CONNECTIVITIES FOR A PRETOPOLOGY
AND ITS INVERSE

Monique Dalud-Vincent¹, Michel Lamure²§

¹FMEPS - Max Weber Center, University Lyon 2
5 Avenue Pierre Mendes-France, B.P. 11 Sociologie
69676 Bron cedex, FRANCE
²EAM 4128, University Lyon 1
11 Rue Guillaume Paradin
69372 Lyon cedex 08, FRANCE

Abstract: In this paper, we present inverse of a given pretopology and we exhibit some equivalences related to connectivity and strong connectivity.

AMS Subject Classification: 54A05, 54B05, 54B15
Key Words: topology, graph theory, connectivity

1. Introduction

We already presented how pretopology generalizes both graph theory and topology ([1]). We also established links between on one hand pretopology and matroids and, on the other hand between pretopology and hypergraphs ([6]).

Here, we present results about strong connectivity ([4]) and connectivity ([7]) related to a given pretopology and its inverse.

Received: February 28, 2013

§Correspondence author
2. Different Types of Pretopological Spaces ([1], [3], [4])

**Definition 1.** Let $X$ be a non-empty set. $\mathcal{P}(X)$ denotes the family of subsets of $X$. We call pseudoclosure on $X$ any mapping $a$ from $\mathcal{P}(X)$ onto $\mathcal{P}(X)$ such as:

$$a(\emptyset) = \emptyset, \forall A \subset X, A \subset a(A)$$

$(X, a)$ is then called pretopological space.

We can define four different types of pretopological spaces.

1. $(X, a)$ is a $\mathcal{V}$ type pretopological space if and only if $\forall A \subset X, \forall B \subset X, A \subset B \Rightarrow a(A) \subset a(B)$

2. $(X, a)$ is a $\mathcal{V}_D$ type pretopological space if and only if $\forall A \subset X, \forall B \subset X, a(A \cup B) = a(A) \cup a(B)$.

3. $(X, a)$ is a $\mathcal{V}_s$ type pretopological space if and only if $\forall A \subset X, a(A) = \bigcup_{x \in A} a(\{x\})$

4. $(X, a)$ a $\mathcal{V}_D$ type pretopological space, is a topological space if and only if $\forall A \subset X, a(a(A)) = a(A)$

**Property 2.** If $(X, a)$ is a $\mathcal{V}_s$ space then $(X, a)$ is a $\mathcal{V}_D$ space. If $(X, a)$ is a $\mathcal{V}_D$ space then $(X, a)$ is a $\mathcal{V}$ space.

**Example 3.** Let $X$ be a non-empty set and $\mathcal{R}$ be a binary relationship defined on $X$. The pretopology of ascending-descending is defined by its pseudoclosure, denoted $a_{ad}$ according to:

$$\forall A \subset X, a_{ad}(A) = \{x \in X/\mathcal{R}^{-1}(x) \cap A \neq \emptyset \land \mathcal{R}(x) \cap A \neq \emptyset\} \cup A$$

This leads us to a $\mathcal{V}$ type pretopological space, which is not a $\mathcal{V}_D$ one nor a $\mathcal{V}_s$ one.

3. Different Pretopological Spaces Defined from a Space $(X,a)$ and Closures [3][5]

**Definition 4.** Let $(X, a)$ be a $\mathcal{V}$ type pretopological space. Let $A \subset X$. $A$ is a closed subset of $X$ if and only if $A = a(A)$.
We denote for any subset \( A \) of \( X \):

\[
a^0(A) = A \text{ and } \forall n, \ n \geq 1, \ a^n(A) = a(a^{n-1}(A)).
\]

We define the closure of \( A \) as the subset of \( X \), denoted \( F_a(A) \), which is the smallest closed subset which contains \( A \). Then, \( F' \), the inverse of the closure generated by \( a \), is defined as:

\[
\forall A \subset X, F'(A) = \{ x \in X / F_a(\{ x \}) \cap A \neq \emptyset \}
\]

We note \( a'' = F'_aF_a \) the composition of \( F'_a \) and \( F_a \), and \( F''_a \) denotes the closure according to \( a'' \).

**Remark 5.** \( F_a(A) \) is the intersect of all closed subsets which contain \( A \). In the case where \((X, a)\) is a "general" pretopological space (i.e. is not a \( \mathcal{V} \) space nor a \( \mathcal{V}_D \) space, nor a \( \mathcal{V}_s \) space, and nor a topological space), the closure may not exist.

**Proposition 6.** Let \((X, a)\) be a \( \mathcal{V} \) space. Let \( A \subset X \). If one of the two following conditions is fulfilled:

- \( X \) is a finite space
- \( a \) defines a \( \mathcal{V}_s \) space

then

\[
F_a(A) = \bigcup_{n \geq 0} a^n(A)
\]

**Remark 7.** If \( a \) defines a \( \mathcal{V} \) type space, then \( a^n, F_a, a'', F''_a \) also define \( \mathcal{V} \) type spaces and \( F'_a \) defines a \( \mathcal{V}_s \) type space. If \( a \) defines a \( \mathcal{V}_s \) type space then \( a^n, F_a, a'', F''_a, F'_a \) also define \( \mathcal{V}_s \) type spaces.

**Definition 8.** Let \((X, a)\) a \( \mathcal{V} \) pretopological space. Let \( A \subset X \). The mapping \( a' \), called inverse of pseudoclosure \( a \), is defined as follows:

\[
\forall A \subset X, a'(A) = \{ x \in X / F_a(\{ x \}) \cap A \neq \emptyset \}
\]

\( a' \) is a pseudoclosure defined on \( \mathcal{P}(X) \)

We denote \( F_{a'} \) the closure according to \( a' \) et \( F'_{a'} \) the inverse of the closure according to \( a' \) (i.e. of \( F_{a'} \)). We also denote \( F''_{a'} \) the closure according to \( (a')'' = F'_{a'}F_{a'} \).
4. Global Results Related to \((X, a)\) et \((X, a')\) [3]

**Remark 9.** \(\forall x \in X, \forall y \in y, y \in a\{x\} \iff x \in a\{y\}\).

*Proof.* \(y \in a'\{x\} \iff a\{y\} \cap \{x\} \neq \emptyset \iff x \in a\{y\}\) according to definition. \(\square\)

**Remark 10.** Let \((X, a)\) a pretopological space. If \((X, a)\) is a \(\mathcal{V}\) type space, then \((X, a')\) is \(\mathcal{V}s\) type space.

*Proof.* \(\forall A \subset X, a'(A) = \{x \in X/a\{x\} \cap A \neq \emptyset\}\)
\(\forall A \subset X, a'(A) = \cup_{y \in A}\{x \in X/y \in a\{x\}\} = \cup_{y \in A}a'(\{y\}).\) \(\square\)

**Proposition 11.** Let \((X, a)\) a pretopological space. Let \(x \in X, y \in X\) and \(n\) an integer.

1. If \((X, a)\) is of \(\mathcal{V}\) type, \(x \in (a')^n\{y\} \Rightarrow y \in a^n\{x\}\).

2. If \((X, a)\) is of \(\mathcal{V}s\) type, \(x \in (a')^n\{y\} \Leftrightarrow y \in a^n\{x\}\).

*Proof.* 1. If \((X, a)\) is of \(\mathcal{V}\) type then \((X, a')\) is of \(\mathcal{V}s\) type. Then \(x \in (a')^n\{y\} \leftrightarrow \exists x_0, x_1, \ldots, x_n\) such as:
\(x_0 = y, x_n = x\) with \(\forall j = 0, \ldots, n - 1, x_{j+1} \in a'\{x_j\}\) (previous remark), which implies \(y \in a^n\{x\}\).

2. \((X, a)\) is \(\mathcal{V}s\) type, we get an equivalence instead of an implication. \(\square\)

**Proposition 12.** Let \((X, a)\) pretopological space.

1. If \((X, a)\) is of \(\mathcal{V}\) type,
\[\forall A \subset X, (a')'(A) = \bigcup_{y \in A} a\{y\} \subset a(A)\]

2. If \((X, a)\) is \(\mathcal{V}s\) type, \((a')' = a\).

*Proof.* 1.
\[
(a')'(A) = \{x \in X/a'(\{x\}) \cap A \neq \emptyset\},
\[
(a')'(A) = \bigcup_{y \in A} \{x \in X/y \in a'(\{x\})\}
\]
\[=(a')'a(x \in X/x \in a\{y\})\]
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(a')'(A) \bigcup_{y \in A} a(\{y\}) \subset a(A) (a defines a \mathcal{V} type space).

2. If \((X, a)\) is of \mathcal{V}_s\ type then \(\bigcup_{y \in A} a(\{y\}) = a(A)\) which leads to the result. \qed

**Proposition 13.** Let \((X, a)\) a pretopological space. Let \(n\) an integer.

1. If \((X, a)\) is of \mathcal{V}\ type, \(\forall A \subset X, (a')^n(A) = \bigcup_{y \in A} (a')^n(\{y\}) \subset (a^n)'(A)\).

2. If \((X, a)\) is of \mathcal{V}_s\ type, \((a')^n = (a^n)'.\)

**Proof.** 1. If \((X, a)\) is of \mathcal{V}\ type then \((X, a')\) is of \mathcal{V}_s\ type then \((a')^n(A) = \bigcup_{y \in A} (a')^n(\{y\})\). By using recurrence, let us show that \((a')^n(A) \subset (a^n)'(A)\).

We get \((a')^{n+1}(A) = a'((a')^n(A)) \subset a'(a^n)'(A)\) (because the property is true for \(n\))

\[ (a')^{n+1}(A) \subset a'\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \]

and

\[ a'(\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\}) \subset \{y \in X/a(\{y\}) \cap \{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \neq \emptyset\} \]

\[ \subset \{y \in X/a^{n+1}(\{y\}) \cap A \neq \emptyset\} \subset (a^{n+1})'(A). \]

2. By using recurrence, let us show that \((a')^n(A) = (a^n)'(A)\). This is true for \(n = 1\). Let us suppose it is true for \(n\).

We get \((a')^{n+1}(A) = a'((a')^n(A)) = a'(a^n)'(A)\) (because the property is true for \(n\))

\[ (a')^{n+1}(A) = a'\{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \]

So,

\[ (a')^{n+1}(A) = \{y \in X/a(\{y\}) \cap \{x \in X/a^n(\{x\}) \cap A \neq \emptyset\} \neq \emptyset\} \]

and

\[ (a')^{n+1}(A) = \{y \in X/a^{n+1}(\{y\}) \cap A \neq \emptyset\} = (a^{n+1})'(A). \]

**Proposition 14.** Let \((X, a)\) a pretopological space.

1. If \((X, a)\) is of \mathcal{V}\ type, \(\forall A \subset X\), we get:
• $F_{a'}(A) \subset F'_{a}(A)$
• $F'_{a'}(A) \subset F_{a}(A)$
• $F^\prime_{a'}(A) \subset F^\prime_{a}(A)$.

2. If $(X, a)$ is of $\mathcal{V}_s$ type, $\forall A \subset X$, we get:
• $F_{a'}(A) = F'_{a}(A)$
• $F'_{a'}(A) = F_{a}(A)$
• $F^\prime_{a'}(A) = F^\prime_{a}(A)$.

Proof. (i) If $(X, a)$ is of $\mathcal{V}$ type then $(X, a')$ is of $\mathcal{V}_s$ type then $F_{a'}(A) = \bigcup_{n \geq 0} (a')^n(A) \subset \bigcup_{n \geq 0} (a^n)'(A)$. Moreover

$$
\bigcup_{n \geq 0} (a^n)'(A) \subset \bigcup_{n \geq 0} \{x \in X/a^n \cap A \neq \emptyset\}
$$

(see Proposition 13.1) and by definition

$F_{a'}(A) \subset \{x \in X/F_{a}(\{x\}) \cap A \neq \emptyset\} \subset F'_{a}(A)$.

$F'_{a'}(A) = \{x \in X/F_{a'}(\{x\}) \cap A \neq \emptyset\}$ (by definition)

$F_{a'}(A) = \bigcup_{\{x \in X/y \in F'_{a}(\{x\})\} \cap A } \{x \in X/y \in F'_{a}(\{x\})\}$ (Proposition 14.1)

We have

$$\bigcup_{y \in A} \{x \in X/y \in F'_{a}(\{x\})\} \subset \bigcup_{y \in A} \{x \in X/x \in F_{a}(\{y\})\}
$$

(see [5]) and

$$\bigcup_{y \in A} \{x \in X/x \in F_{a}(\{y\})\} \subset \bigcup_{y \in A} F_{a}(\{y\}) \subset F_{a}(A))
$$

((X, a) is of $\mathcal{V}$ type).

We have $F^\prime_{a'} = F_{(a')^\prime}$ (by definition) . But $(a')^\prime(A) = F_{a}F_{a'}(A)$ (by definition).

$F^\prime_{a'}(A) = F_{(a')^\prime}(A) \subset F_{F_{a}F_{a'}}(A)$.

And $F_{F_{a}F_{a'}}(A) = F_{F_{a}F_{a'}}(A)$ (see [2]) $= F_{a}(A)$ (by definition).

(ii) If $(X, a)$ is of $\mathcal{V}_s$ type, then inclusions in part (i) become equalities, which implies the result. □
Definition 15. Let \((X,a)\) is of \(\mathcal{V}\) type. Let \(A \subset X\). We define the induced pretopology on \(A\) by: \(\forall C \subset A, a_A(C) = a(C) \cap A, (A, a_A)\), or more simply, \(a_A\) is said pretopological subspace of \((X,a)\).

Proposition 16. Let \((X, a)\) a pretopological space of \(\mathcal{V}\) type. Let \(A \subset X\) with \(A \neq \emptyset\).

Let \(C \subset A, (a_A)'(C) = (a')_A(C)\).

Proof. \((a_A)'(C) = \{x \in A/a_A(\{x\}) \cap C \neq \emptyset\}\)
\((a_A)'(C) = \{x \in A/a(\{x\}) \cap C \cap A \neq \emptyset\}\)
\((a_A)'(C) = a'(C \cap A) \cap A = a'(C) \cap A = (a')_A(C)\) (car \(C \subset A\)).

5. Strong Connectivity in \((X,a)\) and \((X,a')\) [3]

Definition 17. Let \((X, a)\) is of \(\mathcal{V}\) type. Let \(A\) and \(B\) two non empty subsets of \(X\). There exists a path in \((X, a)\) from \(B\) to \(A\) if and only if \(B \subset F(A)\).

Proposition 18. Let \((X, a)\) a pretopological space. Let \(x \in X, y \in X\).

\(i-\) If \((X, a)\) is of \(\mathcal{V}\) type, if there exists a path from \(\{x\}\) to \(\{y\}\) in \((X, a')\) then there exists a path from \(\{y\}\) to \(\{x\}\) in \((X, a)\).

\(ii-\) If \((X, a)\) is of \(\mathcal{V}_s\) type, existence of a path from \(\{x\}\) to \(\{y\}\) in \((X, a')\) is equivalent to existence of a path from \(\{y\}\) to \(\{x\}\) in \((X, a)\).

Proof. \((i)\) If \((X, a)\) is of \(\mathcal{V}\) type, then \((X, a')\) is of \(\mathcal{V}_s\) type and there exists a path from \(\{x\}\) to \(\{y\}\) in \((X, a')\) which is equivalent to existence of a sequence \(x_0, ..., x_n\) of elements of \(X\) such as \(x_0 = y, x_n = x\) with \(\forall j = 0, ..., n - 1, x_{j+1} \in a'(\{x_j\})\) (see [6]). In other words, there exists a sequence \(x_0, ..., x_n\) of elements of \(X\) such as \(x_0 = y, x_n = x\) with \(\forall j = 0, ..., n - 1, x_{j+1} \in a(\{x_j\})\) (previous remark), which implies there exists a path from \(\{y\}\) to \(\{x\}\) in \((X, a)\) (see [6]).

\((ii)\) If \((X, a)\) is of \(\mathcal{V}_s\) type, we get equivalence (see [6]).

Remark 19. The converse of the \((i)\) is not true generally speaking.

Example 20. Let \((X, a)\) a pretopological space with \(X = \{a, b, c, d\}\) and \(a\) the pseudoclosure of ascending-descending as defined on the following graph (figure 1): \(a \in F_a(\{c\})\), then there exists a path from \(\{a\}\) to \(\{c\}\) in \((X, a)\) but \(c \notin F_a'(\{a\})\).

Indeed, \(a'(\{a\}) = \{x \in X/a(\{x\}) \cap \{a\} \neq \emptyset\}\)
\(a'(\{a\}) = \{x \in X/a \in a(\{x\})\} = \{a, b\} \) and
\( a'(\{a, b\}) = \{x \in X/a(\{x\}) \cap \{a, b\} \neq \emptyset\} = \{a, b\} \).

So \( F_{a'}(\{a\}) = \{a, b\} \) and \( c \notin F_{a'}(\{a\}) \).

**Proposition 21.** Let \((X, a)\) is of \(V\) type.

\((X, a)\) strongly connected \(\Leftrightarrow\) \(\forall A \subset X, A \neq \emptyset, \forall B \subset X, B \neq \emptyset, \) there exists a path from \(B\) to \(A\) in \((X, a)\).

**Definition 22.** Let \((X, a)\) is of \(V\) type. Let \(A \subset X\).

\(A\) is a pretopological subspace strongly connected of \(X\) if and only if \((A, a_A)\) is strongly connected as a pretopological space.

**Proposition 23.** Let \((X, a)\) a pretopological space.

Let \(A \subset X, A \neq \emptyset\).

(i) If \((X, a)\) is of \(V\) type then \(A\) subspace strongly connected of \((X, a')\) implies \(A\) subspace strongly connected of \((X, a)\).

(ii) If \((X, a)\) is of \(V_s\) type then \(A\) subspace strongly connected of \((X, a')\) is equivalent to \(A\) subspace strongly connected of \((X, a)\).

**Proof.** (i) A subspace strongly connected of \((X, a')\)

\[\forall x \in A, \forall y \in A, \text{ there exists a path from } \{y\} \text{ to } \{x\} \text{ in } (A, (a')_A) \text{ (see [6])}\]

\[\forall x \in A, \forall y \in A, \text{ there exists a path from } \{y\} \text{ to } \{x\} \text{ in } (A, (a_A)') \text{ (Proposition 16)}, \text{ therefore } \forall x \in A, \forall y \in A, \text{ there exists a path from } \{x\} \text{ to } \{y\} \text{ in } (A, a_A). \]

So \(A\) is a subspace strongly connected of \((X, a)\) (see [6]).

(ii) We get equivalences if \((X, a)\) is of \(V_s\) type (Proposition 18.ii). \(\square\)
Definition 24. Let \((X, a)\) is of \(V\) type. Let \(A \subset X, A \neq \emptyset\). \((A, a_A)\) is a greatest subspace strongly connected of \((X, a)\) if and only if \((A, a_A)\) is a subspace strongly connected of \((X, a)\) and \(\forall B, A \subset B \subset X\) and \(A \neq B, (B, a_B)\) is not a subspace strongly connected of \((X, a)\).

Proposition 25. Let \((X, a)\) is of \(V_s\) type. Let \(A \subset X, A \neq \emptyset\). \((A, a_A)\) is a greatest subspace strongly connected of \((X, a)\) \(\iff (A, a_A)\) is the greatest subspace strongly connected of \((X, a')\).

Proof. obvious from proposition 18.ii

Conclusion. Decomposing a pretopological space \((X, a)\) of \(V_s\) type into greatest strongly connected subspaces is equivalent to decomposing the pretopological space \((X, a')\) into greatest strongly connected subspaces.

6. Connectivity in \((X, a)\) and \((X, a')\) [3]

Definition 26. Let \((X, a)\) of \(V\) type. Let \(A\) and \(B\) two non empty subsets of \(X\). There exists a chain in \((X, a)\) from \(B\) to \(A\) if and only if \(B \subset F^* (A)\).

Proposition 27. Let \((X, a)\) a pretopological space. Let \(x \in X, y \in X\).

(i) If \((X, a)\) of \(V\) type, there exists a chain from \(\{y\}\) to \(\{x\}\) in \((X, a')\), which implies there exists one chain from \(\{y\}\) to \(\{x\}\) in \((X, a)\).

(ii) If \((X, a)\) of \(V_s\) type, there exists a chain from \(\{y\}\) to \(\{x\}\) in \((X, a')\) is equivalent to there exists one chain from \(\{y\}\) to \(\{x\}\) in \((X, a)\).

Proof. (i)If \((X, a)\) is of \(V\) type then \((X, a')\) is of \(V_s\) type, so there exists a chain from \(\{y\}\) to \(\{x\}\) in \((X, a')\)
\[
\iff \text{there exists a sequence } x_0, ..., x_n \text{ of elements of } X \text{ such as: } \\
x_0 = x, x_n = y \text{ with } \forall j = 0, ..., n - 1, x_{j+1} \in a'(\{x_j\}) \text{ or } x_j \in a'({x_{j+1}}) \text{ (see [6])} \\
\iff \text{there exists a sequence } x_0, ..., x_n \text{ of elements of } X \text{ such as: } \\
x_0 = x, x_n = y \text{ with } \forall j = 0, ..., n - 1, x_{j+1} \in a(\{x_j\}) \text{ or } x_j \in a(\{x_{j+1}\}) \text{ (previous remark) which implies there exists a chain from } \{y\} \text{ to } \{x\} \text{ in } (X, a) \text{ (see [6])}. \\
\]
(ii) If \((X, a)\) is of \(V_s\) type, we get an equivalence (see [6]).

Remark 28. The converse of (i) is not true generally speaking.
Example 29. Let \((X, a)\) a pretopological space with \(X = \{a, b, c, d\}\) and \(a\) the pseudoclosure of ascending-descending as defined on the graph (figure 1).

\[c \in F_a(\{a\}) \text{ then } c \in F''_a(\{a\}) \text{ so there exists one chain from } \{c\} \text{ to } \{a\} \text{ in } (X, a) \text{ but } c \notin F''_{a'}(\{a\})\]. Indeed, \(F_a(\{a\}) = \{a, b\} \text{ and } F'_{a'}F''_{a'}(\{a\}) = F'_{a'}(\{a, b\}) = \{x \in X/F'_{a'}(\{x\}) \cap \{a, b\} \neq \emptyset\} \text{ with } F_{a'}(\{c\}) = F_{a'}(\{d\}) = \{c, d\} \text{. then } F'_{a'}F'_{a'}(\{a\}) = \{a, b\} = F''_{a'}(\{a\}) \text{ and } c \notin F'_{a'}(\{a\})\).

Definition 30. Let \((X, a)\) a pretopological space of \(\mathcal{V}\) type. \((X, a)\) is connected if and only if \(\forall C \subset X, C \neq \emptyset, F(C) = X \text{ or } F(X - F(C)) \cap F(C) \neq \emptyset\).

Proposition 31. Let \((X, a)\) be a \(\mathcal{V}\) type pretopological space. If \(x \in X\) and \(y \in X\), there exists a chain in \((X, a)\) from \(\{y\}\) to \(\{x\}\) then \((X, a)\) is connected.

(see [6])

Definition 32. Let \((X, a)\) a pretopological space of \(\mathcal{V}\) type. Let \(A \subset X\). \(A\) is a connected subspace of \((X, a)\) if and only if \((A, a_A)\) is connected as a pretopological space.

Proposition 33. Let \((X, a)\) a pretopological space. Let \(A \subset X, A \neq \emptyset\).

(i) If \((X, a)\) of \(\mathcal{V}\) type, then if \(A\) is a connected subspace of \((X, a')\) then \(A\) is a connected subspace of \((X, a)\).

(ii) If \((X, a)\) of \(\mathcal{V}_s\) type, then \(A\) connected subspace of \((X, a')\) is equivalent to \(A\) connected subspace of \((X, a)\).

Proof. (i) A connected subspace of \((X, a')\)

\[\Leftrightarrow \forall x \in A, \forall y \in A, \text{ there exists a chain from } \{y\} \text{ to } \{x\} \text{ in } (A, (a')_A) \text{ (see [6]) } \Leftrightarrow \forall x \in A, \forall y \in A, \text{ there exists one chain from } \{y\} \text{ to } \{x\} \text{ in } (A, (a_A') \text{ (Proposition 16) so } \forall x \in A, \forall y \in A, \text{ there exists one chain from } \{y\} \text{ to } \{x\} \text{ in } (A, (a_A) \text{ (Proposition 27.i) } \text{ hence } A \text{ is a connected subspace of } (X, a) \text{ (Proposition 31).}

(ii) We get equivalences if \((X, a)\) is of \(\mathcal{V}_s\) type (Proposition 27.ii et [6]).

Definition 34. Let \((X, a)\) a pretopological space of \(\mathcal{V}\) type. Let \(A \subset X, A \neq \emptyset\). \((A, a_A)\) is a greatest connected subspace of \((X, a)\) if and only if \((A, a_A)\) is a connected subspace of \((X, a)\) and \(\forall B, A \subset B \subset X \text{ and } A \neq B, (B, a_{AB})\) is not a connected subspace of \((X, a)\).

Proposition 35. Let \((X, a)\) a pretopological space of \(\mathcal{V}_s\) type. Let \(A \subset X, An \neq \emptyset\).

\(A\) is a greatest connected subspace of \(de (X, a)\) if and only if \(A\) is a greatest connected subset of \((X, a')\).
Proof. Obvious from Proposition 33.ii.

Conclusion. Decomposing a pretopological space \((X, a)\) of \(V_s\) type into greatest connected subspaces is equivalent to decomposing the pretopological space \((X, a')\) into greatest connected subspaces.

7. Application

Definition 36. Let \(X\) a non empty set and \(R\) a binary relationship defined on \(X\). Pretopology of ascending is characterized by:

\[
\forall A \subset X, a_a(A) = \{x \in X/R^{-1}(x) \cap A \neq \emptyset\} \cup A
\]

with \(R^{-1}(x) = \{y \in X/yRx\}\).

Pretopology of descending is characterized by:

\[
\forall A \subset X, a_d(A) = \{x \in X/R(x) \cap A \neq \emptyset\} \cup A
\]

with

\[
R(x) = \{y \in X/xy\}. 
\]

\(a_a\) and \(a_d\) lead to \(V_s\) type spaces (see [8]).

Remark 37. \(a'_a = a_d\)

Proof. \(a'_a(A) = \{x \in X/a_a(\{x\}) \cap A \neq \emptyset\}\) with

\[
a_a(\{x\}) = \{y \in X/R^{-1}(y) \cap \{y\} \neq \emptyset\} \cup \{x\} = \{y \in X/x \in R^{-1}(y)\} \cup \{x\} = \{y \in X/xy\} \cup \{x\} = R(x) \cup \{x\},
\]

so

\[
a'_a(A) = \{x \in X/R(x) \cap A \neq \emptyset\} \cup A = a_d(A).
\]

Conclusion. Decomposing a pretopological space \((X, a_a)\) into greatest (strongly) connected subspaces is equivalent to decomposing the pretopological space \((X, a_d)\) into greatest (strongly) connected subspaces.
References


