THE DIRICHLET SERIES FOR
POWERS OF MAPS ON NATURAL NUMBERS

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Abstract: In this paper we study the Dirichlet series of natural powers of
maps with $O_n(T) = n^a$, $n \in \mathbb{N}$ and $a$ is a nonnegative integer and we also find
the abscissa of convergence of Dirichlet series.

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vergence

1. Introduction

A Dirichlet series is any series of the form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $s$ and $a_n$ are complex numbers, $n = 1, 2, 3, \cdots$. If $a_n = 1$ for all $n$ then
the Dirichlet series is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is the Riemann zeta function, see details in [1].
\[ d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s} \]
if \( T \) is a continuous mapping \( T : X \to X \), when \( X \) is a compact metric space and \( O_n(T) = n^a \) for all \( n \geq 1 \) and for some integer \( a \geq 0 \).

In 2011, Pakapongpun, A. studied orbit Dirichlet series of a prime power of maps \( d_{T^p}(s) \) where \( O_n(T) = n^a \) and \( p \) is a prime number, see all detail in [2].

In 2012, Rakporn Dokchan and Apisit Pakapongpun studied the special cases of the number theoretic of Dirichlet series, the detail is in [4].

In this paper has been improved to the general form of Dirichlet series for powers of maps \( d_{T^m}(T) \) where \( m \) is a positive integer.

## 2. Preliminary Notes

**Definition 2.1.** Let \( T \) be a map. A *closed orbit* \( \tau \) of length \( |\tau| \) is a set of the form
\[ \{x, Tx, T^2x, \ldots, T^{|\tau|}x = x\} \]
with cardinality \( |\tau| \). The number of points of period \( n \) is
\[ F_n(T) = \sum_{d|n} dO_d(T), \]
where \( O_n(T) \) is the number of closed orbits of length \( n \) under \( T \), and assume \( O_n(T) < \infty \) for all \( n \geq 1 \). From Möbius inversion formula
\[ O_n(T) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right)F_d(T) \]
which is \( \mu : \mathbb{N} \to \mathbb{R} \) is the Möbius function.

The orbits of length \( n \) under the iterate \( T^p \) for some prime \( p \) and \( T^m \) for integer \( m \) are shown in the next two theorem, see more detail in [1].

The number of orbits of length \( n \) under an iterate \( T^p \) for some prime \( p \) will be shown in the next theorem, see the proof theorem 3.1 in [5].

**Theorem 2.2.** Let \( p \) be a prime. Then
\[ O_n(T^p) = \left\{ \begin{array}{ll}
pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\
pO_{pn}(T) & \text{if } p \mid n.
\end{array} \right. \]
Theorem 2.3. Let \( m = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t} \) and \( S = \{p_1, p_2, \ldots, p_t\} \), let \( D_n \) be the set of \( p \) such that \( p \in S \) and \( p_i \mid n \) for each \( n \). Define \( r_p \) is the maximum power \( k \) such that \( p^k \mid m \). If \( O_n(T) = n^k, p_1 p_2 \cdots p_j \nmid n \) and \( p_{j+1} p_{j+2} \cdots p_t \nmid n \) then

\[
O_n(T^{p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t}}) = \sum_{0 \leq i_1 \leq r_1} p_1^{i_1} \cdots \sum_{0 \leq i_j \leq r_j} p_j^{i_j} O_n p_1^{i_1} \cdots p_j^{i_j} p_{j+1}^{r_{j+1}} \cdots p_t^{r_t}(T)
\]

see detail the proof theorem 3.2 in [5].

3. Main Results

Theorem 3.1. Let \( m = p_1^{r_1} p_2^{r_2} \cdots p_t^{r_t} \) and \( S = \{p_1, p_2, \ldots, p_t\} \). For each \( n \) let \( D_n \) be the set of \( p \) such that \( p \in S \) and \( p_i \mid n \). Define \( r_p \) is the maximum power \( k \) such that \( p^k \mid m \). If \( O_n(T) = n^k \) then

\[
d_T(n) = \zeta(s-k) \prod_p \left(1 + p^{2(k+1)} + \cdots + p^{(k+1)(r_p-1)}\right) \left(1 - \frac{1}{p^{s-k}}\right) + p^{(k+1)r_p}
\]

and the abscissa of convergence \( d_T(n) \) is \( k + 1 \).

Proof. From theorem 2.3, we have

\[
\sum_{n=1}^{\infty} \frac{O_n(T^m)}{n^s} = \sum_{n=1}^{\infty} \frac{\left(\prod_{p \in D_n} p^{r_p}\right) \sum_{p \in S-D_p} \left(\prod_{p \in S-D_n} p^{i_p}\right) O_n(\prod_{p \in S-D_n} p^{i_p}) (\prod_{p \in D_n})^r(T)}{n^s} = \sum_{n=1}^{\infty} \frac{\left(\prod_{p \in D_n} p^{r_p}\right) \sum_{p \in S-D_p} \left(\prod_{p \in S-D_n} p^{i_p}\right) \prod_{p \in S-D_n} p^{k_{ip}} \prod_{p \in D_n} p^{k_{rp}}}{n^{s-k}} = \sum_{n=1}^{\infty} \frac{\sum_{p \in S-D_n} \prod_{p \in S-D_n} p^{(k+1)ip} \prod_{p \in D_n} p^{(k+1)r_p}}{n^{s-k}} = \sum_{n=1}^{\infty} \frac{\prod_{p \in S-D_n} (1 + p^{2(k+1)} + \cdots + p^{(k+1)r_p}) \prod_{p \in D_n} p^{(k+1)r_p}}{n^{s-k}}
\]
\[
\prod_{p \notin S} \left( 1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \ldots \right)
\]

\[
\prod_{p \in S} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)r_p} \right) + \frac{p^{(k+1)r_p}}{p^{s-k}} + \frac{p^{(k+1)r_p}}{p^{2(s-k)}} + \ldots
\]

\[
\prod_{p \notin S} \left( 1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \ldots \right)
\]

\[
\prod_{p \in S} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)(r_p-1)} \right) + p^{(k+1)r_p} \left( 1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \ldots \right)
\]

Now consider the product above. Let \( T \subset S \). The expression above is equal to

\[
\sum_{T \subset S} \prod_{p \in T} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)(r_p-1)} \right)
\]

\[
\prod_{p \in S-T} p^{(k+1)r_p} \prod_{p} \left( 1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \ldots \right)
\]

\[
\prod_{p \in S-T} p^{(k+1)r_p} \prod_{p} \left( 1 + \frac{1}{p^{s-k}} + \frac{1}{p^{2(s-k)}} + \ldots \right)
\]

\[
= \zeta(s-k) \sum_{T \subset S} \prod_{p \in T} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)(r_p-1)} \right) \left( 1 - \frac{1}{p^{s-k}} \right) \prod_{p \in S-T} p^{(k+1)r_p}
\]

\[
= \zeta(s-k) \prod_{p \in S} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)(r_p-1)} \right) \left( 1 - \frac{1}{p^{s-k}} \right) + p^{(k+1)r_p}
\]

Here \( \prod_{p \in S} \left( 1 + p^{2(k+1)} + \ldots + p^{(k+1)(r_p-1)} \right) \left( 1 - \frac{1}{p^{s-k}} \right) + p^{(k+1)r_p} \) is absolutely convergent for any value \( s \). So, the abscissa of convergence \( d_{T^n} \) is \( s = k+1 \). \( \square \)

**Example 3.2.** Let \( d_T(s) = \zeta(s-2) = \sum_{n=1}^{\infty} \frac{1}{n^{s-2}} \). Then

\[
d_{T^{10}}(s) = \zeta(s-2)(9 - \frac{1}{2^{s-2}})(123 - \frac{1}{5^{s-2}}),
\]

it is obvious, the abscissa of convergence \( d_{T^{10}} \) is 3.
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References


