ON COMPLETE α -IDEALS IN ORDERED SEMIGROUPS

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Abstract: In this note, we introduce and study α -ideals and complete α -ideals in ordered semigroups (po-semigroups). The results obtained generalize the results on semigroups without order.

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1. Preliminaries

Let S be a semigroup. A left (respectively, right, two-sided) ideal A of S is said to be *complete* if $SA = A$ (respectively, $AS = A$, $SA = AS = A$). This was introduced and studied by Fabrici [3]. Catino [2] generalized the results by using the concepts which are called α -ideals and *complete α -ideals*. The purpose of this paper is to define and study α -ideals and complete α -ideals in ordered semigroups. The results of the authors mentioned before become then special cases.

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Now, we recall some certain definitions and results used throughout the paper.

Let $(S, \cdot)$ be a semigroup. If A, B are non-empty subsets of S , we let

$$AB = \{xy \mid x \in A, y \in B\}.$$

The power set of S , that is the set of all subsets of the set S , will be denoted by $P(S)$. If X is a set, then the free monoid over an alphabet X will be denoted by X^* . For $\alpha \in \{0, 1\}^*$, define $f_\alpha^S : P(S) \rightarrow P(S)$ as follows: $f_\alpha^S(A) = \emptyset$ if α is the empty word and

$$f_\alpha^S(A) = A_1 A_2 \cdots A_n$$

if $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$, $\alpha_i \in \{0, 1\}$, where

$$A_i = \begin{cases} A & \text{if } \alpha_i = 1; \\ S & \text{if } \alpha_i = 0. \end{cases}$$

If $A = \{a\}$, we write $f_\alpha^S(a)$ instead of $f_\alpha^S(\{a\})$.

Hereafter, we let $\Lambda = \{0, 1\}^* \setminus \{1\}$ and let $\alpha \in \Lambda$ (Hence α is not the empty word.). If $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$ where $\alpha_i \in \{0, 1\}$, then the *length* of α , denoted by $l(\alpha)$, is n .

Lemma 1. ([2], p.155) *Let $(S, \cdot)$ be a semigroup, and let A, B be non-empty subsets of S . If $\alpha, \beta \in \Lambda$, then the following conditions hold:*

- (1) *if $A \subseteq B$ then $f_\alpha^S(A) \subseteq f_\alpha^S(B)$;*
- (2) *$f_{\alpha\beta}^S(A) = f_\alpha^S(A)f_\beta^S(A)$;*
- (3) *$Af_\alpha^S(A) \subseteq f_\alpha^S(A)$, $f_\alpha^S(A)A \subseteq f_\alpha^S(A)$;*
- (4) *$f_\alpha^S(A)f_\alpha^S(A) \subseteq f_\alpha^S(A)$;*
- (5) *$f_\alpha^S(A \cup f_\alpha^S(A)) \subseteq f_\alpha^S(A)$;*
- (6) *$f_\alpha^S(A \cup A^2 \cup \cdots \cup A^k \cup f_\alpha^S(A)) \subseteq f_\alpha^S(A)$ for any positive integer k .*

A semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is *compatible* with the semigroup operation, meaning that for $x, y, z \in S$,

$$x \leq y \Rightarrow zx \leq zy, \quad xz \leq yz,$$

is called an *ordered semigroup* [1]. If A is a non-empty subset of S , we let

$$[A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

If A, B are non-empty subsets of an ordered semigroup $(S, \cdot, \leq)$, then the following conditions hold [4]:

- (1) $A \subseteq [A]$;
- (2) $A \subseteq B \Rightarrow [A] \subseteq [B]$;
- (3) $[A][B] \subseteq [AB]$;
- (4) $[A \cup B] = [A] \cup [B]$;
- (5) $[[A]] = [A]$.

By a *subsemigroup* of an ordered semigroup $(S, \cdot, \leq)$ we mean a non-empty subset A of S such that $AA \subseteq A$, i.e. $xy \in A$ for all $x, y \in A$. Notice that if A is a subsemigroup of S , then $[A]$ is a subsemigroup of S . Indeed: $[A][A] \subseteq [AA] \subseteq [A]$.

Lemma 2. *Let $(S, \cdot, \leq)$ be an ordered semigroup, and let A be a non-empty subset of S . If $\alpha \in \Lambda$, then*

$$f_\alpha^S([A]) \subseteq (f_\alpha^S(A)).$$

Proof. We will proceed by induction on the length of α . This is easy to see if $l(\alpha) = 1$. Suppose that the condition holds for all word $\beta \in \Lambda$ such that $l(\beta) < n$. There are two cases to consider:

Case 1: $\alpha = 1\beta$. Then $\beta \in \Lambda$, and so, by Lemma 1, we have

$$f_\alpha^S([A]) = [A]f_\beta^S([A]) \subseteq [A](f_\beta^S(A)) \subseteq (Af_\beta^S(A)) = (f_\alpha^S(A)).$$

Case 2: $\alpha = 0\beta$. If $\beta \in \Lambda$, in the same manners as Case 1, we have the condition holds. Suppose that $\beta \notin \Lambda$. Then $\alpha = \gamma 1$ for some $\gamma \in \Lambda$, and thus

$$f_\alpha^S([A]) = f_\gamma^S([A])(A) \subseteq (f_\gamma^S(A))(A) \subseteq (f_\gamma^S(A)A) = (f_\alpha^S(A)).$$

This completes the proof. □

2. Complete α -Ideals of Ordered Semigroups

We introduce the following two main concepts of this paper:

Definition 3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A subsemigroup A of S is called an α -ideal of S if

- (i) $f_\alpha^S(A) \subseteq A$;
- (ii) $[A] = A$, i.e. if $x \in A$ and $y \in S$ such that $y \leq x$ then $y \in A$.

Definition 4. Let $(S, \cdot, \leq)$ be an ordered semigroup. An α -ideal A of S is said to be *complete* if $(f_\alpha^S(A)) = A$.

Theorem 5. Let $(S, \cdot, \leq)$ be an ordered semigroup. If A is a non-empty subset of S , then

$$(A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A))$$

is the smallest α -ideal of S containing A .

Proof. Assume that A is a non-empty subset of S . Then, by Lemma 1, we have

$$\begin{aligned} & (A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A))(A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A)) \\ \subseteq & ((A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A))(A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A))) \\ = & ((A^2 \cup A^3 \cup \dots \cup A^{l(\alpha)} \cup A f_\alpha^S(A)) \cup \\ & (A^3 \cup A^4 \cup \dots \cup A^{l(\alpha)+1} \cup A^2 f_\alpha^S(A)) \cup \dots \cup \\ & (f_\alpha^S(A)A \cup f_\alpha^S(A)A^2 \cup \dots \cup f_\alpha^S(A)A^{l(\alpha)-1} \cup f_\alpha^S(A)f_\alpha^S(A))) \\ \subseteq & (A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A)). \end{aligned}$$

Therefore $(A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A))$ is a subsemigroup of S , and so, by Lemma 1(6) and Lemma 2, this is an α -ideal of S containing A . If B is an α -ideal of S containing A , by Lemma 1 (1), then $f_\alpha^S(A) \subseteq f_\alpha^S(B) \subseteq B$. Hence $(A \cup A^2 \cup \dots \cup A^{l(\alpha)-1} \cup f_\alpha^S(A)) \subseteq B$. □

Theorem 6. Let $(S, \cdot, \leq)$ be an ordered semigroup. Then every α -ideal of S is complete if and only if $a \in (f_\alpha^S(a))$ for all $a \in S$.

Proof. Assume that every α -ideal of S is complete. If $a \in S$, by Theorem 5, Lemma 1(6) and Lemma 2, then

$$a \in (\{a\} \cup \{a^2\} \cup \dots \cup \{a^{l(\alpha)-1}\} \cup f_\alpha^S(a))$$

$$\begin{aligned}
&= (f_\alpha^S((\{a\} \cup \{a^2\} \cup \dots \cup \{a^{l(\alpha)-1}\} \cup f_\alpha^S(a))) \\
&\subseteq (f_\alpha^S(a)].
\end{aligned}$$

Conversely, assume that $a \in (f_\alpha^S(a)]$ for all $a \in S$. Let A be an α -ideal of S . If $a \in A$, by Lemma 1, then $a \in (f_\alpha^S(a)] \subseteq (f_\alpha^S(A)]$, and so A is complete. $\square$

Theorem 7. *Let $(S, \cdot, \leq)$ be an ordered semigroup. If A, B are complete α -ideals of S , then*

$$((A \cup B) \cup (A \cup B)^2 \cup \dots \cup (A \cup B)^{l(\alpha)-1} \cup f_\alpha^S(A \cup B)],$$

the smallest α -ideal of S containing $A \cup B$, is complete.

Proof. Assume that A, B are complete α -ideals of S . That is $(f_\alpha^S(A)] = A$ and $(f_\alpha^S(B)] = B$. We have

$$A \cup B = (f_\alpha^S(A)] \cup (f_\alpha^S(B)] = (f_\alpha^S(A) \cup f_\alpha^S(B)] \subseteq (f_\alpha^S(A \cup B)],$$

and thus $(A \cup B)^k \subseteq (f_\alpha^S(A \cup B)]$ for all positive integer k since Lemma 1. Hence, by Lemma 2,

$$\begin{aligned}
&((A \cup B) \cup (A \cup B)^2 \cup \dots \cup (A \cup B)^{l(\alpha)-1} \cup f_\alpha^S(A \cup B)] \\
&\subseteq (f_\alpha^S(A \cup B)] \\
&\subseteq (f_\alpha^S((A \cup B) \cup (A \cup B)^2 \cup \dots \cup (A \cup B)^{l(\alpha)-1} \cup f_\alpha^S(A \cup B))) \\
&\subseteq (f_\alpha^S(((A \cup B) \cup (A \cup B)^2 \cup \dots \cup (A \cup B)^{l(\alpha)-1} \cup f_\alpha^S(A \cup B)))).
\end{aligned}$$

$\square$

Define α as follows:

- (i) if $\alpha = 0^n$ for some positive integer n then $\alpha = 0$;
- (ii) if $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$ such that $\alpha_j \neq \alpha_1$ for some $j > 1$, then

$$\alpha = \begin{cases} \alpha_1 \alpha_j \alpha_n & \text{if } \alpha_n \neq \alpha_j; \\ \alpha_1 \alpha_j & \text{if } \alpha_n = \alpha_j. \end{cases}$$

Then $\alpha \in \{0, 10, 01, 101, 010\}$.

Using Lemmas 6, 7 and 8 in [2], we obtain Lemmas 8, 9 and 10, respectively.

Lemma 8. *Let $(S, \cdot, \leq)$ be an ordered semigroup. If A, B are complete α -ideals of S , then $(f_\alpha^S(AB)] \subseteq (f_{\alpha^*}^S(AB)]$.*

Lemma 9. Let $(S, \cdot, \leq)$ be an ordered semigroup. If A, B are complete α -ideals of S and $\alpha \neq 0$, then $(f_{\alpha^*}^S(AB)) \subseteq (AB)$.

Lemma 10. Let $(S, \cdot, \leq)$ be an ordered semigroup, and let A be a complete α -ideal of S . If $\alpha \neq 0$, then $(f_{\alpha^*}^S(A)) = A$.

Now, we prove the following result:

Theorem 11. Let $(S, \cdot, \leq)$ be an ordered semigroup, and let A, B be complete α -ideals of S . If $\alpha \neq 0$, then (AB) is an α -ideal of S .

Proof. Assume that $\alpha \neq 0$, then $\alpha \in \{10, 01, 101, 010\}$. Suppose that $\alpha = 010$. By Lemma 10, we have (AB) is a subsemigroup of S . Indeed: $(AB)(AB) \subseteq (ABAB) = (ABA(SBS)) \subseteq (ASA(SBS)) \subseteq (A(SASBS)) \subseteq (A(SBS)) = (AB)$. Similarly, if $\alpha \in \{10, 01, 101\}$, then (AB) is a subsemigroup of S . By Lemmas 2, 8, 9, it follows that

$$f_{\alpha}^S((AB)) \subseteq (f_{\alpha}^S(AB)) \subseteq (f_{\alpha^*}^S(AB)) \subseteq (AB).$$

Therefore, (AB) is an α -ideal of S . □

References

- [1] G. Birkhoff, *Lattice Theory*, Volume 25, Providence, RI, Amer. Math. Soc., Coll. Publ. (1967).
- [2] F. Catino, On complete α -ideals in semigroups, *Czechoslovak Mathematical Journal*, **40** (1990), 155-158.
- [3] I. Fabrici, On complete ideals in semigroups, *Mathematica Slovaca*, **18** (1968), 34-39.
- [4] N. Kehayopulu, M. Tsingelis, On left regular ordered semigroups, *Southeast Asian Bulletin of Mathematics*, **25** (2002), 609-615.