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CONNECTIVITIES FOR A PRETOPOLOGY AND ITS INVERSE

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Abstract: In this paper, we present inverse of a given pretopology and we exhibit some equivalences related to connectivity and strong connectivity.

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1. Introduction

We already presented how pretopology generalizes both graph theory and topology ([1]). We also established links between on one hand pretopology and matroids and, on the other hand between pretopology and hypergraphs ([6]).

Here, we present results about strong connectivity ([4]) and connectivity ([7]) related to a given pretopology and its inverse.

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2. Different Types of Pretopological Spaces ([1], [3], [4])

Definition 1. Let X be a non-empty set. $\mathcal{P}(X)$ denotes the family of subsets of X. We call pseudoclosure on X any mapping a from $\mathcal{P}(X)$ onto $\mathcal{P}(X)$ such as:

$$a(\emptyset) = \emptyset, \forall A \subset X, A \subset a(A)$$

(X, a) is then called pretopological space.

We can define four different types of pretopological spaces.

1. (X, a) is a $\mathcal{V}$ type pretopological space if and only if $\forall A \subset X, \forall B \subset X, A \subset B \Rightarrow a(A) \subset a(B)$
2. (X, a) is a $\mathcal{V}_D$ type pretopological space if and only if $\forall A \subset X, \forall B \subset X, a(A \cup B) = a(A) \cup a(B)$.
3. (X, a) is a $\mathcal{V}_s$ type pretopological space if and only if $\forall A \subset X, a(A) = \bigcup_{x \in A} a(\{x\})$
4. (X, a) a $\mathcal{V}_D$ type pretopological space, is a topological space if and only if $\forall A \subset X, a(a(A)) = a(A)$

Property 2. *If (X, a) is a $\mathcal{V}_s$ space then (X, a) is a $\mathcal{V}_D$ space. If (X, a) is a $\mathcal{V}_D$ space then (X, a) is a $\mathcal{V}$ space.*

Example 3. *Let X be a non-empty set and $\mathcal{R}$ be a binary relationship defined on X. The pretopology of ascending-descending is defined by its pseudoclosure, denoted a_{ad} according to:*

$$\forall A \subset X, a_{ad}(A) = \{x \in X / \mathcal{R}^{-1}(x) \cap A \neq \emptyset \wedge \mathcal{R}(x) \cap A \neq \emptyset\} \cup A$$

This leads us to a $\mathcal{V}$ type pretopological space, which is not a $\mathcal{V}_D$ one nor a $\mathcal{V}_s$ one.

3. Different Pretopological Spaces Defined from a Space (X,a) and Closures [3][5]

Definition 4. Let (X, a) be a $\mathcal{V}$ type pretopological space. Let $A \subset X$. A is a closed subset of X if and only if $A = a(A)$.

We denote for any subset A of X:

$$a^0(A) = A \text{ and } \forall n,\ n \geq 1,\ a^n(A) = a(a^{n-1}(A)).$$

We define the closure of A as the subset of X, denoted $F_a(A)$, which is the smallest closed subset which contains A. Then, F', the inverse of the closure generated by a, is defined as:

$$\forall A \subset X, F'(A) = \{x \in X / F_a(\{x\}) \cap A \neq \emptyset\}$$

We note $a" = F'_a F_a$ the composition of F'_a and F_a, and $F"_a$ denotes the closure according to $a"$.

Remark 5. *$F_a(A)$ is the intersect of all closed subsets which contain A. In the case where (X, a) is a "general" pretopological space (i.e. is not a $\mathcal{V}$ space nor a $\mathcal{V}_D$ space, nor a $\mathcal{V}_s$ space, and nor a topological space), the closure may not exist.*

Proposition 6. *Let (X, a) be a $\mathcal{V}$ space. Let $A \subset X$. If one of the two following conditions is fulfilled:*
X is a finite space
a defines a $\mathcal{V}_s$ space
then

$$F_a(A) = \bigcup_{n \geq 0} a^n(A)$$

Remark 7. *If a defines a $\mathcal{V}$ type space, then $a^n, F_a, a", F"_a$ also define $\mathcal{V}$ type spaces and F'_a defines a $\mathcal{V}_s$ type space. If a defines a $\mathcal{V}_s$ type space then $a^n, F_a, a", F"_a, F'_a$ also define $\mathcal{V}_s$ type spaces.*

Definition 8. Let (X, a) a $\mathcal{V}$ pretopological space. Let $A \subset X$. The mapping a', called inverse of pseudoclosure a, is defined as follows:

$$\forall A \subset X, a'(A) = \{x \in X / F_a(\{x\}) \cap A \neq \emptyset\}$$

a' is a pseudoclosure defined on $\mathcal{P}(X)$

We denote $F_{a'}$ the closure according to a' et $F'_{a'}$ the inverse of the closure according to a' (i.e. of $F_{a'}$) . We also denote $F"_{a'}$ the closure according to $(a')" = F'_{a'} F_{a'}$.

4. Global Results Related to (X, a) et (X, a') [3]

Remark 9. $\forall x \in X, \forall y \in y, y \in a(\{x\}) \Leftrightarrow x \in a(\{y\})$.

Proof. $y \in a'(\{x\}) \Leftrightarrow a(\{y\}) \cap \{x\} \neq \emptyset \Leftrightarrow x \in a(\{y\})$ according to definition. □

Remark 10. *Let (X, a) a pretopological space. If (X, a) is a $\mathcal{V}$ type space, then (X, a') is $\mathcal{V}_s$ type space.*

Proof. $\forall A \subset X, a'(A) = \{x \in X / a(\{x\}) \cap A \neq \emptyset\}$
$\forall A \subset X, a'(A) = \cup_{y \in A}\{x \in X / y \in a(\{x\})\} = \cup_{y \in A} a'(\{y\})$. □

Proposition 11. *Let (X, a) a pretopological space. Let $x \in X, y \in X$ and n an integer.*

1. *If (X, a) is of $\mathcal{V}$ type, $x \in (a')^n(\{y\}) \Rightarrow y \in a^n(\{x\})$.*
2. *If (X, a) is of $\mathcal{V}_s$ type, $x \in (a')^n(\{y\}) \Leftrightarrow y \in a^n(\{x\})$.*

Proof. 1. If (X, a) is of $\mathcal{V}$ type then (X, a') is of $\mathcal{V}_s$ type. Then $x \in (a')^n(\{y\}) \Leftrightarrow \exists x_0, x_1, .., x_n$ such as:
$x_0 = y, x_n = x$ with $\forall j = 0, .., n-1, x_{j+1} \in a'(\{x_j)\}$ (previous remark),which implies $y \in a^n(\{x\})$.

2. (X, a) is $\mathcal{V}_s$ type, we get an equivalence instead of an implication. □

Proposition 12. *Let (X, a) pretopological space.*

1. *If (X, a) is of $\mathcal{V}$ type,*

$$\forall A \subset X, (a')'(A) = \bigcup_{y \in A} a(\{y\}) \subset a(A)$$

2. *If (X, a) is $\mathcal{V}_s$ type, $(a')' = a$.*

Proof. 1.

$$\begin{aligned}
(a')'(A) =& \{x \in X / a'(\{x\}) \cap A \neq \emptyset\}, \\
(a')'(A) =& \bigcup_{y \in A} \{x \in X / y \in a'(\{x\})\} \\
=& (a')'(A) \bigcup_{y \in A} \{x \in X / x \in a(\{y\})\}
\end{aligned}$$

$$=(a')'(A) \bigcup_{y\in A} a(\{y\}) \subset a(A) \text{ (a defines a $\mathcal{V}$ type space).}$$

2. If (X,a) is of $\mathcal{V}_s$ type then $\cup_{y\in A} a(\{y\}) = a(A)$ which leads to the result. □

Proposition 13. *Let (X,a) a pretopological space. Let n an integer.*

1. *If (X,a) is of $\mathcal{V}$ type, $\forall A \subset X, (a')^n(A) = \bigcup_{y\in A}(a')^n(\{y\}) \subset (a^n)'(A)$.*
2. *If (X,a) is of $\mathcal{V}_s$ type, $(a')^n = (a^n)'$.*

Proof. 1. If (X,a) is of $\mathcal{V}$ type then (X,a') is of $\mathcal{V}_s$ type then $(a')^n(A) = \bigcup_{y\in A}(a')^n(\{y\})$. By using recurrence, let us show that $(a')^n(A) \subset (a^n)'(A)$. This is true for $n=1$. Let us suppose it is true for n.

We get $(a')^{n+1}(A) = a'((a')^n(A)) \subset a'(a^n)'(A)$ (because the property is true for n)

$$(a')^{n+1}(A) \subset a'(\{x \in X / a^n(\{x\}) \cap A \neq \emptyset\})$$

and

$$a'(\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\}) \subset \{y \in X/a(\{y\}) \cap \{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \neq \emptyset\}$$
$$\subset \{y \in X/a^{n+1}(\{y\}) \cap A \neq \emptyset\} \subset (a^{n+1})'(A).$$

2. By using recurrence, let us show that $(a')^n(A) = (a^n)'(A)$. This is true for $n=1$. Let us suppose it is true for n.

We get $(a')^{n+1}(A) = a'((a')^n(A)) = a'(a^n)'(A)$ (because the property is true for n)

$$(a')^{n+1}(A) = a'(\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\}).$$

So,

$$(a')^{n+1}(A) = \{y \in X/a(\{y\}) \cap \{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \neq \emptyset\}$$

and

$$(a')^{n+1}(A) = \{y \in X/a^{n+1}(\{y\}) \cap A \neq \emptyset\} = (a^{n+1})'(A).$$ □

Proposition 14. *Let (X,a) a pretopological space.*

1. *If (X,a) is of $\mathcal{V}$ type, $\forall A \subset X$, we get:*

- $F_{a'}(A) \subset F'_a(A)$
- $F'_{a'}(A) \subset F_a(A)$
- $F"_{a'}(A) \subset F"_a(A)$.

2. If (X, a) is of $\mathcal{V}_s$ type, $\forall A \subset X$, we get:

- $F_{a'}(A) = F'_a(A)$
- $F'_{a'}(A) = F_a(A)$
- $F"_{a'}(A) = F"_a(A)$.

Proof. (i) If (X, a) is of $\mathcal{V}$ type then (X, a') is of $\mathcal{V}_s$ type then $F_{a'}(A) = \bigcup_{n\geq 0}(a')^n(A) \subset \bigcup_{n\geq 0}(a^n)'(A)$. Moreover

$$\bigcup_{n\geq 0}(a^n)'(A) \subset \bigcup_{n\geq 0}\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\}$$

(see Proposition 13.1) and by definition

$F_{a'}(A) \subset \{x \in X/F_a(\{x\}) \cap A \neq \emptyset\} \subset F'_a(A)$.

$F'_{a'}(A) = \{x \in X/F_{a'}(\{x\}) \cap A \neq \emptyset\}$ (by definition)

$F'_{a'}(A) = \bigcup_{y\in A}\{x \in X/y \in F'_a(\{x\})\}$ (Proposition 14.1)

We have

$$\bigcup_{y\in A}\{x \in X/y \in F'_a(\{x\})\} \subset \bigcup_{y\in A}\{x \in X/x \in F_a(\{y\})\}$$

(see [5]) and

$$\bigcup_{y\in A}\{x \in X/x \in F_a(\{y\})\} \subset \bigcup_{y\in A} F_a(\{y\}) \subset F_a(A))$$

((X, a) is of $\mathcal{V}$ type).

We have $F"_{a'} = F_{(a')"}$ (by definition) . But $(a')"(A) = F'_aF_{a'}(A)$ (by definition).

$(a')"(A) \subset F'_{a'}F'_a(A)$ (Proposition 14.1) and then:

$F"_{a'}(A) = F_{(a')"}(A) \subset F_{F_aF'_a}(A)$.

And $F_{F_aF'_a}(A) = F_{F'_aF_a}(A)$ (see [2]) $= F"_a(A)$ (by definition).

(ii) If (X, a) is of $\mathcal{V}_s$ type, then inclusions in part (i) become equalities, which implies the result. □

Definition 15. Let (X, a) is of $\mathcal{V}$ type. Let $A \subset X$. We define the induced pretopology on A by a by: $\forall C \subset A, a_A(C) = a(C) \cap A$, (A, a_A), or more simply, a_A is said pretopological subspace of (X, a) .

Proposition 16. *Let (X, a) a pretopological space of $\mathcal{V}$ type. Let $A \subset X$ with $A \neq \emptyset$.*

Let $C \subset A, (a_A)'(C) = (a')_A(C)$.

Proof. $(a_A)'(C) = \{x \in A / a_A(\{x\}) \cap C \neq \emptyset\}$
$(a_A)'(C) = \{x \in A / a(\{x\}) \cap C \cap A \neq \emptyset\}$
$(a_A)'(C) = a'(C \cap A) \cap A = a'(C) \cap A = (a')_A(C)$ (car $C \subset A$). □

5. Strong Connectivity in (X,a) and (X,a') [3]

Definition 17. Let (X, a) is of $\mathcal{V}$ type. Let A and B two non empty subsets of X. There exists a path in (X, a) from B to A if and only if $B \subset F(A)$.

Proposition 18. *Let (X, a) a pretopological space. Let $x \in X, y \in X$. i- If (X, a) is of $\mathcal{V}$ type , if there exists a path from $\{x\}$ to $\{y\}$ in (X, a') then there exists a path from $\{y\}$ to $\{x\}$ in (X, a).*

ii- If (X, a) is of $\mathcal{V}_s$ type, existence of a path from $\{x\}$ to $\{y\}$ in (X, a') is equivalent to existence of a path from $\{y\}$ to$\{x\}$ in (X, a).

Proof. (i) If (X, a) is of $\mathcal{V}$ type, then (X, a') is of $\mathcal{V}_s$ type and there exists a path from $\{x\}$ to $\{y\}$ in (X, a') which is equivalent to existence of a sequence $x_0, ..., x_n$ of elements of X such as $x_0 = y, x_n = x$ with $\forall j = 0, ..., n-1, x_{j+1} \in a'(\{x_j\})$ (see [6]). In other words, there exists a sequence $x_0, ..., x_n$ of elements of X such as $x_0 = y, x_n = x$ with $\forall j = 0, ..., n-1, x_{j+1} \in a(\{x_j\})$ (previous remark), which implies there exists a path from $\{y\}$ to $\{x\}$ in (X, a) (see [6]).

(ii) If (X, a) is of $\mathcal{V}_s$ type, we get equivalence (see [6]). □

Remark 19. *The converse of the (i) is not true generally speaking.*

Example 20. *Let (X, a) a pretopological space with $X = \{a, b, c, d\}$ and a the pseudoclosure of ascending-descending as defined on the following graph (figure 1): $a \in F_a(\{c\})$, then there exists a path from $\{a\}$ to $\{c\}$ in (X, a) but $c \notin F_{a'}(\{a\})$.*

Indeed, $a'(\{a\}) = \{x \in X / a(\{x\}) \cap \{a\} \neq \emptyset\}$
$a'(\{a\}) = \{x \in X / a \in a(\{x\})\} = \{a, b\}$ and

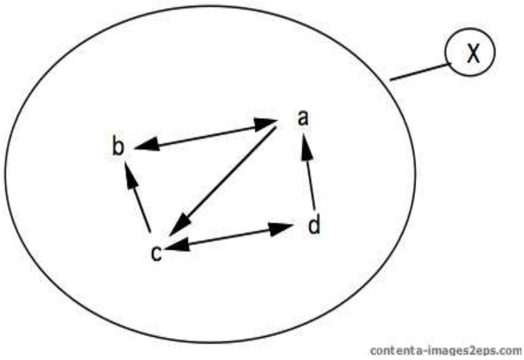

Figure 1: Graph

$a'(\{a,b\}) = \{x \in X/a(\{x\}) \cap \{a,b\} \ \neq \emptyset\} = \{a,b\}$.
So $F_{a'}(\{a\}) = \{a,b\}$ *and* $c \notin F_{a'}(\{a\})$.

Proposition 21. *Let* (X,a) *is of* $\mathcal{V}$ *type.*
(X,a) *strongly connected* $\Leftrightarrow \forall A \subset X, A \neq \emptyset, \forall B \subset X, B \neq \emptyset$, *there exists a path from* B *to* A *in* (X,a).

Definition 22. Let (X,a) is of $\mathcal{V}$ type. Let $A \subset X$.
A is a pretopological subspace strongly connected of X if and only if (A, a_A) is strongly connected as a pretopological space.

Proposition 23. *Let* (X,a) *a pretopological space.*
Let $A \subset X, A \neq \emptyset$.
(i) If (X,a) *is of* $\mathcal{V}$ *type then* A *subspace strongly connected of* (X,a') *implies* A *subspace strongly connected of* (X,a).
(ii) If (X,a) *is of* $\mathcal{V}_s$ *type then* A *subspace strongly connected of* (X,a') *is equivalent to* A *subspace strongly connected of* (X,a) .

Proof. (i) A subspace strongly connected of (X,a')
$\Leftrightarrow$
$\forall x \in A, \forall y \in A$, there exists a path from $\{y\}$ to $\{x\}$ in $(A,(a')_A)$ (see [6])
$\Leftrightarrow$
$\forall x \in A, \forall y \in A$, there exists a path from $\{y\}$ to $\{x\}$ in $(A,(a_A)')$ (Proposition 16), therefore $\forall x \in A, \forall y \in A$, there exists a path from $\{x\}$ to $\{y\}$ in (A, a_A). So A is a subspace strongly connected of (X,a) (see [6]).
(ii) We get equivalences if (X,a) is of $\mathcal{V}_s$ type (Proposition 18.ii). $\square$

Definition 24. Let (X, a) is of $\mathcal{V}$ type. Let $A \subset X, A \neq \emptyset$. (A, a_A) is a greatest subspace strongly connected of (X, a) if and only if (A, a_A) is a subspace strongly connected of (X, a) and $\forall B, A \subset B \subset X$ and $A \neq B, (B, a_B)$ is not a subspace strongly connected of (X, a).

Proposition 25. *Let (X, a) is of $\mathcal{V}_s$ type. Let $A \subset X, A \neq \emptyset$.*
(A, a_A) is a greatest subspace strongly connected of $(X, a) \Leftrightarrow (A, a_A)$ is the greatest subspace strongly connected of (X, a').

Proof. obvious from proposition 18.ii □

Conclusion. Decomposing a pretopological space (X, a) of $\mathcal{V}_s$ type into greatest strongly connected subspaces is equivalent to decomposing the pretopological space (X, a') into greatest strongly connected subspaces.

6. Connectivity in (X, a) and (X, a') [3]

Definition 26. Let (X, a) of $\mathcal{V}$ type. Let A and B two non empty subsets of X. There exists a chain in (X, a) from B to A if and only if $B \subset F"(A)$.

Proposition 27. *Let (X, a) a pretopological space . Let $x \in X, y \in X$.*
(i) If (X, a) of $\mathcal{V}$ type, there exists a chain from $\{y\}$ to $\{x\}$ in (X, a'), which implies there exists one chain from $\{y\}$ to $\{x\}$ in (X, a).
(ii) If (X, a) of $\mathcal{V}_s$ type, there exists a chain from $\{y\}$ to $\{x\}$ in (X, a') is equivalent to there exists one chain from $\{y\}$ to $\{x\}$ in (X, a).

Proof. (i)If (X, a) is of $\mathcal{V}$ type then (X, a') is of $\mathcal{V}_s$ type, so there exists a chain from $\{y\}$ to $\{x\}$ in (X, a')
$\Leftrightarrow$
there exists a sequence $x_0, ..., x_n$ of elements of X such as:
$x_0 = x, x_n = y$ with $\forall j = 0, ..., n-1, x_{j+1} \in a'(\{x_j\})$ or $x_j \in a'(\{x_{j+1}\})$ (see [6])
$\Leftrightarrow$
there exists a sequence $x_0, ..., x_n$ of elements of X such as:
$x_0 = x, x_n = y$ with $\forall j = 0, ..., n-1, x_{j+1} \in a(\{x_j\})$ or $x_j \in a(\{x_{j+1}\})$
(previous remark) which implies there exists a chain from $\{y\}$ to $\{x\}$ in (X, a) (see [6]).
(ii) If (X, a) is of $\mathcal{V}_s$ type, we get an equivalence (see [6]). □

Remark 28. *The converse of (i) is not true generally speaking.*

Example 29. *Let (X, a) a pretopological space with $X = \{a, b, c, d\}$ and a the pseudoclosure of ascending-descending as defined on the graph (figure 1).*

$c \in F_a(\{a\})$ then $c \in F"_a(\{a\})$ so there exists one chain from $\{c\}$ to $\{a\}$ in (X, a) but $c \notin F"_{a'}(\{a\})$. Indeed, $F_{a'}(\{a\}) = \{a, b\}$ and $F'_{a'}F_{a'}(\{a\}) = F'_{a'}(\{a, b\}) = \{x \in X / F_{a'}(\{x\}) \cap \{a, b\} \neq \emptyset\}$ with $F_{a'}(\{c\}) = F_{a'}(\{d\}) = \{c, d\}$. then $F'_{a'}F_{a'}(\{a\}) = \{a, b\} = F"_{a'}(\{a\})$ and $c \notin F'_{a'}(\ a\})$.

Definition 30. Let (X, a) a pretopological space of $\mathcal{V}$ type. (X, a) is connected if and only if $\forall C \subset X, C \neq \emptyset, F(C) = X$ or $F(X - F(C)) \cap F(C) \neq \emptyset$.

Proposition 31. *Let (X, a) be a $\mathcal{V}$ type pretopological space. If $x \in X$ and $y \in X$, there exists a chain in (X, a) from $\{y\}$ to $\{x\}$ then (X, a) is connected.*

(see [6])

Definition 32. Let (X,a) a pretopological space of $\mathcal{V}$ type. Let $A \subset X$. A is a connected subspace of (X, a) if and only if (A, a_A) is connected as a pretopological space.

Proposition 33. *Let (X, a) a pretopological space. Let $A \subset X, A \neq \emptyset$.*

(i) If (X, a) of $\mathcal{V}$ type, then if A is a connected subspace of (X, a') then A is a connected subspace of (X, a).

(ii) If (X, a) of $\mathcal{V}_s$ type, then A connected subspace of (X, a') is equivalent to A connected subspace of (X, a).

Proof. (i) A connected subspace of (X, a')
$\Leftrightarrow$
$\forall x \in A, \forall y \in A$, there exists a chain from $\{y\}$ to $\{x\}$ in $(A, (a')_A)$ (see [6]) $\Leftrightarrow$ $\forall x \in A, \forall y \in A$, there exists one chain from $\{y\}$ to $\{x\}$ in $(A, (a_A)')$ (Proposition 16) so $\forall x \in A, \forall y \in A$, there exists one chain from $\{y\}$ to $\{x\}$ in (A, a_A) (Proposition 27.i) hence$A$ is a connected subspace of (X, a) (Proposition 31).

(ii) We get equivalences if (X, a) is of $\mathcal{V}_s$ type (Proposition 27.ii et [6]). □

Definition 34. Let (X, a) a pretopological space of $\mathcal{V}$ type . Let $A \subset X, A \neq \emptyset$. (A, a_A) is a greatest connected subspace of (X, a) if and only if (A, a_A) is a connected subspace of (X, a) and $\forall B, A \subset B \subset X$ and $A \neq B, (B, a_B)$ is not a connected subspace of (X, a).

Proposition 35. *Let (X,a) a pretopological space of $\mathcal{V}_s$ type . Let $A \subset X, An \neq \emptyset$.*

A is a greatest connected subspace of de (X, a) if and only if A is a greatest connected subset of (X, a') .

Proof. Obvious from Proposition 33.ii. □

Conclusion. Decomposing a pretopological space (X, a) of $\mathcal{V}_s$ type into greatest connected subspaces is equivalent to decomposing the pretopological space (X, a') into greatest connected subspaces.

7. Application

Definition 36. Let X a non empty set and $\mathcal{R}$ a binary relationship defined on X. Pretopology of ascending is characterized by:

$$\forall A \subset X, a_a(A) = \{x \in X / \mathcal{R}^{-1}(x) \cap A \neq \emptyset\} \cup A \tag{1}$$

with $\mathcal{R}^{-1}(x) = \{y \in X / y\mathcal{R}x\}$.

Pretopology of descending is characterized by:

$$\forall A \subset X, a_d(A) = \{x \in X / \mathcal{R}(x) \cap A \neq \emptyset\} \cup A \tag{2}$$

with

$$\mathcal{R}(x) = \{y \in X / x\mathcal{R}y\}.$$

a_a and a_d lead to $\mathcal{V}_s$ type spaces (see [8]).

Remark 37. $a'_a = a_d$

Proof. $a'_a(A) = \{x \in X / a_a(\{x\}) \cap A \neq \emptyset\}$ with

$$\begin{aligned} a_a(\{x\}) = \{y \in X / \mathcal{R}^{-1}(y) \cap \{x\} \neq \emptyset\} \cup \{x\} &= \{y \in X / x \in \mathcal{R}^{-1}(y)\} \cup \{x\} \\ &= \{y \in X / y \in \mathcal{R}(x)\} \cup \{x\}\} = \mathcal{R}(x) \cup \{x\}, \end{aligned}$$

so

$$a'_a(A) = \{x \in X / \mathcal{R}(x) \cap A \neq \emptyset\} \cup A = a_d(A).$$

□

Conclusion. Decomposing a pretopological space (X, a_a) into greatest (strongly) connected subspaces is equivalent to decomposing the pretopological space (X, a_d) into greatest (strongly) connected subspaces.

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