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JOIN PRESERVING MAPS AND RESIDUATED CONNECTIONS

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Abstract: In this paper, we investigate the relations between join preserving (meet preserving ,join-meet, meet-join) maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices.

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1. Introduction

Hájek [4] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [1], [2] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X\times Y}$ on a complete residuated lattice. Zhang [9], [10] introduced the fuzzy complete lattice which is defined by join and meet on fuzzy posets. It is an important mathematical tool for algebraic structure of fuzzy contexts [1]-[3], [5]-[8].

In this paper, we investigate the relations between join preserving (meet preserving, join-meet, meet-join) maps and residuated (dual residuated, residuated, Galois, dual Galois) connections in complete residuated lattices.

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Definition 1.1. (see [1], [4]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, ^* 0, 1)$ is a complete residuated lattice with the law of double negation;i.e. $x^{**} = x$.

Lemma 1.2. (see [1], [4]) *For each $x, y, z, x_i, y_i \in L$, we have the following properties.*

(1) *If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.*

(2) $x \rightarrow (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \rightarrow y_i)$.

(3) $(\bigvee_{i\in\Gamma} x_i) \rightarrow y = \bigwedge_{i\in\Gamma}(x_i \rightarrow y)$.

(4) $\bigwedge_{i\in\Gamma} y_i^* = (\bigvee_{i\in\Gamma} y_i)^*$ *and* $\bigvee_{i\in\Gamma} y_i^* = (\bigwedge_{i\in\Gamma} y_i)^*$.

(5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

(6) $x \odot y = (x \rightarrow y^*)^*$ *and* $x \rightarrow y = y^* \rightarrow x^*$.

(7) $x \odot (x \rightarrow y) \leq y$.

(8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

(9) $x \leq y \rightarrow z$ *iff* $y \leq x \rightarrow z$.

Definition 1.3. (see [9], [10]) Let X be a set. A function $e_X : X \times X \rightarrow L$ is called:

(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,

(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,

(E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

If e satisfies (E1) and (E2), (X, e_X) is a fuzzy preorder set. If e satisfies (E1), (E2) and (E3), (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

Remark 1.4. (1) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x\in X}(A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 1.2 (8).

Definition 1.5. (see [1], [3]) Let (X, e_X) and (Y, e_Y) be a fuzzy poset and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is called a Galois connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(x, g(y)).$$

(2)(e_X, f, g, e_Y) is called a dual Galois connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(g(y), x).$$

(3) (e_X, f, g, e_Y) is called a residuated connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(x, g(y)).$$

(4) (e_X, f, g, e_Y) is called a dual residuated connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(g(y), x).$$

Definition 1.6. (see [5], [9], [10]) Let (X, e_X) be a fuzzy poset and $A \in L^X$.

A point x_0 is called a join of A, denoted by $x_0 = \sqcup A$, if it satisfies

(J1) $A(x) \le e_X(x, x_0)$,

(J2) $\bigwedge_{x \in X}(A(x) \to e_X(x, y)) \le e_X(x_0, y)$.

A point x_1 is called a meet of A, denoted by $x_1 = \sqcap A$, if it satisfies

(M1) $A(x) \le e_X(x_1, x)$,

(M2) $\bigwedge_{x \in X}(A(x) \to e_X(y, x)) \le e_X(y, x_1)$.

The pair (X, e_X) is called a fuzzy complete lattice if for all $A \in L^X$, $\sqcup A$ and $\sqcap A$ exist.

Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be a fuzzy poset and $\eta : L^X \to L^Y$ a map. We define $\eta^{\to} : L^{L^X} \to L^{L^Y}, \eta^{\leftarrow} : L^{L^Y} \to L^{L^X}$ as

$$\eta^{\to}(\mathcal{U})(B) = \bigvee_{\eta(A)=B} \mathcal{U}(A), \quad \eta^{\leftarrow}(\mathcal{V})(A) = \mathcal{V}(\eta(A)).$$

(1) η is a join preserving map if $\eta(\sqcup \mathcal{U}) = \sqcup \eta^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(2) η is a meet preserving map if $\eta(\sqcap \mathcal{U}) = \sqcap \eta^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(3) η is a meet-join map if $\eta(\sqcap \mathcal{U}) = \sqcup \eta^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(4) η is a join-meet map if $\eta(\sqcup\mathcal{U}) = \sqcap\eta^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

In this paper, $1_x : X \to Y$ is a characteristic function,i.e. $1_x(x) = 1$, otherwise $1_x(y) = 0$. Moreover $\uparrow A, \downarrow B : L^X \to L^X$ is defined as $\uparrow A(C) = e_{L^X}(A, C)$, $\downarrow B(C) = e_{L^X}(C, B)$.

Remark 1.7. (see [5]) Let (L, e_L) be a fuzzy poset and $A \in L^L$.

(1) Since x_0 is a join of A iff $\bigwedge_{x\in L}(A(x) \to e_L(x, y)) = \bigwedge_{x\in L}(A(x) \to (x \Rightarrow y)) = \bigvee_{x\in L}(x \odot A(x)) \to y = e_L(x_0, y) = x_0 \to y$, then $x_0 = \sqcup A = \bigvee_{x\in L}(x \odot A(x))$.

(2) Since x_0 is a join of A iff $\bigwedge_{x\in L}(A(x) \to e_L(x, y)) = \bigwedge_{x\in L}(A(x) \to (y \to x)) = \bigwedge_{x\in L}(y \to (A(x) \to x)) = y \to \bigwedge_{x\in L}(A(x) \to x) = y \to \sqcap A$, then $\sqcap A = \bigwedge_{x\in L}(A(x) \to x)$.

Remark 1.8. (see [5]) Let (L^X, e_{L^X}) be a fuzzy poset and $\mathcal{U} \in L^{L^X}$.

(1) Since $\bigwedge_{A\in L^X}(\mathcal{U}(A) \to e_{L^X}(A, B)) = e_{L^X}(\bigvee_{A\in L^X}(\mathcal{U}(A) \odot A), B) = e_{L^X}(\sqcup\mathcal{U}, B)$, then $\sqcup\mathcal{U} = \bigvee_{A\in L^X}(\mathcal{U}(A) \odot A)$.

(2) Since $\bigwedge_{A\in L^X}(\mathcal{U}(A) \to e_{L^X}(B, A) = \bigwedge_{A\in L^X} e_{L^X}(B, (\mathcal{U}(A) \to A)) = e_{L^X}(B, \bigwedge_{A\in L^X}(\mathcal{U}(A) \to A))$, then $\sqcap\mathcal{U} = \bigwedge_{A\in L^X}(\mathcal{U}(A) \to A)$.

Theorem 1.9. (see [5]) *Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets and $\mathcal{U} \in L^{L^X}$. Then the following statements hold.*

(1) $\eta(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \eta(A_i)$ and $\eta(\alpha \odot A) = \alpha \to \eta(A)$ iff $\eta(\sqcup\mathcal{U}) = \sqcap\eta^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(2) $\phi(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \phi(A_i)$ and $\phi(\alpha \odot A) = \alpha \odot \phi(A)$ iff $\phi(\sqcup\mathcal{U}) = \sqcup\phi^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(3) $\omega(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \omega(A_i)$ and $\omega(\alpha \to A) = \alpha \odot \omega(A)$ iff $\omega(\sqcap\mathcal{U}) = \sqcup\omega^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

(4) $\psi(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \psi(A_i)$ and $\psi(\alpha \to A) = \alpha \to \psi(A)$ iff $\psi(\sqcap\mathcal{U}) = \sqcap\psi^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$.

2. Join Preserving Maps and Residuated Connections

Theorem 2.1. *Let (L^X, e_{L^X}) and (L^Y, e_{L^Y}) be fuzzy posets and $\mathcal{U} \in L^{L^X}$. Then the following statements hold.*

(1) $\eta(\sqcup\mathcal{U}) = \sqcap\eta^{\rightarrow}(\mathcal{U})$ *for all* $\mathcal{U} \in L^{L^X}$ *iff there exists* $\theta : L^Y \to L^X$ *with*

$$\begin{aligned}\theta(B) &= \bigvee\{A \in L^X \mid B \leq \eta(A)\} = \sqcup\eta^{\leftarrow}(\uparrow B)\\ &= \bigvee_{C\in L^X}(e_{L^Y}(B, \eta(C)) \odot C)\end{aligned}$$

such that $(e_{L^X}, \eta, \theta, e_{L^Y})$ *is a Galois connection.*

(2) $\phi(\sqcup\mathcal{U}) = \sqcup\phi^{\rightarrow}(\mathcal{U})$ *for all* $\mathcal{U} \in L^{L^X}$ *iff there exists* $\gamma : L^Y \to L^X$ *with*

$$\begin{aligned}\gamma(B) &= \bigvee\{A \in L^X \mid \phi(A) \leq B\} = \sqcup\phi^{\leftarrow}(\downarrow B)\\ &= \bigvee_{C\in L^X}(e_{L^Y}(\phi(C), B) \odot C)\end{aligned}$$

such that $(e_{L^X}, \phi, \gamma, e_{L^Y})$ *is a residuated connection.*

(3) $\omega(\sqcap\mathcal{U}) = \sqcup\omega^{\rightarrow}(\mathcal{U})$ *for all* $\mathcal{U} \in L^{L^X}$ *iff there exists* $\rho : L^Y \to L^X$ *with*

$$\begin{aligned}\rho(B) &= \bigwedge\{A \in L^X \mid \omega(A) \leq B\} = \sqcap\omega^{\leftarrow}(\downarrow B)\\ &= \bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C)\end{aligned}$$

such that $(e_{L^X}, \omega, \rho, e_{L^Y})$ *is an dual Galois connection.*

(4) *If* $\psi(\sqcap\mathcal{U}) = \sqcap\psi^{\rightarrow}(\mathcal{U})$ *for all* $\mathcal{U} \in L^{L^X}$*, then there exists* $\tau : L^Y \to L^X$ *with*

$$\tau(B) = \sqcap\psi^{\leftarrow}(\uparrow B) = \bigwedge_{C\in L^X}(e_{L^Y}(B, \psi(C)) \to C) = \bigwedge\{A \in L^X \mid \psi(A) \geq B\}$$

such that $(e_{L^X}, \psi, \tau, e_{L^Y})$ *is a dual residuated connection.*

Proof. (1) ($\Rightarrow$) Define $\theta : L^Y \to L^X$ as $\theta(B) = \bigvee\{C \in L^X \mid B \leq \eta(C)\}$. By Theorem 1.9 (1), since $\eta(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma}\eta(A_i)$ and $\eta(\alpha \odot A) = \alpha \to \eta(A)$, for $C = \bigvee_{x\in X}(C(x) \odot 1_x)$, we have

$$\eta(C)(y) = \bigwedge_{x\in X}(C(x) \to \eta(1_x)(y)).$$

Thus,

$$\begin{aligned}\theta(B)(x) &= \bigvee\{C(x) \mid \eta(C) \geq B\}\\ &= \bigvee\{C(x) \mid \bigwedge(C(x) \to \eta(1_x)(y)) \geq B(y)\}\\ &= \bigvee\{C(x) \mid \bigwedge(B(y) \to \eta(1_x)(y)) \geq C(x)\} \quad \text{(by Lemma 1.2 (9))}\\ &= \bigwedge(B(y) \to \eta(1_x)(y)).\end{aligned}$$

$$\begin{aligned}
& e_{L^Y}(B,\eta(A)) \\
&= \bigwedge_{y\in Y}(B(y)\to \eta(\bigvee_{x\in X}(A(x)\odot 1_x))(y)) \\
&= \bigwedge_{y\in Y}(B(y)\to \bigwedge_{x\in X}(A(x)\to \eta(1_x)(y))) \\
&= \bigwedge_{y\in Y}\bigwedge_{x\in X}(B(y)\to (A(x)\to \eta(1_x)(y))) \\
&= \bigwedge_{x\in X}(A(x)\to \bigwedge_{y\in Y}(B(y)\to \eta(1_x)(y)))\ \ \text{(by Lemma 1.2 (5))} \\
&= \bigwedge_{x\in X}(A(x)\to \theta(B)(x)) = e_{L^X}(A,\theta(B)).
\end{aligned}$$

Then $(e_{L^X},\eta,\theta,e_{L^Y})$ is a Galois connection.

By the definition of $\sqcup\eta^{\leftarrow}(\uparrow B)$, we have $\sqcup\eta^{\leftarrow}(\uparrow B) = \bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)$ from:

$$\begin{aligned}
e_{L^X}(\sqcup\eta^{\leftarrow}(\uparrow B),D) &= \bigwedge_{C\in L^Y}(\eta^{\leftarrow}(\uparrow B)(C)\to e_{L^X}(C,D)) \\
&= \bigwedge_{C\in L^Y}(e_{L^X}(\eta^{\leftarrow}(\uparrow B)(C)\odot C,D) \\
&= \bigwedge_{C\in L^Y} e_{L^X}(e_{L^Y}(B,\eta(C))\odot C,D) \\
&= e_{L^X}(\bigvee_{C\in L^Y}(e_{L^Y}(B,\eta(C))\odot C),D).
\end{aligned}$$

Finally, we will show that $\theta(B) = \bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)$. Since $B\odot (e_{L^Y}(B,\eta(C))\le \eta(C)$, by Theorem 1.9 (1),

$$\eta(\bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)) = \bigwedge_{C\in L^X}(e_{L^Y}(B,\eta(C))\to \eta(C)\ge B.$$

By the definition of θ, $\theta(B)\ge \bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)$. Since $(e_{L^X},\eta,\theta,e_{L^Y})$ is a Galois connection,

$$e_{L^Y}(B,\eta(\theta(B))) = e_{L^X}(\theta(B),\theta(B)) = 1.$$

Hence, $\theta(B) = e_{L^Y}(B,\eta(\theta(B)))\odot\theta(B)\le \bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)$. Thus $\theta(B) = \bigvee_{C\in L^X}(e_{L^Y}(B,\eta(C))\odot C)$.

($\Leftarrow$) Put $B_0 = \sqcap\eta^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}
e_{L^Y}(B,B_0) &= \bigwedge_{C\in L^Y}(\eta^{\rightarrow}(\mathcal{U})(C)\to e_{L^Y}(B,C)) \\
&= \bigwedge_{C\in L^Y}((\bigvee_{\eta(A)=C}\mathcal{U}(A)\to e_{L^Y}(B,\eta(A))) \\
&= \bigwedge_{A\in L^X}(\mathcal{U}(A)\to e_{L^Y}(B,\eta(A))) \\
&= \bigwedge_{A\in L^X}(\mathcal{U}(A)\to e_{L^X}(A,\theta(B))) \\
&= e_{L^X}(\sqcup\mathcal{U},\theta(B)) = e_{L^Y}(B,\eta(\sqcup\mathcal{U})).
\end{aligned}$$

Hence $B_0 = \sqcap\eta^{\rightarrow}(\mathcal{U}) = \eta(\sqcup\mathcal{U})$.

(2) ($\Rightarrow$) Define $\gamma : L^Y \to L^X$ as $\gamma(B) = \bigvee\{C\in L^X \mid \phi(C)\le B\}$. By Theorem 1.9(2), since $\phi(\bigvee_{i\in\Gamma}A_i) = \bigvee_{i\in\Gamma}\phi(A_i)$ and $\phi(\alpha\odot A) = \alpha\odot\phi(A)$, for $C = \bigvee_{x\in X}(C(x)\odot 1_x)$, we have

$$\phi(C)(y) = \bigvee_{x\in X}(C(x)\odot\phi(1_x)(y)).$$

Thus

$$\begin{aligned}\gamma(B)(x) &= \bigvee\{C(x) \mid \phi(C) \le B\}\\ &= \bigvee\{C(x) \mid \bigvee(C(x)\odot\phi(1_x)(y)) \le B(y)\}\\ &= \bigvee\{C(x) \mid C(x) \le \bigwedge_{y\in Y}(\phi(1_x)(y)\to B(y))\}\\ &= \bigwedge_{y\in Y}(\phi(1_x)(y)\to B(y)).\end{aligned}$$

$$\begin{aligned}e_{L^Y}(\phi(A),B) &= \bigwedge_{y\in Y}(\phi(\bigvee_{x\in X}(A(x)\odot 1_x))(y)\to B(y))\\ &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A(x)\odot\phi(1_x)(y)\to B(y))\\ &= \bigwedge_{y\in Y}\bigwedge_{x\in X}(A(x)\to(\phi(1_x)(y)\to B(y)))\\ &= \bigwedge_{x\in X}(A(x)\to\bigwedge_{y\in Y}(\phi(1_x)(y)\to B(y)))\\ &= \bigwedge_{x\in X}(A(x)\to\gamma(B)(x)) = e_{L^X}(A,\gamma(B)).\end{aligned}$$

By the definition of $\sqcup\phi^{\leftarrow}(\downarrow B)$, we have $\sqcup\phi^{\leftarrow}(\downarrow B) = \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot C)$ from:

$$\begin{aligned}e_{L^X}(\sqcup\phi^{\leftarrow}(\downarrow B),D) &= \bigwedge_{C\in L^Y}(\phi^{\leftarrow}(\downarrow B)(C)\to e_{L^X}(C,D))\\ &= \bigwedge_{C\in L^Y}(e_{L^X}(\phi^{\leftarrow}(\downarrow B)(C)\odot C,D)\\ &= \bigwedge_{C\in L^Y} e_{L^X}(e_{L^Y}(\phi(C),B)\odot C,D)\\ &= e_{L^X}(\bigvee_{C\in L^Y}(e_{L^Y}(\phi(C),B)\odot C),D).\end{aligned}$$

Finally, we will show that $\gamma(B) = \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot C)$. Since

$$\phi(\bigvee_{C\in L^X}(e_{L^Y}(B,F(C))\odot C)) = \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot\phi(C) \le B,$$

by the definition of γ, $\gamma(B) \ge \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot C)$. Since $(e_{L^X},\phi,\gamma,e_{L^Y})$ is a residuated connection, then

$$e_{L^Y}(\phi(\gamma(B)),B) = e_{L^X}(\gamma(B),\gamma(B)) = 1.$$

Hence $\gamma(B) = e_{L^Y}(\phi(\gamma(B)),B)\odot\gamma(B) \le \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot C)$. Thus, $\gamma(B) \le \bigvee_{C\in L^X}(e_{L^Y}(\phi(C),B)\odot C)$.

($\Leftarrow$) Put $B_1 = \sqcup\phi^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}e_{L^Y}(B_1,B) &= \bigwedge_{C\in L^Y}(\phi^{\rightarrow}(\mathcal{U})(C)\to e_{L^Y}(C,B))\\ &= \bigwedge_{C\in L^Y}((\bigvee_{\phi(A)=C}\mathcal{U}(A)\to e_{L^Y}(\phi(A),B))\\ &= \bigwedge_{A\in L^X}(\mathcal{U}(A)\to e_{L^Y}(\phi(A),B))\\ &= \bigwedge_{A\in L^X}(\mathcal{U}(A)\to e_{L^X}(A,\rho(B)))\\ &= e_{L^X}(\sqcup\mathcal{U},\rho(B)) = e_{L^Y}(\phi(\sqcup\mathcal{U}),B).\end{aligned}$$

Hence $B_1 = \phi(\sqcup\mathcal{U}) = \sqcup\phi^{\rightarrow}(\mathcal{U})$.

(3) ($\Rightarrow$) Define $\rho : L^Y \to L^X$ as $\rho(B) = \bigwedge\{C \in L^X \mid \omega(C) \leq B\}$. By Theorem 1.9(3), since $\omega(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \omega(A_i)$ and $\omega(\alpha \to A) = \alpha \odot \omega(A)$, for $C = \bigwedge_{x\in X}(1_x \to C(x))$, we have $\omega(\bigwedge_{x\in X}(1_x \to C(x))) = \omega(\bigwedge_{x\in X}(C^*(x) \to 1^*_x)) = \bigvee(C^*(x) \odot \omega(1^*_x)(y))$. Thus

$$\begin{aligned} \rho(B)(x) &= \bigwedge\{C(x) \mid \omega(C) \leq B\} \\ &= \bigwedge\{C(x) \mid \omega(\bigwedge_{x\in X}(1_x \to C(x))) \leq B\} \\ &= \bigwedge\{C(x) \mid \bigvee(C^*(x) \odot \omega(1^*_x)(y) \leq B(y)\} \\ &= \bigwedge\{C(x) \mid C^*(x) \leq \bigwedge_{y\in Y}(\omega(1^*_x)(y) \to B(y))\} \\ &= \bigvee_{y\in Y}(\omega(1^*_x)(y) \odot B^*(y)) \text{ (by Lemma 1.2 (6))}. \end{aligned}$$

Since $a^* \odot b \to c = (a^* \odot b \odot c^*)^* = c^* \odot b \to a$, we have

$$\begin{aligned} e_{L^Y}(\omega(A), B) &= \bigwedge_{y\in Y}(\omega(\bigwedge_{x\in X}(1_x \to A(x))) \to B(y)) \\ &= \bigwedge_{y\in Y}(\bigvee_{x\in X}(A^*(x) \odot \omega(1^*_x)(y) \to B(y)) \\ &= \bigwedge_{y\in Y} \bigwedge_{x\in X}((B^*(y) \odot \omega(1^*_x)(y) \to A(x)) \\ &= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B^*(y) \odot \omega(1^*_x)(y) \to A(x)) \\ &= \bigwedge_{x\in X}(\rho(B)(x) \to A(x)) = e_{L^X}(\rho(B), A). \end{aligned}$$

By the definition of $\sqcap\omega^{\leftarrow}(\downarrow B)$, we have $\sqcap\omega^{\leftarrow}(\downarrow B) = \bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C)$ from:

$$\begin{aligned} e_{L^X}(D, \sqcup\omega^{\leftarrow}(\downarrow B)) &= \bigwedge_{C\in L^Y}(\omega^{\leftarrow}(\downarrow B)(C) \to e_{L^X}(D, C)) \\ &= \bigwedge_{C\in L^Y}(e_{L^X}(\omega^{\leftarrow}(\downarrow B)(C) \odot D, C) \\ &= \bigwedge_{C\in L^Y} e_{L^X}(e_{L^Y}(\omega(C), B) \odot D, C) \\ &= e_{L^X}(D, \bigwedge_{C\in L^Y}(e_{L^Y}(\omega(C), B) \to C)). \end{aligned}$$

Finally, we will show that $\rho(B) = \bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C)$. Since

$$\omega(\bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C) = \bigvee_{C\in L^X}(e_{L^Y}(\omega(C), B) \odot \omega(C) \leq B,$$

by the definition of ρ, $\rho(B) \leq \bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C)$. Since $(e_{L^X}, \omega, \rho, e_{L^Y})$ is an dual Galois connection,

$$e_{L^Y}(\omega(\rho(B)), B) = e_{L^X}(\rho(B), \rho(B)) = 1.$$

Hence, $\rho(B) = e_{L^Y}(\omega(\rho(B)), B) \to \rho(B) \geq \bigwedge_{C\in L^X}(e_{L^Y}(\omega(C), B) \to C)$.

($\Leftarrow$) Put $B_1 = \sqcup\omega^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned} e_{L^Y}(B_1, B) &= \bigwedge_{C\in L^Y}(\omega^{\rightarrow}(\mathcal{U})(C) \to e_{L^Y}(C, B)) \\ &= \bigwedge_{C\in L^Y}((\bigvee_{\omega(A)=C}\mathcal{U}(A) \to e_{L^Y}(\omega(A), B)) \\ &= \bigwedge_{C\in L^Y} \bigwedge_{\omega(A)=C}(\mathcal{U}(A) \to e_{L^Y}(\omega(A), B)) \\ &= \bigwedge_{A\in L^X}(\mathcal{U}(A) \to e_{L^X}(\rho(B), A)) \\ &= e_{L^X}(\rho(B), \sqcap\mathcal{U}) = e_{L^Y}(\omega(\sqcap\mathcal{U}), B). \end{aligned}$$

Hence $B_1 = \omega(\sqcap\mathcal{U}) = \sqcup\omega^{\to}(\mathcal{U})$.

(4) ($\Rightarrow$) Define $\tau : L^Y \to L^X$ as $\tau(B) = \bigwedge\{C \in L^X \mid \psi(C) \geq B\}$. By Theorem 1.9(4), since $\psi(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma}\psi(A_i)$ and $\psi(\alpha \to A) = \alpha \to \psi(A)$, for $C = \bigwedge_{x\in X}(1_x \to C(x))$, we have $\psi(\bigwedge_{x\in X}(1_x \to C(x))) = \psi(\bigwedge_{x\in X}(C^*(x) \to 1^*_x)) = \bigwedge_{x\in X}(C^*(x) \to \psi(1^*_x))$.

Since $\psi(\bigwedge_{x\in X}(1_x \to C(x))) = \bigwedge_{x\in X}(C^*(x) \to \psi(1^*_x))$, we have

$$\begin{aligned}\tau(B)(x) &= \bigwedge\{C(x) \mid \psi(C) \geq B\}\\ &= \bigwedge\{C(x) \mid \bigwedge_{x\in X}(C^*(x) \to \psi(1^*_x)) \geq B\}\\ &= \bigwedge\{C(x) \mid \bigwedge_{y\in Y}(B(y) \to \psi(1^*_x)) \geq C^*(x)\}\\ &= \bigwedge\{C(x) \mid \bigvee_{y\in Y}(B(y) \odot \psi(1^*_x)^*(y)) \leq C(x)\}\\ &= \bigvee_{y\in Y}(B(y) \odot \psi(1^*_x)^*(y)).\end{aligned}$$

$$\begin{aligned}&e_{L^X}(\tau(B), A)\\ &= \bigwedge_{x\in X}(\tau(B)(x) \to A(x))\\ &= \bigwedge_{x\in X}(\bigvee_{y\in Y}(B(y) \odot \psi(1^*_x)^*(y)) \to A(x))\\ &= \bigwedge_{x\in X}\bigwedge_{y\in Y}(B(y) \to (\psi(1^*_x)^*(y) \to A(x)))\\ &= \bigwedge_{y\in Y}(B(y) \to \bigwedge_{x\in X}(A^*(x) \to \psi(1^*_x)(y))) \text{ (by Lemma 1.2(6))}\\ &= \bigwedge_{y\in Y}(B(y) \to \psi(A)(y)) = e_{L^X}(B, \psi(A)).\end{aligned}$$

By the definition of $\sqcap\psi^{\leftarrow}(\uparrow B)$, we have

$$\sqcap\psi^{\leftarrow}(\uparrow B) = \bigwedge_{C\in L^X}(e_{L^Y}(B, \psi(C)) \to C)$$

from:

$$\begin{aligned}e_{L^X}(D, \sqcap\psi^{\leftarrow}(\uparrow B)) &= \bigwedge_{C\in L^Y}(\psi^{\leftarrow}(\uparrow B)(C) \to e_{L^X}(D, C))\\ &= \bigwedge_{C\in L^Y}(e_{L^X}(\psi^{\leftarrow}(\uparrow B)(C) \odot D, C)\\ &= \bigwedge_{C\in L^Y} e_{L^X}(e_{L^Y}(B, \psi(C)) \odot D, C)\\ &= e_{L^X}(D, \bigwedge_{C\in L^Y}(e_{L^Y}(B, \psi(C)) \to C)).\end{aligned}$$

Finally, we will show that

$$\tau(B) = \bigwedge_{C\in L^X}(e_{L^Y}(B, \psi(C)) \to C).$$

Since $e_{L^Y}(B, \psi(C)) \odot B \leq \psi(C)$ implies $B \leq e_{L^Y}(B, \psi(C)) \to \psi(C)$, we have

$$\psi(\bigwedge_{C\in L^X}(e_{L^Y}(B, \psi(C)) \to C) = \bigwedge_{C\in L^X}(e_{L^Y}(B, \psi(C)) \to \psi(C) \geq B.$$

By the definition of τ,

$$\tau(B) \le \bigwedge_{C\in L^X} (e_{L^Y}(\psi(C), B) \to C).$$

Since $(e_{L^X}, \psi, \tau, e_{L^Y})$ is a dual residuated connection,

$$e_{L^X}(\psi(\tau(B)), B) = e_{L^Y}(\tau(B), \tau(B)).$$

Hence, $\tau(B) = e_{L^Y}(\psi(\tau(B)), B) \to \tau(B)) \ge \bigwedge_{C\in L^X}(e_{L^Y}(\psi(C), B) \to C)$. Thus $\tau(B) = \bigwedge_{C\in L^X}(e_{L^Y}(\psi(C), B) \to C)$.

($\Leftarrow$) Put $B_0 = \sqcap\phi^{\to}(\mathcal{U})$. Then

$$\begin{aligned} e_{L^Y}(B, B_0) &= \textstyle\bigwedge_{C\in L^Y}(\psi^{\to}(\mathcal{U})(C) \to e_{L^Y}(B, C)) \\ &= \textstyle\bigwedge_{C\in L^Y}((\bigvee_{\psi(A)=C}\mathcal{U}(A) \to e_{L^Y}(B, \psi(A))) \\ &= \textstyle\bigwedge_{A\in L^X}(\mathcal{U}(A) \to e_{L^Y}(B, \psi(A))) \\ &= \textstyle\bigwedge_{A\in L^X}(\mathcal{U}(A) \to e_{L^X}(\tau(B), A)) \\ &= e_{L^X}(\tau(B), \sqcap\mathcal{U}) = e_{L^Y}(B, \psi(\sqcap\mathcal{U})). \end{aligned}$$

Hence $B_0 = \psi(\sqcap\mathcal{U}) = \sqcap\psi^{\to}(\mathcal{U})$.

Theorem 2.2. *Let $R \in L^{X\times Y}$ be a fuzzy relation. Define maps $\eta_R, \phi_R, \omega_R, \psi_R : L^X \to L^Y$ as follows:*

$$\begin{aligned} \eta_R(A)(y) &= \textstyle\bigwedge_{x\in X}(A(x) \to R(x, y)), \\ \phi_R(A)(y) &= \textstyle\bigvee_{x\in X}(A(x) \odot R(x, y)), \\ \omega_R(A)(y) &= \textstyle\bigvee_{x\in X}(A^*(x) \odot R(x, y)), \\ \psi_R(A)(y) &= \textstyle\bigwedge_{x\in X}(R(x, y) \to A(x)). \end{aligned}$$

Then the following statements hold.

(1) There exists $\theta_R : L^Y \to L^X$ with $\theta_R(B)(x) = \bigwedge_{y\in Y}(B(y) \to R(x, y))$ such that $(e_{L^X}, \eta_R, \theta_R, e_{L^Y})$ is a Galois connection.

(2) There exists $\gamma_R : L^Y \to L^X$ with $\gamma_R(B)(x) = \bigwedge_{y\in Y}(R(x, y) \to B(y))$ such that $(e_{L^X}, \phi_R, \gamma_R, e_{L^Y})$ is a residuated connection.

(3) There exists $\rho_R : L^Y \to L^X$ with $\rho_R(B)(x) = \bigvee_{y\in Y}(R(x, y) \odot B^*(y))$ such that $(e_{L^X}, \omega_R, \rho_R, e_{L^Y})$ is an dual Galois connection.

(4) There exists $\tau_R : L^Y \to L^X$ with $\tau_R(B)(y) = \bigvee_{y\in Y}(B(y) \odot R(x, y))$ such that $(e_{L^X}, \psi_R, \tau_R, e_{L^Y})$ is a dual residuated connection.

Proof. (1) Since $\eta_R(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \eta_R(A_i)$ and $\eta_R(\alpha\odot A) = \bigwedge_{x\in X}((\alpha\odot A(x)) \to R(x,y)) = \alpha \to \bigwedge_{x\in X}(A(x) \to R(x,y)) = \alpha \to \eta_R(A)$, by Theorem 1.9(1), we have $\eta_R(\sqcup\mathcal{U}) = \sqcap\eta_R^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$. By Theorem 2.1 (1), there exists θ_R such that $(e_{L^X}, \eta_R, \theta_R, e_{L^Y})$ is a Galois connection with

$$\begin{aligned}\theta_R(B)(x) &= \sqcup\eta_R^{\leftarrow}(\uparrow B)(x) = \bigvee_{C\in L^X}(e_{L^Y}(B, \eta_R(C)) \odot C(x)\\ &= \bigvee\{A(x) \mid B \le \eta_R(A)\} = \bigwedge_{y\in Y}(B(y) \to \eta_R(1_x)(y))\\ &= \bigwedge_{y\in Y}(B(y) \to R(x,y)).\end{aligned}$$

(2) Since $\phi_R(\bigvee_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \phi_R(A_i)$ and $\phi_R(\alpha\odot A) = \alpha\odot\phi_R(A)$, by Theorem 1.9(2), we have $\phi_R(\sqcup\mathcal{U}) = \sqcup\phi_R^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$. By Theorem 2.1(2), there exists γ_R such that $(e_{L^X}, \phi_R, \gamma_R, e_{L^Y})$ is a residuated connection with

$$\begin{aligned}\gamma_R(B)(x) &= \sqcup\phi_R^{\leftarrow}(\downarrow B)(x) = \bigvee_{C\in L^X}(e_{L^Y}(\phi_R(C), B) \odot C(x))\\ &= \bigvee\{A(x) \mid \phi_R(A) \le B\} = \bigwedge_{y\in Y}(\phi_R(1_x)(y) \to B(y))\\ &= \bigwedge_{y\in Y}(R(x,y) \to B(y)).\end{aligned}$$

(3) Since $\omega_R(\bigwedge_{i\in\Gamma} A_i) = \bigvee_{i\in\Gamma} \omega_R(A_i)$ and $\omega_R(\alpha\to A) = \bigvee_{x\in X}((\alpha\to A)^*(x)\odot R(x,y)) = \bigvee_{x\in X}((\alpha\odot A^*(x)\odot R(x,y)) = \alpha\odot\omega_R(A)$, by Theorem 1.9(3), $\omega_R(\sqcap\mathcal{U}) = \sqcup\omega_R^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$. By Theorem 2.1(3), there exists ρ_R such that $(e_{L^X}, \omega_R, \rho_R, e_{L^Y})$ is an dual Galois connection with

$$\begin{aligned}\rho(B)(x) &= \sqcap\omega_R^{\leftarrow}(\downarrow B)(x) = \bigwedge_{C\in L^X}(e_{L^Y}(\omega_R(C), B) \to C(x))\\ &= \bigwedge\{A(x) \mid \omega_R(A) \le B\} = \bigvee_{y\in Y}(\omega_R(1_x^*)(y) \odot B^*(y))\\ &= \bigvee_{y\in Y}(R(x,y) \odot B^*(y)).\end{aligned}$$

(4) Since $\psi_R(\bigwedge_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \psi_R(A_i)$ and $\psi_R(\alpha\to A) = \bigwedge_{x\in X}(R(x,y) \to (\alpha\to A(x))) = \bigwedge_{x\in X}(\alpha\to (R(x,y)\to A(x))) = \alpha\to\psi_R(A)$, by Theorem 1.9(4), $\psi_R(\sqcap\mathcal{U}) = \sqcap\psi_R^{\to}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$. By Theorem 2.1(4), there exists $\tau_R : L^Y \to L^X$ such that $(e_{L^X}, \psi_R, \tau_R, e_{L^Y})$ is a dual residuated connection with

$$\begin{aligned}\tau_R(B)(x) &= \sqcap\psi_R^{\leftarrow}(\uparrow B)(x) = \bigwedge_{C\in L^X}(e_{L^Y}(B, \psi_R(C)) \to C(x))\\ &= \bigwedge\{A \in L^X \mid \psi_R(A) \ge B\} = \bigvee_{y\in Y}(B(y) \odot \psi_R(1_x^*)^*(y))\\ &= \bigvee_{y\in Y}(B(y) \odot R(x,y)).\end{aligned}$$

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