varphi(T_3) \neq \emptyset.\] Therefore \(T_3 \in B(T_5)\). We have \(\cup B(T_5) = T_3 \cup T_4 = T_5\). By Theorem 2.1, we conclude that \(\alpha\) is the regular element of the \(B_X(D)\).

Now we calculate the number of regular elements \(\alpha\), satisfying the hypothesis of Theorem 3.4. Let \(\alpha \in B_X(D)\) be a regular element which is quasinormal representation of the form \(\alpha = \bigcup_{i=1}^{8} (Y^\alpha_i \times T_i)\) and \(V(D, \alpha) = Q\). Then there exist a complete \(\alpha-\text{isomorphism } \varphi : Q \rightarrow D' = \{\varphi(T_1), \varphi(T_2), \ldots, \varphi(T_8)\}\) satisfying the hypothesis of Theorem 3.4. So, \(\alpha \in R_{\varphi}(Q, D')\). We will denote \(\varphi(T_i) = \overline{T_i}, i = 1, 2, \ldots, 8\). Diagram of the \(D' = \{\overline{T_1}, \overline{T_2}, \overline{T_3}, \overline{T_4}, \overline{T_5}, \overline{T_6}, \overline{T_7}, \overline{T_8}\}\) is shown in Figure 3.2. Then the Equation (3.6) reduced to below equation.

\[
\begin{align*}
Y^\alpha_1 \supseteq & \overline{T_1} \\
Y^\alpha_1 \cup Y^\alpha_2 \supseteq & \overline{T_2} \\
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \supseteq & \overline{T_3} \\
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_4 \supseteq & \overline{T_4} \\
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \cup Y^\alpha_4 \cup Y^\alpha_5 \cup Y^\alpha_6 \supseteq & \overline{T_6}, \\
Y^\alpha_1 \cup Y^\alpha_2 \cup Y^\alpha_3 \cup Y^\alpha_4 \cup Y^\alpha_5 \cup Y^\alpha_7 \supseteq & \overline{T_7} \\
Y^\alpha_1 \cap \varphi(T_2) \neq & \emptyset, Y^\alpha_3 \cap \varphi(T_3) \neq \emptyset, \\
Y^\alpha_4 \cap \varphi(T_4) \neq & \emptyset, Y^\alpha_5 \cap \varphi(T_5) \neq \emptyset, \\
Y^\alpha_6 \cap \varphi(T_6) \neq & \emptyset, Y^\alpha_7 \cap \varphi(T_7) \neq \emptyset.
\end{align*}
\]

(3.9)

On the other hand, the image of the sets in Lemma 3.2 under the \(\alpha-\text{isomorphism } \varphi\)

\[
\overline{T_1}, (\overline{T_3} \cap \overline{T_4}) \setminus \overline{T_1}, \overline{T_4} \setminus \overline{T_3}, \overline{T_5} \setminus \overline{T_4}, (\overline{T_7} \cap \overline{T_6}) \setminus \overline{T_5}, \overline{T_7} \setminus \overline{T_6}, \overline{T_6} \setminus \overline{T_7}, X \setminus \overline{T_8}
\]
are also pairwise disjoint sets and union of these sets equals $X$.

**Lemma 3.5.** For every $\alpha \in R_\varphi(Q, D')$, there exists an ordered system of disjoint mappings which is defined \{$(T_1, (T_3 \cap T_4) \setminus T_1, T_4 \setminus T_3, T_3 \setminus T_4, (T_7 \cap T_6) \setminus T_5, T_7 \setminus T_6, T_6 \setminus T_7, X \setminus T_8)$\). Also, ordered systems are different which correspond to different binary relations.

**Proof.** Let $f_\alpha : X \to D$ be a mapping satisfying the condition $f_\alpha(t) = t\alpha$ for all $t \in X$. We consider the restrictions of the mapping $f_\alpha$ as $f_1\alpha$, $f_2\alpha$, $f_3\alpha$, $f_4\alpha$, $f_5\alpha$, $f_6\alpha$, $f_7\alpha$, $f_8\alpha$ on the sets $T_1, (T_3 \cap T_4) \setminus T_1, T_4 \setminus T_3, T_3 \setminus T_4, (T_7 \cap T_6) \setminus T_5, T_7 \setminus T_6, T_6 \setminus T_7, X \setminus T_8$ respectively.

Now, considering the definition of the sets $Y_\alpha^i$, $i = 1, 2, \ldots, 8$, together with the Equation (3.9) we have

\[ t \in T_1 \Rightarrow t \in Y_1^\alpha \Rightarrow t\alpha = T_1 \Rightarrow f_1\alpha(t) = T_1, \forall t \in T_1. \]
\[ t \in (T_3 \cap T_4) \setminus T_1 \Rightarrow t \in (T_3 \cap T_4) \subseteq Y_1^\alpha \cup Y_2^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2\} \]
\[ \Rightarrow f_2\alpha(t) \in \{T_1, T_2\}, \forall t \in (T_3 \cap T_4) \setminus T_1. \]

Since $Y_2^\alpha \cap \overline{T}_2 \neq \emptyset$, there is an element $t_2 \in Y_2^\alpha \cap \overline{T}_2$. Then $t_2\alpha = T_2$ and $t_2 \in \overline{T}_2$. If $t_2 \in T_1$ then $t_2 \in \overline{T}_1 \subseteq Y_1^\alpha$. Therefore, $t_2\alpha = T_2$ which is in contradiction with the equality $t_2\alpha = T_2$. So $f_2\alpha(t_2) = T_2$ for some $t_2 \in \overline{T}_2 \setminus T_1$.

\[ t \in \overline{T}_4 \setminus T_3 \Rightarrow t \in \overline{T}_4 \setminus T_3 \subseteq \overline{T}_4 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2, T_4\} \]
\[ \Rightarrow f_3\alpha(t) \in \{T_1, T_2, T_4\}, \forall t \in \overline{T}_4 \setminus T_3. \]

$Y_4^\alpha \cap \overline{T}_4 \neq \emptyset$ so there is an element $t_4 \in Y_4^\alpha \cap \overline{T}_4$. Then $t_4\alpha = T_4$ and $t_4 \in \overline{T}_4$. If $t_4 \in \overline{T}_3$ then $t_4 \in \overline{T}_3 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha$. Thus $t_4\alpha \in \{T_1, T_2, T_3\}$ which is in contradiction with the equality $t_4\alpha = T_4$. So there is an element $t_4 \in \overline{T}_4 \setminus T_3$ with $f_3\alpha(t_4) = T_4$.

\[ t \in T_3 \setminus T_4 \Rightarrow t \in T_3 \setminus T_4 \subseteq T_3 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2, T_3\} \]
\[ \Rightarrow f_4\alpha(t) \in \{T_1, T_2, T_3\}, \forall t \in T_3 \setminus T_4. \]

Since $Y_3^\alpha \cap \overline{T}_3 \neq \emptyset$, there is an element $t_3$ with $t_3\alpha = T_3$ and $t_3 \in \overline{T}_3$. If $t_3 \in \overline{T}_4$ then $t_3 \in \overline{T}_4 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_4^\alpha$. Therefore, $t_3\alpha \in \{T_1, T_2, T_4\}$ which contradicts to the equality $t_3\alpha = T_3$. So there is an element $t_3 \in \overline{T}_3 \setminus T_4$ with $f_4\alpha(t_3) = T_3$.

\[ t \in (T_7 \cap T_6) \setminus T_5 \Rightarrow t \in (T_7 \cap T_6) \setminus T_5 \subseteq T_7 \cap T_6 \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2, T_3, T_4, T_5\} \]
\[ \Rightarrow f_5\alpha(t) \in \{T_1, T_2, T_3, T_4, T_5\}, \forall t \in (T_7 \cap T_6) \setminus T_5. \]
\[ t \in \overline{T_7} \setminus \overline{T_6} \Rightarrow t \in \overline{T_7} \setminus \overline{T_6} \subseteq \overline{T_7} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_7^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2, T_3, T_4, T_5, T_7\} \]
\[ \Rightarrow f_{6\alpha}(t) \in \{T_1, T_2, T_3, T_4, T_5, T_7\}, \forall t \in \overline{T_7} \setminus \overline{T_6}. \]

Also, there is an element \( t_7 \in Y_7^\alpha \cap \overline{T_7} \) since \( Y_7^\alpha \cap \overline{T_7} \neq \emptyset \). Then \( t_7\alpha = T_7 \) and \( t_7 \in \overline{T_7} \). If \( t_7 \in \overline{T_6} \) then \( t_7 \in \overline{T_6} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \). So \( t_7\alpha \in \{T_1, T_2, T_3, T_4, T_5, T_6\} \). However this contradicts to \( t_7\alpha = T_7 \). So \( f_{6\alpha}(t_7) = T_7 \) for some \( t_7 \in \overline{T_7} \setminus \overline{T_6} \).

\[ t \in \overline{T_6} \setminus \overline{T_7} \Rightarrow t \in \overline{T_6} \setminus \overline{T_7} \subseteq \overline{T_6} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \]
\[ \Rightarrow t\alpha \in \{T_1, T_2, T_3, T_4, T_5, T_6\} \]
\[ \Rightarrow f_{7\alpha}(t) \in \{T_1, T_2, T_3, T_4, T_5, T_6\}, \forall t \in \overline{T_6} \setminus \overline{T_7}. \]

Similarly there is an element \( t_6 \) with \( t_6\alpha = T_6 \) and \( t_6 \in \overline{T_6} \) since \( Y_6^\alpha \cap \overline{T_6} \neq \emptyset \). If \( t_6 \in \overline{T_7} \) then \( t_6 \in \overline{T_7} \subseteq Y_1^\alpha \cup Y_2^\alpha \cup Y_3^\alpha \cup Y_4^\alpha \cup Y_5^\alpha \cup Y_6^\alpha \). Therefore, \( t_6\alpha \in \{T_1, T_2, T_3, T_4, T_5, T_7\} \) which is in contradiction with the equality \( t_6\alpha = T_6 \). So \( f_{7\alpha}(t_6) = T_6 \) for some \( t_6 \in \overline{T_6} \setminus \overline{T_7} \).

\[ t \in X \setminus \overline{T_8} \Rightarrow t \in X \setminus \overline{T_8} \subseteq X = \bigcup_{i=1}^8 Y_i^\alpha \Rightarrow t\alpha \in Q \Rightarrow f_{8\alpha}(t) \in Q, \forall t \in X \setminus \overline{T_8}. \]

Therefore, for every binary relation \( \alpha \in R_\varphi(Q, D') \) there exists an ordered system \((f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha}, f_{7\alpha}, f_{8\alpha})\).

On the other hand, suppose that for \( \alpha, \beta \in R_\varphi(Q, D') \) which \( \alpha \neq \beta \), be obtained \( f_\alpha = (f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha}, f_{7\alpha}, f_{8\alpha}) \) and \( f_\beta = (f_{1\beta}, f_{2\beta}, f_{3\beta}, f_{4\beta}, f_{5\beta}, f_{6\beta}, f_{7\beta}, f_{8\beta}) \). If \( f_\alpha = f_\beta \), we get

\[ f_\alpha = f_\beta \Rightarrow f_\alpha(t) = f_\beta(t), \forall t \in X \Rightarrow t\alpha = t\beta, \forall t \in X \Rightarrow \alpha = \beta \]

which contradicts to \( \alpha \neq \beta \). Therefore different binary relations’ ordered systems are different. \( \square \)

**Lemma 3.6.** Let \( f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8) \) be ordered system from
Then $\beta = \bigcup_{x \in X} (\{x\} \times f(x)) \in B_X(D)$ is regular and $\varphi$ is complete $\beta$–isomorphism $\theta$. So $\beta \in R_\varphi(Q, D')$.

Proof. First we see that $V(D, \beta) = Q$. Considering $V(D, \beta) = \{Y \beta \mid Y \in D\}$, the properties of $f$ mapping, $T_i \beta = \bigcup_{x \in T_i} x \beta$ and $D' \subseteq D$, we get

\begin{align*}
T_1 \in Q & \Rightarrow T_1 \beta = T_1 \Rightarrow T_1 \in V(D, \beta), \\
T_2 \in Q & \Rightarrow T_2 \beta = T_1 \cup T_2 \Rightarrow T_2 \in V(D, \beta), \\
T_3 \in Q & \Rightarrow T_3 \beta = T_1 \cup T_2 \cup T_3 = T_3 \Rightarrow T_3 \in V(D, \beta), \\
T_4 \in Q & \Rightarrow T_4 \beta = T_1 \cup T_2 \cup T_3 \cup T_4 = T_4 \Rightarrow T_4 \in V(D, \beta), \\
T_5 \in Q & \Rightarrow T_5 \beta = (T_3 \cup T_4) \beta = T_3 \cup T_4 = T_5 \Rightarrow T_5 \in V(D, \beta), \\
T_6 \in Q & \Rightarrow T_6 \beta = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_6 = T_6 \Rightarrow T_6 \in V(D, \beta), \\
T_7 \in Q & \Rightarrow T_7 \beta = T_1 \cup T_2 \cup T_3 \cup T_4 \cup T_5 \cup T_7 = T_7 \Rightarrow T_7 \in V(D, \beta), \\
T_8 \in Q & \Rightarrow T_8 \beta = (T_6 \cup T_7) \beta = T_6 \cup T_7 = T_8 \Rightarrow T_8 \in V(D, \beta).
\end{align*}

Then $Q \subseteq V(D, \beta)$. Also,

\begin{align*}
Z \in V(D, \beta) & \Rightarrow Z = Y \beta, \exists Y \in D \\
& \Rightarrow Z = Y \beta = \bigcup_{y \in Y} y \beta = \bigcup_{y \in Y} f(y) \in Q
\end{align*}

since $f(y) \in Q$ and $Q$ is closed set-theoretic union. Therefore, $V(D, \beta) \subseteq Q$. Hence $V(D, \beta) = Q$.

Also, $\beta = \bigcup_{T \in V(X^*, \beta)} \left( Y_T^\beta \times T \right)$ is quasinormal representation of $\beta$ since $\emptyset \notin Q$. From the definition of $\beta$, $f(x) = x \beta$ for all $x \in X$. It is easily seen that
\[ V(X^*, \beta) = V(D, \beta) = Q. \] We get \( \beta = \bigcup_{i=1}^{8} (Y_i^{\beta} \times T_i) \).

On the other hand
\[
t \in \overline{T}_1 \Rightarrow t\beta = f(t) = T_1 \Rightarrow t \in Y_1^{\beta} \Rightarrow \overline{T}_1 \subseteq Y_1^{\beta},
\]
\[
t \in \overline{T}_2 = \overline{T}_1 \cup ((\overline{T}_3 \cap \overline{T}_4) \setminus \overline{T}_1) \Rightarrow t\beta = f(t) \in \{T_1, T_2\} \Rightarrow t \in Y_1^{\beta} \cup Y_2^{\beta} \Rightarrow \overline{T}_2 \subseteq Y_1^{\beta} \cup Y_2^{\beta}
\]
\[
t \in \overline{T}_3 = \overline{T}_1 \cup ((\overline{T}_3 \cap \overline{T}_4) \setminus \overline{T}_1) \cup (\overline{T}_3 \setminus \overline{T}_4) \Rightarrow t\beta = f(t) \in \{T_1, T_2, T_3\} \Rightarrow t \in Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta} \Rightarrow \overline{T}_4 \subseteq Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta},
\]
\[
t \in \overline{T}_4 = \overline{T}_1 \cup ((\overline{T}_3 \cap \overline{T}_4) \setminus \overline{T}_1) \cup (\overline{T}_4 \setminus \overline{T}_3) \Rightarrow t\beta = f(t) \in \{T_1, T_2, T_4\} \Rightarrow t \in Y_1^{\beta} \cup Y_2^{\beta} \cup Y_4^{\beta} \Rightarrow \overline{T}_4 \subseteq Y_1^{\beta} \cup Y_2^{\beta} \cup Y_4^{\beta},
\]
\[
t \in \overline{T}_6 = (\overline{T}_6 \setminus \overline{T}_7) \cup ((\overline{T}_7 \setminus \overline{T}_6) \setminus \overline{T}_5) \cup \overline{T}_3 \cup \overline{T}_4 \Rightarrow t\beta = f(t) \in \{T_1, T_2, T_3, T_4, T_5, T_6\} \Rightarrow t \in Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta} \cup Y_4^{\beta} \cup Y_5^{\beta} \cup Y_6^{\beta} \Rightarrow \overline{T}_6 \subseteq Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta} \cup Y_4^{\beta} \cup Y_5^{\beta} \cup Y_6^{\beta},
\]
\[
t \in \overline{T}_7 = (\overline{T}_7 \setminus \overline{T}_6) \cup ((\overline{T}_7 \setminus \overline{T}_6) \setminus \overline{T}_5) \cup \overline{T}_3 \cup \overline{T}_4 \Rightarrow t\beta = f(t) \in \{T_1, T_2, T_3, T_4, T_5, T_7\} \Rightarrow t \in Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta} \cup Y_4^{\beta} \cup Y_5^{\beta} \cup Y_7^{\beta} \Rightarrow \overline{T}_6 \subseteq Y_1^{\beta} \cup Y_2^{\beta} \cup Y_3^{\beta} \cup Y_4^{\beta} \cup Y_5^{\beta} \cup Y_7^{\beta},
\]

Also, by using \( f_2(t_2) = T_2, \ \exists t_2 \in \overline{T}_2 \setminus \overline{T}_1, \) we obtain \( Y_2^{\beta} \cap \overline{T}_2 \neq \emptyset. \) Similarly, from properties of \( f_3, f_4, f_6, f_7, \) be seen \( Y_3^{\beta} \cap \overline{T}_3 \neq \emptyset, Y_4^{\beta} \cap \overline{T}_4 \neq \emptyset, Y_6^{\beta} \cap \overline{T}_6 \neq \emptyset \) and \( Y_7^{\beta} \cap \overline{T}_7 \neq \emptyset. \) Therefore the mapping \( \varphi : Q \rightarrow D' = \{\overline{T}_1, \overline{T}_2, \ldots, \overline{T}_8\} \) to be defined \( \varphi(T_i) = \overline{T}_i \) satisfy the conditions in (3.9) for \( \beta. \) Hence \( \varphi \) is complete \( \beta- \)isomorphism because of \( \varphi(T) \beta = \overline{T}_\beta = T, \) for all \( T \in V(D, \beta). \) By Theorem 3.4, \( \beta \in R_\varphi(Q, D'). \)

Therefore, there is one to one correspondence between the elements of \( R_\varphi(Q, D') \) and the set of ordered systems of disjoint mappings.

**Theorem 3.7.** Let \( X \) be a finite set and \( Q \) be XI- semilattice. If
\[
D' = \{\overline{T}_1, \overline{T}_2, \overline{T}_3, \overline{T}_4, \overline{T}_5, \overline{T}_6, \overline{T}_7, \overline{T}_8\}
\]
is \( \alpha- \) isomorphic to \( Q \) and \( \Omega(Q) = m_0, \) then
\[
|R(D')| = m_0 \cdot 4 \cdot (2|\overline{T}_3 \setminus \overline{T}_4| - |\overline{T}_2 \setminus \overline{T}_1| - 1) \cdot (3|\overline{T}_4 \setminus \overline{T}_3| - 2|\overline{T}_4 \setminus \overline{T}_3|)
\]
\[ \cdot \left( 3 | T_3 \setminus T_4 | - 2 | T_3 \setminus T_4 | \right) \cdot 5 | T_7 \cap T_6 \setminus T_5 | \cdot \left( 6 | T_7 \setminus T_6 | - 5 | T_7 \setminus T_6 | \right) \]
\[ \cdot \left( 6 | T_6 \setminus T_7 | - 5 | T_6 \setminus T_7 | \right) \cdot 8 | T_8 | \]

**Proof.** Lemma 3.5 and Lemma 3.6 show us that the number of the ordered system of disjoint mappings \((f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha}, f_{7\alpha}, f_{8\alpha})\) is equal to \(|R_\phi(Q, D')|\), which \(\alpha \in BX(D)\) regular element, \(V(D, \alpha) = Q\) and \(\phi : Q \rightarrow D'\) is a complete \(\alpha\)-isomorphism.

From the Theorem 2.2, the number of the mappings \(f_{1\alpha}, f_{2\alpha}, f_{3\alpha}, f_{4\alpha}, f_{5\alpha}, f_{6\alpha}, f_{7\alpha}\) and \(f_{8\alpha}\) are respectively as
\[
1, \left( 2 | T_3 \cap T_4 \setminus T_2 | \right) \cdot \left( 2 | T_2 \setminus T_1 | - 1 \right), \left( 3 | T_4 \setminus T_3 | - 2 | T_4 \setminus T_3 | \right), \left( 2 | T_3 \setminus T_4 | - 2 | T_3 \setminus T_4 | \right), \\
5 | T_7 \cap T_6 \setminus T_5 | , \left( 6 | T_7 \setminus T_6 | - 5 | T_7 \setminus T_6 | \right) , \left( 6 | T_6 \setminus T_7 | - 5 | T_6 \setminus T_7 | \right) , 8 | T_8 | .
\]

Now, we determine the number of regular elements
\[
|R_\phi(Q, D')| = \left( 2 | T_3 \cap T_4 \setminus T_2 | \right) \cdot \left( 2 | T_2 \setminus T_1 | - 1 \right) \cdot \left( 3 | T_4 \setminus T_3 | - 2 | T_4 \setminus T_3 | \right) \\
\cdot \left( 3 | T_3 \setminus T_4 | - 2 | T_3 \setminus T_4 | \right) \cdot 5 | T_7 \cap T_6 \setminus T_5 | \cdot \left( 6 | T_7 \setminus T_6 | - 5 | T_7 \setminus T_6 | \right) \\
\cdot \left( 6 | T_6 \setminus T_7 | - 5 | T_6 \setminus T_7 | \right) \cdot 8 | T_8 | .
\]

The number of all automorphisms of the semilattice \(Q\) is \(q = 4\). These are
\[
I_Q = \left( \begin{array}{cccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \end{array} \right) \quad \varphi = \left( \begin{array}{cccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \end{array} \right),
\]
\[
\theta = \left( \begin{array}{cccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \end{array} \right) \quad \tau = \left( \begin{array}{cccccccc} T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \\ T_1 & T_2 & T_3 & T_4 & T_5 & T_6 & T_7 & T_8 \end{array} \right).
\]

Therefore by using Theorem 2.3,
\[
|R(D')| = m_0 \cdot 4 \cdot \left( 2 | T_3 \cap T_4 \setminus T_2 | \right) \cdot \left( 2 | T_2 \setminus T_1 | - 1 \right) \cdot \left( 3 | T_4 \setminus T_3 | - 2 | T_4 \setminus T_3 | \right) \\
\cdot \left( 3 | T_3 \setminus T_4 | - 2 | T_3 \setminus T_4 | \right) \cdot 5 | T_7 \cap T_6 \setminus T_5 | \cdot \left( 6 | T_7 \setminus T_6 | - 5 | T_7 \setminus T_6 | \right) \\
\cdot \left( 6 | T_6 \setminus T_7 | - 5 | T_6 \setminus T_7 | \right) \cdot 8 | T_8 | .
\]

is obtained.

**Example 1.** Let \(X = \{1, 2, 3, 4, 5, 6\}\) and
\[
D = \{ T_1 = \{1\}, T_2 = \{1, 2\}, T_3 = \{1, 2, 3\}, T_4 = \{1, 2, 4\}, T_5 = \{1, 2, 3, 4\}, \ T_6 = \{1, 2, 3, 4, 5\}, T_7 = \{1, 2, 3, 4, 6\}, T_8 = \{1, 2, 3, 4, 5, 6\} \}. \]
$D$ is an $X$–semilattice of unions since $D$ is closed the union of sets. Moreover $D$ satisfies the conditions in (3.1). Then, $D$ is an $XI$–semilattice. Let $D = Q$. Therefore $|\Omega(Q)| = 1$. Besides, the number of all automorphisms of $Q$ is $q = 4$. By using Theorem 3.7

$$|R(Q)| = 1 \cdot 4 \cdot 2^{2\left(\overline{T_3 \cap T_4} \setminus T_2\right)} \cdot 2^{2\left|T_2 \setminus T_1\right|} \cdot \left(3^{\left|T_4 \setminus T_5\right|} - 2^{\left|T_4 \setminus T_3\right|}\right)$$

$$\cdot \left(3^{\left|T_3 \setminus T_4\right|} - 2^{\left|T_3 \setminus T_4\right|}\right) \cdot 5^{\left|T_7 \cap T_6\right| \setminus T_5} \cdot 6^{\left|T_7 \setminus T_6\right| - 5^{\left|T_7 \setminus T_6\right|}}$$

$$= 4$$

is obtained.

References


