FUZZY CONNECTIONS ON UPPER SETS

Yong Chan Kim^{1 §}, Jung Mi Ko²

^{1,2}Department of Mathematics

Gangneung-Wonju National University

Gangneung, Gangwondo 210-702, KOREA

Abstract: In this paper, we investigate residuated (dual residuated, residuated, Galois, dual Galois) connections on upper (lower) sets in complete residuated lattices.

AMS Subject Classification: 03E72, 03G10, 06A15, 06F07

Key Words: complete residuated lattices, upper(lower) sets, isotone (anti-tone) maps, residuated (dual residuated, residuated, Galois, dual Galois) connections

1. Introduction

Hájek [7] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Bělohlávek [1]-[3] developed the notion of fuzzy contexts using Galois connections with $R \in L^{X \times Y}$ on a complete residuated lattice. Georgescue and Popescu [5], [6] introduced the non-commutative fuzzy connection on generalized residuated lattice without commutative conditions. Garcia [4] investigated fuzzy connections categorically. Zhang [11], [12] introduced the notions of fuzzy completeness, join and meet for a fuzzy set. It is an important mathematical tool for algebraic structure of fuzzy contexts [1]-[3], [8]-[12].

In this paper, we investigate residuated (dual residuated, residuated, Galois, dual Galois) connections induced by various function on upper (lower) sets in

Received: February 28, 2013

© 2013 Academic Publications, Ltd.
url: www.acadpubl.eu

[§]Correspondence author

complete residuated lattices. We give their examples.

Definition 1.1. (see [1], [7]) An algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is called a complete residuated lattice if it satisfies the following conditions:

(C1) $L = (L, \leq, \vee, \wedge, 1, 0)$ is a complete lattice with the greatest element 1 and the least element 0;

(C2) $(L, \odot, 1)$ is a commutative monoid;

(C3) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for $x, y, z \in L$.

In this paper, we assume $(L, \wedge, \vee, \odot, \rightarrow, {}^*0, 1)$ is a complete residuated lattice with the law of double negation; i.e. $x^{**} = x$.

Lemma 1.2. (see [1], [7]) For each $x, y, z, x_i, y_i \in L$, we have the following properties.

- (1) If $y \leq z$, $(x \odot y) \leq (x \odot z)$, $x \rightarrow y \leq x \rightarrow z$ and $z \rightarrow x \leq y \rightarrow x$.
- (2) $x \rightarrow (\bigwedge_{i \in \Gamma} y_i) = \bigwedge_{i \in \Gamma} (x \rightarrow y_i)$.
- (3) $(\bigvee_{i \in \Gamma} x_i) \rightarrow y = \bigwedge_{i \in \Gamma} (x_i \rightarrow y)$.
- (4) $\bigwedge_{i \in \Gamma} y_i^* = (\bigvee_{i \in \Gamma} y_i)^*$ and $\bigvee_{i \in \Gamma} y_i^* = (\bigwedge_{i \in \Gamma} y_i)^*$.
- (5) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.
- (6) $x \odot y = (x \rightarrow y^*)^*$ and $x \rightarrow y = y^* \rightarrow x^*$.
- (7) $x \odot (x \rightarrow y) \leq y$ and $y \rightarrow z \leq x \odot y \rightarrow x \odot z$.
- (8) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.
- (9) $x \leq y \rightarrow z$ iff $y \leq x \rightarrow z$.
- (10) $y \rightarrow z \leq (x \rightarrow y) \rightarrow (x \rightarrow z)$ and $y \rightarrow z \leq (z \rightarrow x) \rightarrow (y \rightarrow x)$.
- (11) $(\bigwedge_{i \in \Gamma} x_i) \rightarrow (\bigwedge_{i \in \Gamma} y_i) \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$.
- (12) $(\bigvee_{i \in \Gamma} x_i) \rightarrow (\bigvee_{i \in \Gamma} y_i) \geq \bigwedge_{i \in \Gamma} (x_i \rightarrow y_i)$.

Definition 1.3. (see [1]-[3]) Let X be a set. A function $e_X : X \times X \rightarrow L$ is called fuzzy partially order if it satisfies

- (E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,
- (E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$,
- (E3) if $e_X(x, y) = e_X(y, x) = 1$, then $x = y$.

The pair (X, e_X) is a fuzzy partially order set (simply, fuzzy poset).

A fuzzy set A is called an upper (lower) set if $A(x) \odot e_X(x, y) \leq A(y)$ ($A(x) \odot e_X(y, x) \leq A(y)$) for all $x, y \in X$. $U(L^X)$ ($D(L^X)$) is the collection of upper (lower) sets on L^X .

Remark 1.4. (1) We define a function $e_{L^X} : L^X \times L^X \rightarrow L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$. Then (L^X, e_{L^X}) is a fuzzy poset from Lemma 1.2 (8).

(2) Define a function $e_{U(L^X)} : U(L^X) \times U(L^X) \rightarrow L$ as $e_{U(L^X)}(A, B) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))$. Then $(U(L^X), e_{U(L^X)})$ is a fuzzy poset. Similarly, the pair $(D(L^X), e_{D(L^X)})$ is a fuzzy poset.

Definition 1.5. (see [1]-[3]) Let (X, e_X) and (Y, e_Y) be a fuzzy poset and $f : X \rightarrow Y$ and $g : Y \rightarrow X$ maps.

(1) (e_X, f, g, e_Y) is called a Galois connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(x, g(y)).$$

(2) (e_X, f, g, e_Y) is called a dual Galois connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(g(y), x).$$

(3) (e_X, f, g, e_Y) is called a residuated connection if for all $x \in X, y \in Y$,

$$e_Y(f(x), y) = e_X(x, g(y)).$$

(4) (e_X, f, g, e_Y) is called a dual residuated connection if for all $x \in X, y \in Y$,

$$e_Y(y, f(x)) = e_X(g(y), x).$$

(5) A map $f : (X, e_X) \rightarrow (Y, e_Y)$ is called an isotone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(x), f(z))$.

(6) A map $f : (X, e_X) \rightarrow (Y, e_Y)$ is called an antitone map if for all $x, z \in X$, $e_X(x, z) \leq e_Y(f(z), f(x))$.

Definition 1.6. (see [11], [12]) Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) A point x_0 is called a join (or supremum) of A , denoted by $x_0 = \sqcup A$, if it satisfies

$$(J1) \ A(x) \leq e_X(x, x_0),$$

$$(J2) \ \bigwedge_{x \in X} (A(x) \rightarrow e_X(x, y) \leq e_X(x_0, y)).$$

A point x_1 is called a meet (or infimum) of A , denoted by $x_1 = \sqcap A$, if it satisfies

$$(M1) \ A(x) \leq e_X(x_1, x),$$

$$(M2) \ \bigwedge_{x \in X} (A(x) \rightarrow e_X(y, x) \leq e_X(y, x_1)).$$

The pair (X, e_X) is called a fuzzy complete lattice if for all $A \in L^X$, $\sqcup A$ and $\sqcap A$ exist.

Let (X, e_X) and (Y, e_Y) be fuzzy complete lattice and $f : X \rightarrow Y$ a map.

(2) f is a join preserving map if $f(\sqcup A) = \sqcup f^{\rightarrow}(A)$ for all $A \in L^X$, where $f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$.

(3) f is a meet preserving map if $f(\sqcap A) = \sqcap f^{\rightarrow}(A)$ for all $A \in L^X$.

(4) f is a meet-join map if $f(\sqcap A) = \sqcup f^{\rightarrow}(A)$ for all $A \in L^X$.

(5) f is a join-meet map if $f(\sqcup A) = \sqcap f^{\rightarrow}(A)$ for all $A \in L^X$.

Remark 1.7. (see [11], [12]) Let (X, e_X) be a fuzzy poset and $A \in L^X$.

(1) If x_0 is a join of A , then it is unique because $e_X(x_0, y) = e_X(y_0, y)$ for all $y \in X$, put $y = x_0$ or $y = y_0$, then $e_X(x_0, y_0) = e_X(y_0, x_0) = \top$ implies $x_0 = y_0$. Similarly, if a meet of A exist, then it is unique.

(2) x_0 is a join of A iff $\bigwedge_{x \in X} (A(x) \rightarrow e_X(x, y)) = e_X(x_0, y)$.

(3) x_1 is a meet of A iff $\bigwedge_{x \in X} (A(x) \rightarrow e_X(y, x)) = e_X(y, x_1)$.

2. Fuzzy Connections on Upper Sets

indent

Theorem 2.1. Let (X, e_X) be a fuzzy poset. Then:

(1) $(U(L^X), e_{L^X})$ is a complete.

(2) $(D(L^X), e_{L^X})$ is a complete.

Proof. (1) For all $\mathcal{U} : U(L^X) \rightarrow L$, we have $\sqcup \mathcal{U} = \bigvee_{A \in U(L^X)} \mathcal{U}(A) \odot A$ and $\sqcap \mathcal{U} = \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow A)$ from: by Lemma 1.2 (5),

$$\begin{aligned} e_{U(L^X)}(\sqcup \mathcal{U}, B) &= \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow e_{U(L^X)}(A, B)) \\ &= e_{U(L^X)}(\bigvee_{A \in U(L^X)} (\mathcal{U}(A) \odot A), B), \end{aligned}$$

$$\begin{aligned} e_{U(L^X)}(B, \sqcap \mathcal{U}) &= \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow e_{L^X}(B, A)) \\ &= e_{U(L^X)}(B, \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow A)). \end{aligned}$$

Since

$$\begin{aligned} \bigvee_{A \in U(L^X)} \mathcal{U}(A) \odot A(x) \odot e_X(x, y) &\leq \bigvee_{A \in U(L^X)} \mathcal{U}(A) \odot A(y) \\ \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow A(x)) \odot e_X(x, y) &\leq \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow A(y)) \end{aligned}$$

$\bigvee_{A \in U(L^X)} \mathcal{U}(A) \odot A \in U(L^X)$ and $\sqcap \mathcal{U} = \bigwedge_{A \in U(L^X)} (\mathcal{U}(A) \rightarrow A) \in U(L^X)$.

(2) It similarly proved as (1).

Theorem 2.2. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $f : X \rightarrow Y$ be an isotone map. Define $f_l^\rightarrow, f_u^\rightarrow : L^X \rightarrow L^Y$ as

$$f_l^\rightarrow(A)(y) = \bigvee_{x \in X} (A(x) \odot e_Y(y, f(x))),$$

$$f_u^{\rightarrow}(A)(y) = \bigvee_{x \in X} (A(x) \odot e_Y(f(x), y)),$$

and $f_l^{\leftarrow}, f_u^{\leftarrow} : L^Y \rightarrow L^X$ as

$$f_l^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B(f(z)) \odot e_X(x, z)),$$

$$f_u^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B(f(z)) \odot e_X(z, x)).$$

Then the following properties hold.

(1) $(D(L^X), f_l^{\rightarrow}, f_l^{\leftarrow}, D(L^Y))$ is a residuated connection and $f_l^{\rightarrow}(\sqcup \mathcal{U}) = \sqcup (f_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(f_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{f_l^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(2) $(U(L^X), f_u^{\rightarrow}, f_u^{\leftarrow}, D(L^X))$ is a residuated connection and $f_u^{\rightarrow}(\sqcup \mathcal{U}) = \sqcup (f_u^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(f_u^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{f_u^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(3) $\sqcup f_l^{\rightarrow}(A) \leq f(\sqcup A)$ and $f(\sqcap A) \leq \sqcap f_u^{\rightarrow}(A)$.

(4) $e_{L^X}(A, B) \leq e_{L^Y}(f_l^{\rightarrow}(A), f_l^{\rightarrow}(B))$ and $e_{L^X}(A, B) \leq e_{L^Y}(f_u^{\rightarrow}(A), f_u^{\rightarrow}(B))$.

(5) $e_{L^Y}(C, D) \leq e_{L^X}(f_l^{\leftarrow}(C), f_l^{\leftarrow}(D))$ and $e_{L^Y}(A, B) \leq e_{L^X}(f_u^{\leftarrow}(C), f_u^{\leftarrow}(D))$.

Proof. (1) Since $f_l^{\rightarrow}(A)(y_1) \odot e_Y(y_2, y_1) = \bigvee_{x \in X} (A(x) \odot e_Y(y_1, f(x))) \odot e_Y(y_2, y_1) \leq \bigvee_{x \in X} (A(x) \odot e_Y(y_2, f(x)))$, then $f_l^{\rightarrow}(A) \in D(L^Y)$. Similarly, $f_l^{\leftarrow}(B) \in D(L^X)$. By Lemma 1.2 (5), we have

$$\begin{aligned} e_{D(L^X)}(f_l^{\rightarrow}(A), B) &= \bigwedge_{y \in Y} (f_l^{\rightarrow}(A)(y) \rightarrow B(y)) \\ &= \bigwedge_{y \in Y} (\bigvee_{x \in X} (A(x) \odot e_Y(y, f(x))) \rightarrow B(y)) \\ &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (A(x) \rightarrow (e_Y(y, f(x)) \rightarrow B(y))) \\ &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{y \in Y} (e_Y(y, f(x)) \rightarrow B(y))) \\ &\leq \bigwedge_{x \in X} (A(x) \rightarrow (e_Y(f(x), f(x)) \rightarrow B(f(x)))) \\ &= \bigwedge_{x \in X} (A(x) \rightarrow B(f(x))) \\ &\leq \bigwedge_{x \in X} (A(x) \rightarrow \bigvee_{z \in X} (B(f(z)) \odot e_X(x, z))) \\ &= \bigwedge_{x \in X} (A(x) \rightarrow f_l^{\leftarrow}(B)(x)) \\ &= e_{D(L^X)}(A, f_l^{\leftarrow}(B)). \end{aligned}$$

Let $a \leq e_{D(L^X)}(A, f_l^{\leftarrow}(B)) \leq A(x) \rightarrow f_l^{\leftarrow}(B)(x)$ be given. Then $a \odot A(x) \leq$

$$f_l^{\leftarrow}(B)(x).$$

$$\begin{aligned}
 a \odot f_l^{\rightarrow}(A)(y) &= a \odot \bigvee_{x \in X} (A(x) \odot e_Y(y, f(x))) \\
 &= \bigvee_{x \in X} (a \odot A(x) \odot e_Y(y, f(x))) \\
 &\leq \bigvee_{x \in X} (f_l^{\leftarrow}(B)(x) \odot e_Y(y, f(x))) \\
 &= \bigvee_{x \in X} \bigvee_{z \in X} (B(f(z)) \odot e_X(x, z) \odot e_Y(y, f(x))) \\
 &\leq \bigvee_{x \in X} \bigvee_{z \in X} (B(f(z)) \odot e_Y(f(x), f(z)) \odot e_Y(y, f(x))) \\
 &\leq \bigvee_{z \in X} (B(f(z)) \odot e_Y(y, f(z))) \\
 &\leq B(y).
 \end{aligned}$$

Hence $a \leq f_l^{\rightarrow}(A)(y) \rightarrow B(y)$. Thus, $e_{D(L^X)}(A, f_l^{\leftarrow}(B)) \leq e_{D(L^X)}(f_l^{\rightarrow}(A), B)$.

Put $B_0 = \sqcup(f_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}
 e_{D(L^X)}(B_0, B) &= \bigwedge_{C \in D(L^X)} ((f_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(C) \rightarrow e_{D(L^X)}(C, B)) \\
 &= \bigwedge_{C \in D(L^X)} (\bigvee_{f_l^{\rightarrow}(A)=C} \mathcal{U}(A) \rightarrow e_{D(L^X)}(f_l^{\rightarrow}(A), B)) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(f_l^{\rightarrow}(A), B)) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(A, f_l^{\leftarrow}(B))) \\
 &= e_{D(L^X)}(\sqcup \mathcal{U}, f_l^{\leftarrow}(B)) = e_{D(L^X)}(f_l^{\rightarrow}(\sqcup \mathcal{U}), B).
 \end{aligned}$$

Hence $B_0 = \sqcup(f_l^{\rightarrow})^{\rightarrow}(\mathcal{U}) = f_l^{\rightarrow}(\sqcup \mathcal{U})$.

(2) It is similarly proved as (1).

(3) $\sqcup f_l^{\rightarrow}(A) \leq f(\sqcup A)$ and $f(\sqcap A) \leq \sqcap f_u^{\rightarrow}(A)$ from:

$$\begin{aligned}
 e_{D(L^Y)}(\sqcup f_l^{\rightarrow}(A), f(\sqcup A)) &= \bigwedge_{y \in Y} (f_l^{\rightarrow}(A)(y) \rightarrow e_Y(y, f(\sqcup A))) \\
 &= \bigwedge_{y \in Y} (\bigvee_{x \in X} (A(x) \odot e_Y(y, f(x))) \rightarrow e_Y(y, f(\sqcup A))) \\
 &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (A(x) \rightarrow (e_Y(y, f(x)) \rightarrow e_Y(y, f(\sqcup A)))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{y \in Y} (e_Y(y, f(x)) \rightarrow e_Y(y, f(\sqcup A)))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow e_Y(f(x), f(\sqcup A))) \\
 &\geq \bigwedge_{x \in X} (A(x) \rightarrow e_X(x, \sqcup A)) \\
 &= e_X(\sqcup A, \sqcup A) = 1.
 \end{aligned}$$

$$\begin{aligned}
 e_{D(L^Y)}(f(\sqcap A), \sqcap f_u^{\rightarrow}(A)) &= \bigwedge_{y \in Y} (f_l^{\rightarrow}(A)(y) \rightarrow e_Y(f(\sqcap A), y)) \\
 &= \bigwedge_{y \in Y} (\bigvee_{x \in X} (A(x) \odot e_Y(f(x), y)) \rightarrow e_Y(f(\sqcap A), y)) \\
 &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (A(x) \rightarrow (e_Y(f(x), y) \rightarrow e_Y(f(\sqcap A), y))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{y \in Y} (e_Y(f(x), y) \rightarrow e_Y(f(\sqcap A), y))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow e_Y(f(\sqcap A), f(x))) \\
 &\geq \bigwedge_{x \in X} (A(x) \rightarrow e_X(\sqcap A, x)) \\
 &= e_X(\sqcap A, \sqcap A) = 1.
 \end{aligned}$$

(4) By Lemma 1.2(7,12), we have

$$\begin{aligned} f_l^{\rightarrow}(A)(y) &\rightarrow f_l^{\rightarrow}(B)(y) \\ &= \bigvee_{x \in X} (A(x) \odot e_Y(y, f(x))) \rightarrow \bigvee_{x \in X} (B(x) \odot e_Y(y, f(x))) \\ &\geq \bigwedge_{x \in X} ((A(x) \odot e_Y(y, f(x))) \rightarrow (B(x) \odot e_Y(y, f(x)))) \\ &\geq \bigwedge_{x \in X} (A(x) \rightarrow B(x)) \end{aligned}$$

Hence $e_{L^X}(A, B) \leq e_{L^Y}(f_l^{\rightarrow}(A), f_l^{\rightarrow}(B))$.

Other case and (5) are similarly proved.

Theorem 2.3. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $g : X \rightarrow Y$ be a surjective map with $e_X(x, y) = e_Y(g(y), g(x))$ for each $x, y \in X$. Define $g_l^{\rightarrow}, g_u^{\rightarrow} : L^X \rightarrow L^Y$ as

$$g_l^{\rightarrow}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow e_Y(y, g(x)))$$

$$g_u^{\rightarrow}(A)(y) = \bigwedge_{x \in X} (A(x) \rightarrow e_Y(g(x), y))$$

and $g_l^{\leftarrow}, g_u^{\leftarrow} : L^Y \rightarrow L^X$ as

$$g_l^{\leftarrow}(B)(x) = \bigwedge_{z \in X} (B(g(z)) \rightarrow e_X(x, z))$$

$$g_u^{\leftarrow}(B)(x) = \bigwedge_{z \in X} (B(g(z)) \rightarrow e_X(z, x))$$

Then the following properties hold.

(1) $(D(L^X), g_l^{\rightarrow}, g_l^{\leftarrow}, D(L^Y))$ is a Galois connection and

$$g_l^{\rightarrow}(\sqcup \mathcal{U}) = \sqcap (g_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$$

for all $\mathcal{U} \in L^{L^X}$ where $(g_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{g_l^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(2) $(U(L^X), g_u^{\rightarrow}, g_u^{\leftarrow}, U(L^Y))$ is a Galois connection and

$$g_u^{\rightarrow}(\sqcup \mathcal{U}) = \sqcap (g_u^{\rightarrow})^{\rightarrow}(\mathcal{U})$$

for all $\mathcal{U} \in L^{L^X}$ where $(g_u^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{g_u^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(3) $e_{L^X}(A, B) \leq e_{L^Y}(g_l^{\rightarrow}(B), g_l^{\rightarrow}(A))$ and $e_{L^X}(A, B) \leq e_{L^Y}(g_u^{\rightarrow}(B), g_u^{\rightarrow}(A))$.

(4) $e_{L^Y}(C, D) \leq e_{L^X}(g_l^{\leftarrow}(D), g_l^{\leftarrow}(C))$ and $e_{L^Y}(C, D) \leq e_{L^X}(g_u^{\leftarrow}(D), g_u^{\leftarrow}(C))$.

Proof. (1) Since $g_l^{\rightarrow}(A)(y_1) \odot e_Y(y_2, y_1) \odot A(x) \leq (A(x) \rightarrow e_Y(y_1, g(x))) \odot e_Y(y_2, y_1) \odot A(x) \leq e_Y(y_2, g(x))$, then $g_l^{\rightarrow}(A)(y_1) \odot e_Y(y_2, y_1) \leq g_l^{\rightarrow}(A)(y_2)$; i.e. $g_l^{\rightarrow}(A) \in D(L^Y)$. Similarly, for $B \in D(L^Y)$, we have $g_l^{\leftarrow}(B) \in D(L^X)$.

$$\begin{aligned}
 e_{D(L^Y)}(B, g_l^{\rightarrow}(A)) &= \bigwedge_{y \in Y} (B(y) \rightarrow g_l^{\rightarrow}(A)(y)) \\
 &= \bigwedge_{y \in Y} (B(y) \rightarrow \bigwedge_{x \in X} (A(x) \rightarrow e_Y(y, g(x)))) \\
 &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (A(x) \rightarrow (B(y) \rightarrow e_Y(y, g(x)))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{y \in Y} (B(y) \rightarrow e_Y(y, g(x)))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{z \in X} (B(g(z)) \rightarrow e_Y(g(z), g(x)))) \\
 &= \bigwedge_{x \in X} (A(x) \rightarrow \bigwedge_{z \in X} (B(g(z)) \rightarrow e_X(x, z))) \\
 &= e_{D(L^X)}(A, g_l^{\leftarrow}(B))
 \end{aligned}$$

Put $B_1 = \sqcap (g_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}
 e_{D(L^Y)}(B, B_0) &= \bigwedge_{C \in D(L^X)} ((g_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(C) \rightarrow e_{D(L^X)}(B, C)) \\
 &= \bigwedge_{C \in D(L^X)} (\bigvee_{g_l^{\rightarrow}(A)=C} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(B, g_l^{\rightarrow}(A)))) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(B, g_l^{\rightarrow}(A))) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(A, g_l^{\leftarrow}(B))) \\
 &= e_{D(L^X)}(\sqcup \mathcal{U}, g_l^{\leftarrow}(B)) = e_{D(L^X)}(B, g_l^{\rightarrow}(\sqcup \mathcal{U}))
 \end{aligned}$$

Hence $B_1 = \sqcap (g_l^{\rightarrow})^{\rightarrow}(\mathcal{U}) = g_l^{\rightarrow}(\sqcup \mathcal{U})$.

(2) It is similarly proved as (1).

(3) By Lemma 1.2(10,11), we have

$$\begin{aligned}
 g_l^{\rightarrow}(B)(y) &\rightarrow g_l^{\rightarrow}(A)(y) \\
 &= \bigwedge_{x \in X} (B(x) \rightarrow e_Y(y, g(x))) \rightarrow \bigwedge_{x \in X} (A(x) \rightarrow e_Y(y, g(x))) \\
 &\geq \bigwedge_{x \in X} ((B(x) \rightarrow e_Y(y, g(x))) \rightarrow (A(x) \rightarrow e_Y(y, g(x)))) \\
 &\geq \bigwedge_{x \in X} (A(x) \rightarrow B(x))
 \end{aligned}$$

Other case and (4) are similarly proved.

Theorem 2.4. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $h : X \rightarrow Y$ be a surjective map with $e_X(x, y) = e_Y(h(y), h(x))$ for each $x, y \in X$. Define $h_l^{\rightarrow}, h_u^{\rightarrow} : L^X \rightarrow L^Y$ as

$$h_l^{\rightarrow}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot e_Y(y, h(x))),$$

$$h_u^{\rightarrow}(A)(y) = \bigvee_{x \in X} (A^*(x) \odot e_Y(h(x), y)),$$

and $h_l^{\leftarrow}, h_u^{\leftarrow} : L^Y \rightarrow L^X$ as

$$h_l^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B^*(h(z)) \odot e_X(x, z)),$$

$$h_u^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B^*(h(z)) \odot e_X(z, x)).$$

Then the following properties hold.

(1) $(D(L^X), h_l^{\rightarrow}, h_l^{\leftarrow}, D(L^Y))$ is a dual Galois connection and $h_l^{\rightarrow}(\sqcap \mathcal{U}) = \sqcap(h_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(h_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{h_l^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(2) $(U(L^X), h_u^{\rightarrow}, h_u^{\leftarrow}, U(L^Y))$ is a dual Galois connection and $h_u^{\rightarrow}(\sqcap \mathcal{U}) = \sqcap(h_u^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(h_u^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{h_u^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(3) $e_{L^X}(A, B) \leq e_{L^Y}(h_l^{\rightarrow}(B), h_l^{\rightarrow}(A))$ and $e_{L^X}(A, B) \leq e_{L^Y}(h_u^{\rightarrow}(B), h_u^{\rightarrow}(A))$.

(4) $e_{L^Y}(C, D) \leq e_{L^X}(h_l^{\leftarrow}(D), h_l^{\leftarrow}(C))$ and $e_{L^Y}(C, D) \leq e_{L^X}(h_u^{\leftarrow}(D), h_u^{\leftarrow}(C))$.

Proof. (1) For $A \in D(L^X)$, since

$$\begin{aligned} & h_l^{\rightarrow}(A)(y_1) \odot e_Y(y_2, y_1) \\ &= \bigvee_{x \in X} (A^*(x) \odot e_Y(y_1, f(x))) \odot e_Y(y_2, y_1) \\ &\leq \bigvee_{x \in X} (A^*(x) \odot e_Y(y_2, f(x))) \end{aligned}$$

then $h_l^{\rightarrow}(A) \in D(L^Y)$. Similarly, for $B \in D(L^Y)$, we have $h_l^{\leftarrow}(B) \in D(L^X)$.

$$\begin{aligned} e_{D(L^X)}(h_l^{\rightarrow}(A), B) &= \bigwedge_{y \in Y} (h_l^{\rightarrow}(A)(y) \rightarrow B(y)) \\ &= \bigwedge_{y \in Y} (\bigvee_{x \in X} (A^*(x) \odot e_Y(y, h(x))) \rightarrow B(y)) \\ &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (e_Y(y, h(x)) \rightarrow (A^*(x) \rightarrow B(y))) \\ &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (e_Y(y, h(x)) \rightarrow (B^*(y) \rightarrow A(x))) \\ &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (e_Y(y, h(x)) \odot B^*(y) \rightarrow A(x)) \\ &\leq \bigwedge_{x \in X} (\bigvee_{z \in X} e_Y(h(z), h(x)) \odot B^*(h(z)) \rightarrow A(x)) \\ &\leq \bigwedge_{x \in X} (\bigvee_{z \in X} e_X(x, z) \odot B^*(h(z)) \rightarrow A(x)) \\ &= e_{D(L^X)}(h_l^{\leftarrow}(B), A) \end{aligned}$$

$$\begin{aligned} e_{D(L^X)}(h_l^{\leftarrow}(B), A) &= \bigwedge_{x \in X} (h_l^{\leftarrow}(B)(x) \rightarrow A(x)) \\ &= \bigwedge_{x \in X} (\bigvee_{z \in X} (B^*(h(z)) \odot e_X(x, z)) \rightarrow A(x)) \\ &= \bigwedge_{x \in X} \bigwedge_{z \in X} (e_X(x, z) \rightarrow (B^*(h(z)) \rightarrow A(x))) \\ &= \bigwedge_{x \in X} \bigwedge_{z \in X} (e_X(x, z) \rightarrow (A(x) \rightarrow B(h(z)))) \\ &= \bigwedge_{x \in X} \bigwedge_{z \in X} (e_X(x, z) \odot A^*(x) \rightarrow B(h(z))) \\ &= \bigwedge_{x \in X} \bigwedge_{z \in X} (e_Y(h(z), h(x)) \odot A^*(x) \rightarrow B(h(z))) \\ &= e_{D(L^X)}(h_l^{\leftarrow}(B), A) \end{aligned}$$

Put $B_0 = \sqcup(h_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}
 e_{D(L^X)}(B_0, B) &= \bigwedge_{C \in D(L^X)} ((h_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(C) \rightarrow e_{D(L^X)}(C, B)) \\
 &= \bigwedge_{C \in D(L^X)} (\bigvee_{h_l^{\rightarrow}(A)=C} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(h_l^{\rightarrow}(A), B))) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(h_l^{\rightarrow}(A), B)) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{U(L^X)}(h_l^{\leftarrow}(B), A)) \\
 &= e_{D(L^X)}(h_l^{\leftarrow}(B), \sqcap \mathcal{U}) = e_{U(L^X)}(h_l^{\rightarrow}(\sqcap \mathcal{U}), B)
 \end{aligned}$$

Hence $B_0 = \sqcup(h_l^{\rightarrow})^{\rightarrow}(\mathcal{U}) = h_l^{\rightarrow}(\sqcap \mathcal{U})$.

(2) It is similarly proved as (1).

(3) By Lemma 1.2(6,12), we have

$$\begin{aligned}
 h_l^{\rightarrow}(B)(y) &\rightarrow h_l^{\rightarrow}(A)(y) \\
 &= \bigvee_{x \in X} (B^*(x) \odot e_Y(y, h(x))) \rightarrow \bigvee_{x \in X} (A^*(x) \odot e_Y(y, g(x))) \\
 &\geq \bigwedge_{x \in X} ((B^*(x) \odot e_Y(y, g(x))) \rightarrow (A^*(x) \odot e_Y(y, g(x)))) \\
 &\geq \bigwedge_{x \in X} (B^*(x) \rightarrow A^*(x)) = \bigwedge_{x \in X} (A(x) \rightarrow B(x))
 \end{aligned}$$

Other case and (4) are similarly proved.

Theorem 2.5. Let (X, e_X) and (Y, e_Y) be fuzzy posets and $k : X \rightarrow Y$ be a surjective map with $e_X(x, y) = e_Y(k(y), k(x))$ for each $x, y \in X$. Define $k_l^{\rightarrow}, k_u^{\rightarrow} : L^X \rightarrow L^Y$ as

$$k_l^{\rightarrow}(A)(y) = \bigwedge_{x \in X} (e_Y(y, f(x)) \rightarrow A(x)),$$

$$k_u^{\rightarrow}(A)(y) = \bigwedge_{x \in X} (e_Y(f(x), y) \rightarrow A(x)),$$

and $k_l^{\leftarrow}, k_u^{\leftarrow} : L^Y \rightarrow L^X$ as

$$k_l^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B(f(z)) \odot e_X(x, z)),$$

$$k_u^{\leftarrow}(B)(x) = \bigvee_{z \in X} (B(f(z)) \odot e_X(z, x)).$$

Then the following properties hold.

(1) $(D(L^X), k_l^{\rightarrow}, k_l^{\leftarrow}, U(L^Y))$ is a dual residuated connection and $k_l^{\rightarrow}(\sqcap \mathcal{U}) = \sqcap(k_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(k_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{k_l^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(2) $(U(L^X), k_u^{\rightarrow}, k_u^{\leftarrow}, D(L^Y))$ is a dual residuated connection and $k_u^{\rightarrow}(\sqcap \mathcal{U}) = \sqcap(k_u^{\rightarrow})^{\rightarrow}(\mathcal{U})$ for all $\mathcal{U} \in L^{L^X}$ where $(k_u^{\rightarrow})^{\rightarrow}(\mathcal{U})(B) = \bigvee_{k_u^{\rightarrow}(A)=B} \mathcal{U}(A)$.

(3) $e_{L^X}(A, B) \leq e_{L^Y}(f_l^{\rightarrow}(A), f_l^{\rightarrow}(B))$ and $e_{L^X}(A, B) \leq e_{L^Y}(f_u^{\rightarrow}(A), f_u^{\rightarrow}(B))$.

(4) $e_{L^Y}(C, D) \leq e_{L^X}(f_l^{\leftarrow}(C), f_l^{\leftarrow}(D))$ and $e_{L^Y}(A, B) \leq e_{L^X}(f_u^{\leftarrow}(C), f_u^{\leftarrow}(D))$.

Proof. (1) Since $k_l^{\rightarrow}(A)(y_1) \odot e_Y(y_1, y_2) \odot e_Y(y_2, f(x)) \leq (e_Y(y_1, f(x)) \rightarrow A(x)) \odot e_Y(y_1, f(x)) \leq A(x)$ iff $k_l^{\rightarrow}(A)(y_1) \odot e_Y(y_1, y_2) \leq k_l^{\rightarrow}(A)(y_2)$, then $k_l^{\rightarrow}(A) \in U(L^Y)$. Similarly, $k_l^{\leftarrow}(B) \in D(L^X)$.

$$\begin{aligned}
 e_{D(L^X)}(B, k_l^{\rightarrow}(A)) &= \bigwedge_{y \in Y} (B(y) \rightarrow k_l^{\rightarrow}(A)(y)) \\
 &= \bigwedge_{y \in Y} (B(y) \rightarrow \bigwedge_{x \in X} (e_Y(y, k(x)) \rightarrow A(x))) \\
 &= \bigwedge_{y \in Y} \bigwedge_{x \in X} (B(y) \odot e_Y(y, k(x)) \rightarrow A(x)) \\
 &= \bigwedge_{x \in X} (\bigvee_{y \in Y} B(y) \odot e_Y(y, k(x)) \rightarrow A(x)) \\
 &= \bigwedge_{x \in X} (\bigvee_{z \in X} B(k(z)) \odot e_Y(k(z), k(x)) \rightarrow A(x)) \\
 &= \bigwedge_{x \in X} (\bigvee_{z \in X} B(k(z)) \odot e_X(x, z) \rightarrow A(x)) \\
 &= e_{U(L^X)}(k_l^{\leftarrow}(B), A)
 \end{aligned}$$

Put $B_1 = \sqcap(k_l^{\rightarrow})^{\rightarrow}(\mathcal{U})$. Then

$$\begin{aligned}
 e_{U(L^Y)}(B, B_1) &= \bigwedge_{C \in U(L^Y)} ((k_l^{\rightarrow})^{\rightarrow}(\mathcal{U})(C) \rightarrow e_{U(L^Y)}(B, C)) \\
 &= \bigwedge_{C \in U(L^Y)} (\bigvee_{g_l^{\rightarrow}(A)=C} (\mathcal{U}(A) \rightarrow e_{U(L^Y)}(B, k_l^{\rightarrow}(A))) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{U(L^Y)}(B, k_l^{\rightarrow}(A))) \\
 &= \bigwedge_{A \in D(L^X)} (\mathcal{U}(A) \rightarrow e_{D(L^X)}(k_l^{\leftarrow}(B), A)) \\
 &= e_{D(L^X)}(k_l^{\leftarrow}(B), \sqcap \mathcal{U}) = e_{U(L^Y)}(B, k_l^{\rightarrow}(\sqcap \mathcal{U}))
 \end{aligned}$$

Hence $B_1 = \sqcap(k_l^{\rightarrow})^{\rightarrow}(\mathcal{U}) = k_l^{\rightarrow}(\sqcap \mathcal{U})$.

(2) It is similarly proved as (1).

(3) By Lemma 1.2(10,11), we have

$$\begin{aligned}
 &k_l^{\rightarrow}(A)(y) \rightarrow k_l^{\rightarrow}(B)(y) \\
 &= \bigwedge_{x \in X} (e_Y(y, f(x)) \rightarrow A(x)) \rightarrow \bigwedge_{x \in X} (e_Y(y, f(x)) \rightarrow B(x)) \\
 &\geq \bigwedge_{x \in X} ((e_Y(y, f(x)) \rightarrow A(x)) \rightarrow (e_Y(y, f(x)) \rightarrow B(x))) \\
 &\geq \bigwedge_{x \in X} (A(x) \rightarrow B(x))
 \end{aligned}$$

Other case and (4) are similarly proved.

Example 2.6. Define a binary operation $\odot$ (called Łukasiewicz conjunction) on $L = [0, 1]$ by

$$x \odot y = \max\{0, x + y - 1\}, \quad x \rightarrow y = \min\{1 - x + y, 1\}.$$

(1) Let $(X = \{a, b, c\}, e_X)$ and $(Y = \{x, y, z\}, e_Y)$ be a fuzzy poset with $e_X = (e_X(a, b))$ and $e_Y = (e_Y(x, y))$ as follows:

$$e_X = \begin{pmatrix} 1.0 & 0.7 & 0.4 \\ 0.3 & 1.0 & 0.6 \\ 0.5 & 0.5 & 1.0 \end{pmatrix} \quad e_Y = \begin{pmatrix} 1.0 & 0.8 & 0.6 \\ 0.6 & 1.0 & 0.5 \\ 0.7 & 0.6 & 1.0 \end{pmatrix}$$

We define $f : X \rightarrow Y$ with $f(a) = x, f(b) = f(c) = y$. Then f is an isotone map. It satisfies the conditions of Theorem 2.2. Thus, the properties of Theorem hold.

(2) Let $(X = \{a, b, c\}, e_X)$ and $(Y = \{x, y, z\}, e_Y)$ be a fuzzy poset with $e_X = (e_X(a, b))$ and $e_Y = (e_Y(x, y))$ as follows:

$$e_X = \begin{pmatrix} 1.0 & 0.8 & 0.6 \\ 0.6 & 1.0 & 0.5 \\ 0.7 & 0.6 & 1.0 \end{pmatrix} \quad e_Y = \begin{pmatrix} 1.0 & 0.8 & 0.6 \\ 0.6 & 1.0 & 0.7 \\ 0.5 & 0.6 & 1.0 \end{pmatrix}$$

We define $g : X \rightarrow Y$ with $g(a) = y, g(b) = x, g(c) = z$. Then $e_X(a, b) = e_Y(g(b), g(a))$ for all $a, b \in X$. It satisfies the conditions of Theorems 2.3, 2.4 and 2.5. Thus, the properties of Theorems 2.3, 2.4 and 2.5 hold.

References

- [1] R. Bělohlávek, Fuzzy Galois connections, *Math. Log. Quart.*, **45** (1999), 497-504.
- [2] R. Bělohlávek, Concept lattices and order in fuzzy logic, *Ann. Pure Appl. Logic*, **128**(2004), 277-298.
- [3] R. Bělohlávek, Lattices of fixed points of Galois connections, *Math. Logic Quart.*, **47** (2001), 111-116.
- [4] J.G. Garcia, I.M. Perez, M.A. Vicente, D. Zhang, Fuzzy Galois connections categorically, *Math. Log. Quart.*, **56**(2010), 131-147.

- [5] G. Georgescu, A. Popescue, Non-commutative Galois connections, *Soft Computing*, **7**(2003), 458-467.
- [6] G. Georgescu, A. Popescue, Non-dual fuzzy connections, *Arch. Math. Log.* **43**(2004), 1009-1039.
- [7] P. Hájek, *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, Dordrecht (1998).
- [8] H. Lai, D. Zhang, Complete and directed complete Ω -categories, *Theoretical Computer Science*, **388** (2007), 1-25.
- [9] H. Lai, D. Zhang, Concept lattices of fuzzy contexts, *Int. J. Approx. Reasoning*, **50**(2009), 695-707.
- [10] W. Yao, L.X. Lu, Fuzzy Galois connections on fuzzy posets, *Math. Log. Quart.*, **55**(2009), 105-112.
- [11] Q. Y. Zhang, L. Fan, Continuity in quantitative domains, *Fuzzy Sets and Systems*, **154**(2005), 118-131.
- [12] Q. Y. Zhang, W. X. Xie, Fuzzy complete lattices, *Fuzzy Sets and Systems*, **160** (2009), 2275-2291.